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Exercise Sheet 4

Applied Analysis

Discussion on Thursday 14-11-2013 at 16ct

Exercise 1 (*Compactness everywhere*) (3+1+1+1+3+3+1+1+3+3)

Which of the following subsets A are compact in the metric spaces (M,d)? Give a complete proof for your answer in (a), (e), (f), (i) and (j).

(a) (M,d) is an arbitrary metric space and A a finite subset of M.
(b) (M,d) is an arbitrary metric space and A a closed subset of a compact set $C \subset M$.
(c) $M = \mathbb{R}^n$ and $A = \underline{B(0,1)}$ (always with the usual Euclidean metric $d = d_2$).
(d) $M = \mathbb{R}^n$ and $A = \overline{B(0,1)}$.
(e) $M = \mathbb{R}^2$ and $A = (0,1] \times [0,1]$.
(f) $M = \mathbb{R}$ and $A = \{x^2 + y^5 : 0 \le x \le 1 \text{ and } -3 \le y \le 3\}$.
(g) $M = \mathbb{R}^2$ and $A = \{(x,y) : 0 \le x^2 - y^5 \le 1\}$.
(h) $M = A = \mathbb{R}^n$ with the usual Euclidean metric $d = d_2$.
(i) $M = \ell^2$ (in this case we use always the metric d induced by the norm $\|\cdot\|_2$) and $A = \overline{B(0,1)}$.
(j) $M = \ell^2$ and A a subset with non empty interior $\mathring{A}$.

Exercise 2 (*Normed vector spaces?*) (2+2+2+4+2+2+4)

Is $(X, \|\cdot\|)$ a normed vector space?

(a) $X = [0,1]$ and $\|x\| = |x|$.
(b) $X = C(\mathbb{R})$ the space of continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ and $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$.
(c) $X = C^1([0,1])$ the space of continuously differentiable functions $f\colon [0,1] \to \mathbb{R}$ together with

$$\|f\| = \sup_{x\in[0,1]} \left|f'(x)\right|.$$

(d) $X = C^1([0,1])$ the space of continuously differentiable functions $f\colon [0,1] \to \mathbb{R}$ together with

$$\|f\| = \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} \left|f'(x)\right|.$$

(e) $X = C([0,1])$ the space of continuous functions $f\colon [0,1] \to \mathbb{R}$ together with

$$\|f\| = \int_0^1 f(x)\,dx.$$

(f) $X = C([0,1])$ the space of continuous functions $f\colon [0,1] \to \mathbb{R}$ together with

$$\|f\| = \int_0^1 f(x)^2\,dx.$$

(g) $X = C([0,1])$ the space of continuous functions $f\colon [0,1] \to \mathbb{R}$ together with

$$\|f\| = \int_0^1 |f(x)|\,dx.$$

please turn over!

Exercise 3 (*Multiple Choice*) (8)

Which of the following statements are true? Give a short argument for all of your answers. We will us the notation

$$\|x\|_p := \begin{cases} \sqrt[p]{\sum_{i=1}^n |x_i|^p} & \text{, for } p \in (0, \infty) \\ \max_{i=1,\dots,n} |x_i| & \text{, for } p = \infty. \end{cases}$$

for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $p \in (0, \infty]$. Furthermore we denote by (M, d) a metric space and by N a subset of M. We already know that $(N, d|_{N \times N})$ is a metric space again.

(a) Let $X \subset N$ be given. Then X is open in (M, d) if and only if X is open in $(N, d|_{N\times N})$.
☐ true ☐ false

(b) Let $X \subset N$ be given. Then X is compact in (M, d) if and only if X is compact in $(N, d|_{N\times N})$.
☐ true ☐ false

(c) $(\mathbb{R}^n, \|\cdot\|_p)$ is a normed vector space for $p \in [1, \infty]$.
☐ true ☐ false

(d) $\mathbb{R}^n$ is complete in the norm $\|\cdot\|_p$ for $p \geq 1$.
☐ true ☐ false

(e) $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$ for $0 < p < 1$.
☐ true ☐ false

(f) The identity function from $(\mathbb{R}^n, \|\cdot\|_p)$ to $(\mathbb{R}^n, \|\cdot\|_\infty)$ is continuous ($p \in [1, \infty]$).
☐ true ☐ false

(g) Given two norms $\|\cdot\|$ and $\|\cdot\|'$ on some vector space V. Then $\|v\|_{\text{new}} = \|v\| + \|v\|'$ is a norm too.
☐ true ☐ false

(h) The normed space in exercise 2 (g) is complete.
☐ true ☐ false