University of Ulm

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Applied Analysis: Mock Exam

1. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and $f\colon X \to Y$ a map.

(a) Show that f is continuous if f is Lipschitz continuous.

(b) Show that a Lipschitz continuous linear map $T\colon X \to Y$ is a bounded linear operator.

(c) Let $K \subset X$ be compact. Show that for every $x \in X$ there exists some $y_0 \in K$ such that

$$\|x - y_0\|_X = \inf_{y \in K} \|x - y\|_X.$$

2. Let $f\colon [0,1] \to \mathbb{R}$.

(a) Show that the inverse images under f of disjoint sets $A, B \subset \mathbb{R}$ are disjoint.

(b) Let $\Omega = [0,1]$ and $\Sigma = \sigma(\{[0,1/4] \cup (3/4,1], [1/4,3/4]\})$. Describe all $\Sigma/\mathcal{B}(\mathbb{R})$-measurable functions.

3. Let $(X, \|\cdot\|)$ be a Banach space and let $(x_k)_{k\in\mathbb{N}}$ be an X valued sequence such that $\sum_{k=1}^{\infty} \|x_k\| < \infty$ and such that the real valued sequence $(\|x_k\|)_{k\in\mathbb{N}}$ is monotonically decreasing. Show that $k\|x_k\| \to 0$.

[Hint: Consider the Cauchy criteria for the sequence of the partial sums of $\sum_{k=1}^{\infty} \|x_k\|$ and choose $n, m \geq n_0$ with $m = 2n$.]

4. Formulate the monotone convergence theorem.

5. Let (Ω, Σ, μ) be a measure space and ν another measure on (Ω, Σ) with

$$\mu(A) \geq \nu(A) \text{ for all } A \in \Sigma.$$

(a) Let $f, g\colon \Omega \to \mathbb{R}$ be $\Sigma/\mathcal{B}(\mathbb{R})$-measurable functions. Show that if $f = g$ μ-a.e., then $f = g$ ν-a.e.

(b) Given some non-negative Σ-measurable function $f\colon \Omega \to [0, \infty)$. Show that

$$\int f \, \mathrm{d}\mu \geq \int f \, \mathrm{d}\nu.$$

(c) We define a map $T\colon L^2(\Omega, \Sigma, \mu) \to L^2(\Omega, \Sigma, \nu)$ by $Tf = f$ for all $f \in L^2(\Omega, \Sigma, \mu)$. If follows from (a) and (b) that T is well-defined. You can assume this without proof.

(i) Prove that T is linear.

(ii) Prove that T is continuous.

6. Let us suppose that a set Ω and a subset $\mathcal{E}$ of the power set $\mathcal{P}(\Omega)$ is given.

(a) Define $\sigma(\mathcal{E})$ and $dyn(\mathcal{E})$.

(b) Given some fixed $A \in dyn(\mathcal{E})$. Show that $\mathcal{G}_A := \{B \subset \Omega : A \cap B \in dyn(\mathcal{E})\}$ is a Dynkin system.
[Hint: The identity $A \cap B^c = (A^c \cup (A \cap B))^c$ might be helpful.]

(c) Use part (b) to show Dynkin's π-λ theorem: If $\mathcal{E}$ is stable under intersections, then $dyn(\mathcal{E}) = \sigma(\mathcal{E})$.
You can use without a proof the following fact: Every Dynkin system which is stable under intersections is a σ-algebra.

7. Give an example for each of the following situations. Only state your example, no further explanation is required.

(a) Give an example of a separable normed space where the norm does not come from an inner product, but where there exists an equivalent norm that does come from an inner product.

(b) Give an example of a measure space (Ω, Σ, μ) such that $L^p(\Omega, \Sigma, \mu) \subset L^q(\Omega, \Sigma, \mu)$ for all $1 \leq p < q \leq \infty$.

8. Decide, without an explanation, if the following statements are true or false.

(a) Every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable.

(b) Let (Ω, Σ, μ) be an arbitrary measure space. Every Cauchy sequence in $L^2(\Omega, \Sigma, \mu)$ has an almost everywhere convergent subsequence.

(c) Let $A \in \mathcal{B}(\mathbb{R})$. Then $\lambda(A) = 0$ if and only if A is countable.

(d) $\mathcal{B}(\mathbb{R}) = \mathcal{P}(\mathbb{R})$

(e) $\mathcal{B}(\mathbb{R}) = \sigma(C(\mathbb{R}))$

(f) Let $(\Omega, \Sigma, \mathbb{P})$ be an arbitrary probability space, $A \in \Sigma$ and $\mathcal{E} \subset \Sigma$. Then A is independent of $\mathcal{E}$ if and only if A is independent of $\sigma(\mathcal{E})$.

(g) Let $(X, \|\cdot\|)$ be a normed space and $A \subset X$. Then the closure of A° equals the closure of A.

(h) A nullset is always measurable.

(i) The trigonometric polynomials are dense in $(C([0, 2\pi]), \|\cdot\|_\infty)$.

(j) Given two norms $\|\cdot\|_1, \|\cdot\|_2$ on $\mathbb{R}^d$ where $d \in \mathbb{N}$. Then the compact sets of the normed spaces $(\mathbb{R}^d, \|\cdot\|_1)$ and $(\mathbb{R}^d, \|\cdot\|_2)$ coincide.

(k) Let $f\colon \Omega \to [0, \infty)$ be measurable on (Ω, Σ). Then the uncountable intersection

$$\bigcup_{\varepsilon > 0} \{x \in \Omega : f(x) > \varepsilon\}$$

is measurable.

9. Calculate the following Lebesgue integrals, respectively limits of Lebesgue integrals.

(a) $\displaystyle\int_{\mathbb{N}} \frac{1}{3^n} \, \mathrm{d}\zeta(n)$

(b) $\displaystyle\int_{\mathbb{N}\times\mathbb{R}} \frac{2x^3 \exp(-x^2)}{(1+x^2)^n} \, \mathrm{d}(\zeta \otimes \lambda)(n, x)$

(c) $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}} (1 + |x| + x^2)^{-1} \left(\exp(-n^{-1}|x|) - 1\right) \, \mathrm{d}\lambda(x)$

Here λ is the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and ζ is the counting measure in $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$.