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Exercise Sheet 13

Applied Analysis

Discussion on Thursday 30-1-2014 at 16ct

Exercise 1 (*A new formula for the expected value*) (10)

Let a random variable X on some probability space $(\Omega, \Sigma, \mathbb{P})$ be given with cumulative distribution function $F\colon \mathbb{R} \to [0,1]$. We denote by $X^+ = \mathbb{1}_{\{X \geq 0\}} X$ the positive part of X.

(a) Show that $(1-F)\mathbb{1}_{(0,\infty)}$ is the pointwise limit of the monotonically increasing sequence of simple functions

$$f_n = \sum_{k=1}^{4^n} \left(1 - F\left(\frac{k}{2^n}\right)\right) \mathbb{1}_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right]}.$$

(b) Show that X^+ is the pointwise limit of the monotonically increasing sequence of simple functions

$$g_n = \sum_{k=1}^{4^n} \frac{1}{2^n} \mathbb{1}_{\{X > \frac{k}{2^n}\}}.$$

(c) Prove (for every $n \in \mathbb{N}$)

$$\int_{\mathbb{R}} f_n \, d\lambda = \int_{\Omega} g_n \, d\mathbb{P}.$$

(d) Prove using (a)-(c) and the monotone convergence theorem

$$\mathbb{E}X^+ = \int_{[0,\infty)} (1-F)\, d\lambda.$$

Exercise 2 (*Calculation of some expected values*) (5*+5+5+5)

(a*) Use the definition of the expected value (see Sheet 12) to calculate the expectation for $X\colon \mathbb{N} \to \mathbb{R}$ given by $X(n) = n$ as a random variable on the probability space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mathbb{P})$ with (for fixed $p \in (0,1)$)

$$\mathbb{P}(A) = \sum_{k \in A} (1-p)p^{k-1}$$

for all $A \subset \mathbb{N}$.

(b) Calculate the cumulative distribution function of the random variable in (a).

(c) Calculate the expected value of the random variable in (a) again. This time use the formula given in Exercise 1 (d).

(d) Use Exercise 1 (d) above to calculate the expected value of a random variable Y with cumulative distribution function (for fixed $\alpha > 0$)

$$F_Y(y) = \begin{cases} 0 & \text{, for } y < 0 \\ 1 - e^{-\alpha y} & \text{, for } y \geq 0. \end{cases}$$

please turn over!

Exercise 3 (*Product of measurable spaces and measurability of maps*) (10+15+5*+10*)

(a) Let (Ω, Σ), (Ω_1, Σ_1), and (Ω_2, Σ_2) be measurable spaces and

$$f\colon \Omega \to \Omega_1 \times \Omega_2$$

be a function. Prove that the following properties are equivalent:

i. f is $\Sigma/\Sigma_1 \otimes \Sigma_2$-measurable.

ii. f_1 is Σ/Σ_1-measurable and f_2 is Σ/Σ_2-measurable.

Here $f_1\colon \Omega \to \Omega_1$ and $f_2\colon \Omega \to \Omega_2$ are the coordinate functions of f (i.e. $f(\omega) = (f_1(\omega), f_2(\omega))$ for all $\omega \in \Omega$).

(b) Prove that the following function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R}^2)$-measurable. Here f is given by

$$f(x,y) = \begin{cases} (xy, y-1) & \text{, for } e^x + y \geq 0 \\ (x^2+3, \mathbb{1}_{\mathbb{Q}}(x)) & \text{, for } e^x + y < 0. \end{cases}$$

(c*) Is it true that every function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $f(x,\cdot)$ and $f(\cdot,y)$ are $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable for every fixed $x, y \in \mathbb{R}$ is indeed $\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R})$-measurable?
Hint: You can assume that there exists an $A \subset \mathbb{R}$ which is not Borel measurable (i.e. $A \notin \mathcal{B}(\mathbb{R})$). Is the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \mathbb{1}_A(x) & \text{, for } x = y \\ 0 & \text{, for } x \neq y \end{cases}$$

$\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R})$-measurable? Are $f(\cdot,y)$ and $f(x,\cdot)$ measurable for every fixed $x, y \in \mathbb{R}$?

(d*) Let a not Borel measurable set $A \subset \mathbb{R}$ (i.e. $A \notin \mathcal{B}(\mathbb{R})$) be given. Show that

$$B := \{(x,y) \in \mathbb{R}^2 \colon x = y \in A\}$$

is a null set but not Borel measurable (i.e. $B \notin \mathcal{B}(\mathbb{R}^2)$ but $\lambda^2(C) = 0$ for some $C \supset B$ with $C \in \mathcal{B}(\mathbb{R}^2)$).