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Exercise Sheet 10

Applied Analysis

Discussion on Thursday 9-1-2014 at 16ct

This is also the first mock exam. 100% corresponds to 110 points. In the final exam you are allowed to use a calculator and a double-sided handwritten A4 sheet. This is intended to be solved in 120 minutes.

Exercise 1 (*Three basic properties of metric spaces*) (3+4+5)

Let (M, d) be a metric space.

(a) Give a definition of the following properties of (M, d):

  i. compactness,
  ii. separability, and
  iii. completeness.

(b) Which of the following implications are true? (no proof required)

  i. (M, d) is compact $\Rightarrow$ (M, d) is complete.
  ii. (M, d) is complete $\Rightarrow$ (M, d) is compact.
  iii. (M, d) is compact $\Rightarrow$ (M, d) is separable.
  iv. (M, d) is separable $\Rightarrow$ (M, d) is complete.

(c) Give a counterexample with explanation of one of the wrong implications in (b).

Exercise 2 (*Compactness*) (4+4+5+5)

(a) Which of the following sets are compact? (no proof required)

  i. $(\mathbb{Q}, d)$ where d is the discrete metric.
  ii. $[0, 1] \times \{1\}$ in $(\mathbb{R}^2, d_2)$. Here we denote by d_2 the euclidean metric.
  iii. $(0, 1]$ in $(\mathbb{R}, d_2)$.
  iv. (M, d) a metric space where M is a finite set.

(b) Choose one of your claims in part (a) and prove them.

(c) Show that the closed unit ball in ℓ^∞ is not compact.

(d) Prove that the function $f\colon [0, 1]^2 \to \mathbb{R}$ given by

$$f(x, y) = e^{x^2} + yx + ye^y - x$$

attains its infimum and supremum in $[0, 1]^2$.

Exercise 3 (*Banach's fixed point theorem*) (5+5+5)

(a) Formulate Banach's (classical) fixed point theorem.

(b) Use Banach's fixed point theorem to prove the existence of a unique solution $x^*, y^* \in [-1, 1]$ of

$$10x = x^2 + y$$
$$10y = x^2 + y + 5$$

(c) Calculate the first two decimal digits of x^* and y^* by using the fixed point iteration starting with $x_0 = y_0 = 0$.

please turn over!

Exercise 4 (*Countability*) (6+3)

(a) Which of the following sets are countable? (no proof required)

i. $[0, 1]$

ii. $\mathbb{Z} \times \{0, 1, 2, 3, 4, 5\}$

iii. $\{1, 2, 3, 4, 5, 6\}$

iv. $\mathbb{Q}$

v. $\mathcal{P}(\mathbb{Z}) = \{A : \ A \subset \mathbb{Z}\}$ the power set of $\mathbb{Z}$

vi. $\mathcal{P}_f(\mathbb{N}) := \{A : \ A \subset \mathbb{N} \text{ is finite}\}$

(b) If $A \neq \emptyset$ is uncountable, prove that B with $A \subset B$ is uncountable too.

Exercise 5 (*Linear bounded maps*) (2+5)

If $(x_k) \in \ell^p$ and $(y_k) \in l^q$, then $(x_k y_k) \in \ell^1$ (no proof needed). Here $p, q \in [1, \infty]$ are such that $p^{-1} + q^{-1} = 1$. So for a fixed $(y_k) \in \ell^q$ we get a well-defined function

$$T\colon \ell^p \to \ell^1, \quad T\colon (x_k) \mapsto (x_k y_k).$$

(a) Show that T is linear.

(b) Show that T is bounded.

Exercise 6 (*Measurable functions and σ-algebras*) (3+5+5+3+5)

(a) List all σ-algebras on $\Omega = \{5, 6, 7\}$.

(b) Let $f\colon \Omega_1 \to \Omega_2$ be a function and Σ_2 a σ-algebra on Ω_2. Define $\sigma(f)$.

(c) Let the function $f\colon \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4\}$ given by $f(1) = 1$, $f(2) = f(3) = 2$, $f(4) = f(5) = 3$. We equip the codomain with the σ-algebra $\Sigma = \sigma(\{\{3\}, \{4\}\})$. Write down all the elements of $\sigma(f)$.

(d) Let $f\colon \Omega_1 \to \Omega_2$ be a function, Σ_1 a σ-algebra on Ω_1 and Σ_2 a σ-algebra on Ω_2. Define when f is Σ_1/Σ_2-measurable.

(e) In the situation of part (c): How many functions $g\colon \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4\}$ are $\sigma(f)/\Sigma$-measurable? Give a detailed argumentation.

