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Exercise Sheet 7

Applied Analysis

Discussion on Thursday 5-12-2013 at 16ct

Exercise 1 (5+5+5)

(a) Find a non-convergent Cauchy sequence in $(\mathfrak{c}_{00}, \|\cdot\|_2)$ with the norm

$$\|(x_k)\|_2^2 = \sum_{k=1}^{\infty} |x_k|^2$$

for $(x_k) \in \mathfrak{c}_{00}$.

(b) Let $(\mathbb{R}^N, \|\cdot\|)$ be a normed vector space. If $(x^{(n)})_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}^N$, then the following properties are equivalent:

(i) $(x^{(n)})$ converges in $(\mathbb{R}^N, \|\cdot\|)$ to x.

(ii) Every coordinate sequence $(x_k^{(n)})_{n\in\mathbb{N}}$ converges to x_k for $k = 1, ..., N$.

Prove this equivalence.

(c) Show that the above equivalence is wrong for all the spaces $(\ell^p, \|\cdot\|_p)$ (with $p \in [1, \infty]$) by giving counterexamples.

Exercise 2 (*Continuous functions vanishing at infinity*) (5+5+5)

We denote by

$$C_0(\mathbb{R}) = \left\{ f \colon \mathbb{R} \to \mathbb{R} : \text{ is continuous and } \lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0 \right\}$$

the space of continuous functions, which vanish at infinity and define

$$\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$$

for every $f \in C_0(\mathbb{R})$.

(a) Show that $(C_0(\mathbb{R}), \|\cdot\|_\infty)$ is a normed space.

(b) Is this space complete?

(c) Show that $C_0(\mathbb{R})$ is separable.

Exercise 3 (*Multiple Choice*) (10)

Which of the following statements are true?

(a) The compact sets in a normed space $(\mathbb{R}^N, \|\cdot\|)$ are precisely the bounded and closed sets.
□ true □ false

(b) The compact sets in the normed space $(\ell^1, \|\cdot\|_1)$ are precisely the bounded and closed sets.
□ true □ false

(c) If (K, d) is a compact metric space, then K is separable.
□ true □ false

(d) If (M, d) is a separable metric space, then M is compact.
□ true □ false

(e) $(C(K), \|\cdot\|_\infty)$ for a compact metric space (K, d) is a Polish metric space.
□ true □ false

please turn over!

(f) $(C_b(M), \|\cdot\|_\infty)$ is separable for every metric space (M, d).
□ true □ false

(g) $(\mathcal{F}_b(\Omega), \|\cdot\|_\infty)$ is a Polish metric space for every set Ω.
□ true □ false

(h) $(\mathcal{F}_b(\Omega), \|\cdot\|_\infty)$ is a Polish metric space for every countable set Ω.
□ true □ false

(i) The trigonometric polynomials are dense in $(C([0, 2\pi]), \|\cdot\|_\infty)$.
□ true □ false

(j) The polynomials are dense in $(C([0, 2\pi]), \|\cdot\|_\infty)$.
□ true □ false