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3. Mock Exam
Applied Analysis
There will be no official discussion and we don't provide a solution

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100% corresponds to 100 points (you can achieve 120 points). You are allowed to use a double-sided handwritten A4 sheet. This is intended to be solved in 120 minutes.

Exercise 1 (*Basic properties of metric spaces*) (5+8+7)

All the spaces are equipped with the usual metrics if no metric is specified.

(a) State Banach's (classical) fixed point theorem.

(b) Which of the following sets are **compact**, which are **complete** (no proof required)?

i. The closed unit ball $\overline{B(0,1)}$ of ℓ^∞.

ii. A finite set $X \subset M$ for an arbitrary metric space (M, d).

iii. $[0,1] \times \{(x,y) \in \mathbb{R}^2 : \ e^x y|x| + |y|^3 x \leq 1\} \subset \mathbb{R}^3$.

iv. $\mathbb{Q}$ with the discrete metric.

(c) Prove your claim in (c)iii. **or** (c)iv. (give enough details!).

Exercise 2 (*Integrable and measurable functions*) (5+5+5+5)

Which of the following functions are integrable (prove this!)?

(a) $f(x) = x^3(1 + x^2 + x^6)^{-1}$ on the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$.

(b) $f(n) = n3^{-n}$ on the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \zeta)$.

(c) $f(x) = \frac{(-1)^n}{n^2}$ on the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ with

$$\mu(A) = \sum_{k \in A} k$$

for all $A \subset \mathbb{N}$.

(d) $f(n) = n^{-2}\mathbb{1}_{\mathbb{N}}(n)$ on the measure space $(\mathbb{Z}, \sigma(\mathcal{E}), \zeta)$, where the counting measure is defined by

$$\zeta(A) = |A|$$

for all measurable $A \subset \mathbb{Z}$ and

$$\mathcal{E} = \{\{-n, -n+1, ..., -1, 0, 1, ..., n-1, n\} : \ n \in \mathbb{N}\}.$$

please turn over!

Exercise 3 (*Calculating some Lebesgue integrals*) (5+20)

(a) Given some σ-finite measurable spaces $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$. Give a definition of $\Sigma_1 \otimes \Sigma_2$ and a definition of $\mu_1 \otimes \mu_2$.

(b) Calculate the following Lebesgue integrals respectively limit of Lebesgue integrals (You don't have to prove that the functions are integrable! You can assume this).

i. $\displaystyle\int_{\mathbb{N}} n \, d\mu$ ii. $\displaystyle\int_{\mathbb{Z}} \int_{\mathbb{R}} \frac{nx^4}{(n^4 + x^2 n^6)\, e^{x^2}} \, d\lambda(x) \, d\zeta(n)$

iii. $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}} x^{-2} \cdot \mathbb{1}_{[1,\infty)}(x) \cdot (1 - \cos^n(x)) \, d\lambda(x)$

Here λ is the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, μ is a measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ given by

$$\mu(A) = \sum_{n\in A} n^{-1} 2^{-n}$$

and ζ is the counting measure on $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}))$.

Exercise 4 (*Linear maps*) (5+5+5)

(a) State Fubini's theorem.

(b) We define a function $T\colon L^1([0,1]^2) \to L^1([0,1])$ by

$$Tf = \int_{[0,1]} f(x, \cdot) \, d\lambda(x)$$

Prove that T is linear **and** well-defined.

(c) Prove that T is bounded.

Exercise 5 (*Principle of good sets*) (5+5+10)

Let us suppose that a set Ω and a subset $\mathcal{E}$ of the power set $\mathcal{P}(\Omega)$ is given.

(a) Define $\sigma(\mathcal{E})$ and $\text{dyn}(\mathcal{E})$.

(b) Given some fixed $A \in \text{dyn}(\mathcal{E})$. Show that

$$\mathcal{G}_A = \{B \in \Omega : \ A \cap B \in \text{dyn}(\mathcal{E})\}$$

is a Dynkin system.

Hint: The identity $A \cap B^c = (A^c \cup (A \cap B))^c$ might be helpful.

(c) Use part (b) to show Dynkin's π-λ theorem:

If $\mathcal{E}$ is stable under intersections, then $\text{dyn}(\mathcal{E}) = \sigma(\mathcal{E})$.

You can use without a proof the following fact:

Every Dynkin system which is stable under intersections is a σ-algebra.

please turn over!

Exercise 6 (*Multiple Choice*) (20*)

Decide which of the following statements are true (no proof needed). For every correct answer you get +2 points and for every wrong answer −1 point. The points of this exercise will be rounded up to zero, if the total number is negative.

(a) ℓ^2 is countable.
□ true □ false

(b) $\mathbb{R}$ is countable.
□ true □ false

(c) $\{1, 2, 3, 4, 5, 6, \mathbb{R}, [0, 1]\}$ is countable.
□ true □ false

(d) $\mathcal{P}(A)$ is uncountable for every set A.
□ true □ false

(e) $\mathbb{Q} \times \mathbb{Z} \times \{1, 2, 3, \pi\}$ is countable.
□ true □ false

(f) $(C(K), \|\cdot\|_\infty)$ is a Polish space for every compact metric space (K, d).
□ true □ false

(g) Given a probability space $(\Omega, \Sigma, \mathbb{P})$, a measurable set A and a set $\mathcal{E} \subset \Sigma$ of measurable sets. Then A is independent of $\mathcal{E}$ if and only if A is independent of $\sigma(\mathcal{E})$.
□ true □ false

(h) If U is a closed vector subspace of a Hilbert space H and denote by $P \colon H \to H$ the orthogonal projection on U, then $P \circ P = P$ holds.
□ true □ false

(i) There is a bijective linear map $T \colon \ell^2 \to L^2([0, 1])$ with $\|Tx\|_{L^2([0,1])} = \|x\|_{\ell^2}$ for all $x \in \ell^2$.
□ true □ false

(j) There is a bijective linear map $T \colon \ell^\infty \to \ell^2$ with $\|Tx\|_{\ell^2} = \|x\|_{\ell^\infty}$ for all $x \in \ell^\infty$.
□ true □ false