Exercise 7 (*Calculating Riemann integrals*) (5+8)

(a) Suppose that $f\colon [a, c] \to \mathbb{R}$ is Riemann integrable. If $b \in (a, c)$ is given, then

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Prove this by using the definition of the Riemann integral (You can assume that f is Riemann integrable on $[b, c]$ and $[a, b]$ too).

(b) Calculate the following Riemann integrals:

i. $\displaystyle\int_{-1}^1 e^{x^2+3x^4} x\,dx$

ii. $\displaystyle\int_0^1 x^2 e^{x^3}\,dx$

iii. $\displaystyle\int_0^1 (x^2 + 4x)\,dx$

iv. $\displaystyle\int_{-1}^2 f(x)\,dx$

Here $f\colon [-1, 2] \to \mathbb{R}$ is given by

$$f\colon x \mapsto \begin{cases} -x + 1 & \text{, for } x < 0 \\ 0 & \text{, for } x = 0 \\ x - 1 & \text{, for } x > 0. \end{cases}$$

Exercise 8 (*Independent events*) (5+10)

Let (Ω, Σ, μ) be a probability space. Two sets $A, B \in \Sigma$ are called (stochastically) independent, iff

$$\mu(A \cap B) = \mu(A)\mu(B).$$

Let us suppose that $A \in \Sigma$ and $\mathcal{E} \subset \Sigma$ is given. We say that A is independent of $\mathcal{E}$, iff A, B are independent for all $B \in \mathcal{E}$.

(a) Find a concrete example of the above situation such that A is independent of $\mathcal{E}$ but A is not independent of $\sigma(\mathcal{E})$.

(b) Let us suppose that $\mathcal{E}$ is stable under intersections. Prove that the following properties are equivalent:

  i. A and $\mathcal{E}$ are independent.

  ii. A and $\sigma(\mathcal{E})$ are independent.

Exercise 9 (*Multiple Choice*) (15*)

Decide which of the following statements are true (no proof needed). For every correct answer you get +1 point and for every wrong answer −1 point. The points of this exercise will be rounded up to zero, if the total number is negative.

(a) The trigonometric polynomials are dense in $(C([0, 2\pi]), \|\cdot\|_\infty)$.
☐ true ☐ false

(b) The polynomials are dense in $(C([0, 2\pi]), \|\cdot\|_\infty)$.
☐ true ☐ false

(c) $(C_b(M), \|\cdot\|_\infty)$ is a Polish space if (M, d) is a metric space.
☐ true ☐ false

(d) $(C(M), \|\cdot\|_\infty)$ is a Polish space if (M, d) is a compact metric space.
☐ true ☐ false

(e) $\mathcal{B}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$.
☐ true ☐ false

(f) $A, B \in \mathcal{B}(\mathbb{R})$, then $A \times B \in \mathcal{B}(\mathbb{R}^2)$.
☐ true ☐ false

(g) $\mathcal{B}(\mathbb{R})$ is generated as a σ-algebra by all finite intervals (a, b) with $a < b$.
☐ true ☐ false

(h) $\mathcal{B}(\mathbb{R}^2)$ is generated as a σ-algebra by all open sets in $\mathbb{R}^2$.
☐ true ☐ false

(i) Given two normed spaces $(\mathbb{R}^N, \|\cdot\|)$ and $(\mathbb{R}^N, \|\cdot\|')$. Then the compact subsets of the two metric spaces coincide.
☐ true ☐ false

(j) If (M, d) and (M, d') are metric spaces on same set M. Let us suppose that (x_n) is a convergent sequence in both spaces, then the limits in (M, d) and in (M, d') coincide.
☐ true ☐ false

(k) If (M, d) is a metric space and (x_n) converges to both x and y, then $x = y$.
☐ true ☐ false

(l) The compact subsets in ℓ^2 are precisely the bounded and closed subsets.
☐ true ☐ false

(m) If Σ is a σ-algebra on Ω and $A_i \in \Sigma$ for all $i \in I$ (here I is an arbitrary index set), then $\bigcup_{i \in I} A_i \in \Sigma$.
☐ true ☐ false

(n) If Σ is a σ-algebra on Ω and $A_n \in \Sigma$ for all $n \in \mathbb{N}$, then $\bigcap_{n \in \mathbb{N}} A_n \in \Sigma$.
☐ true ☐ false

(o) The Lebesgue measure λ on $\mathbb{R}$ assigns to every $A \subset \mathbb{R}$ a "length" $\lambda(A) \geq 0$.
☐ true ☐ false