3.9 Integrals depending on a parameter

The main topic of this section is to interchange operations like integration, differentiation and taking limits. This is a topic at the very heart of analysis.

Instead of $[0,1]$ we could use a general metric space here.

Suppose that (Ω, Σ, μ) is a measure space. If we are given a map $f\colon [0,1] \times \Omega \to \mathbb{C}$ such that $f(t,\cdot)$ is integrable for all $t \in [0,1]$, we may define $F(t) := \int_\Omega f(t,x)\,\mathrm{d}\mu(x)$. It is then natural to ask how F depends on the parameter $t \in [0,1]$. In this short section, we use the dominated convergence theorem to prove some results in this direction.

Proposition 3.85. *Let (Ω, Σ, μ) be a measure space. Furthermore, let $f\colon [0,1] \times \Omega \to \mathbb{K}$ be such that the following three properties hold.*

*Properties (a) and (b) say that f is a so-called **Caratheodory function**.*

(a) *$x \mapsto f(t,x)$ is measurable for all $t \in [0,1]$.*
(b) *$t \mapsto f(t,x)$ is continuous for almost all $x \in \Omega$.*
(c) *There exists a $g \in \mathscr{L}^1(\Omega, \Sigma, \mu)$ such that $|f(t,x)| \le g(x)$ for all $(t,x) \in [0,1] \times \Omega$.*

Then $F\colon [0,1] \to \mathbb{C}$ defined by $F(t) = \int_\Omega f(t,x)\,\mathrm{d}\mu(x)$ is continuous.

Proof. Let $t_n \to t$ in $[0,1]$. Then $f(t_n,x) \to f(t,x)$ for almost all $x \in \Omega$ by (b). Since $|f(t_n,x)| \le g(x)$ for all $x \in \Omega$ by assumption and $g \in \mathscr{L}^1(\Omega)$, it follows from the dominated convergence theorem, Theorem 3.75, that

$$F(t_n) = \int_\Omega f(t_n,x)\,\mathrm{d}\mu(x) \to \int_\Omega f(t,x)\,\mathrm{d}\mu(x) = F(t).$$

This proves the continuity of F. □

Proposition 3.86. *Let I be an interval in $\mathbb{R}$ and (Ω, Σ, μ) be a measure space. Furthermore, let $f\colon I \times \Omega \to \mathbb{K}$ be such that the following three properties hold.*

(a) *$x \mapsto f(t,x) \in \mathscr{L}^1(\Omega, \Sigma, \mu)$ for all $t \in I$.*
(b) *$t \mapsto f(t,x)$ is differentiable for all $x \in \Omega$.*
(c) *There exists a $g \in \mathscr{L}^1(\Omega, \Sigma, \mu)$ such that $|\frac{\partial}{\partial t} f(t,x)| \le g(x)$ for all $(t,x) \in I \times \Omega$.*

Then $F\colon I \to \mathbb{K}$ defined by $F(t) = \int_\Omega f(t,x)\,\mathrm{d}\mu(x)$ is differentiable. Moreover, $\frac{\partial}{\partial t} f(t,x)$ is integrable for all $t \in I$ and

$$F'(t) = \frac{d}{dt} \int_\Omega f(t,x)\,\mathrm{d}\mu(x) = \int_\Omega \frac{\partial}{\partial t} f(t,x)\,\mathrm{d}\mu(x).$$

Proof. Fix $t \in I$ and let (t_n) be a sequence in I that converges to t. Define $h_n, h\colon \Omega \to \mathbb{C}$ by $h_n(x) := (t_n - t)^{-1}(f(t_n,x) - f(t,x))$ and $h(x) = \frac{\partial}{\partial t} f(t,x)$. Then h_n is integrable for every $n \in \mathbb{N}$ as a linear combination of integrable functions. Moreover, $h_n(x) \to h(x)$ for all $x \in \Omega$ by assumption. By the mean-value theorem, $h_n(x) = \frac{\partial}{\partial t} f(\xi_n, x)$ for some ξ_n between

t and t_n. In particular, $|h_n| \leq g$. Thus the dominated convergence theorem shows that h is integrable and

$$\frac{F(t_n) - F(t)}{t_n - t} = \int_\Omega h_n(x)\,\mathrm{d}\mu(x) \to \int_\Omega h(x)\,\mathrm{d}\mu(x) = \int_\Omega \frac{\partial}{\partial t} f(t,x)\,\mathrm{d}\mu(x).$$

This finishes the proof. □

3.10 Product measures

In this section we construct a σ-algebra and a corresponding measure on the product of two suitable measure spaces. Our motivation is to extend the theory in order to deal with iterated integrals. The product measure will allow us to write an iterated integral as a single integral with resprect to the product measure.

Definition 3.87. Let (Ω_k, Σ_k) be a measurable space for $k = 1, \dots, n$. The **product (measurable space)** of the spaces (Ω_k, Σ_k) is the measurable space $(\prod_{k=1}^n \Omega_k, \bigotimes_{k=1}^n \Sigma_k)$, where $\prod_{k=1}^n \Omega_i$ is the Cartesian product of the sets Ω_k, i.e., the set of all tuples $(x_1, \dots, x_n)$ where $x_k \in \Omega_k$ and $\bigotimes_{k=1}^n \Sigma_k$ is the σ-algebra generated by the cuboids $A_1 \times \cdots \times A_n$ where $A_k \in \Sigma_k$.

Note that the 'diagonal' $\{(x,x) : x \in \mathbb{R}\}$ is not contained in $\mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathbb{R})$, but it is contained in $\mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}) = \mathscr{B}(\mathbb{R}^2)$. The latter identity is an important exercise based on the principle of good sets.

Exercise 3.88. Let (Ω, Σ) and, for $k = 1, \dots, n$, also (Ω_k, Σ_k) be measure spaces. Let $f_k \colon \Omega \to \Omega_k$ be a function and define $f \colon \Omega \to \prod_{k=1}^n \Omega_k$ by $f(x) = (f_1(x), \dots, f_n(x))$. Show that f is $\Sigma / \bigotimes_{k=1}^n \Sigma_k$-measurable if and only if f_k is Σ/Σ_k-measurable for all $k = 1, \dots, n$.

Observe that the 2-dimensional Lebesgue measure λ_2 is the product measure of the one-dimensional Lebesgue measure with itself on the respective Borel σ-algebras. More precisely, $\lambda_2 = \lambda \otimes \lambda$ on $\mathscr{B}(\mathbb{R}^2) = \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R})$. However, if $\mathcal{M}$ denotes the Lebesgue measurable sets, then $\mathcal{M}(\mathbb{R}) \otimes \mathcal{M}(\mathbb{R}) \subsetneqq \mathcal{M}(\mathbb{R}^2)$. So the product measure space of complete measure spaces does not need to be complete.

In the following, let $(\Omega_i, \Sigma_i, \mu_i)$ be σ-finite measure spaces for $i = 1, 2$. We define a measure $\mu_1 \otimes \mu_2$ on the σ-algebra $\Sigma_1 \otimes \Sigma_2$ which is the product of the measures μ_1 and μ_2 in the sense that

$$\mu_1 \otimes \mu_2(A \times B) = \mu_1(A)\mu_2(B)$$

for all $A \in \Sigma_1$ and $B \in \Sigma_2$. Note that as μ_1 and μ_2 are σ-finite, by Corollary 3.38 there exists at most one such measure.

For a set $Q \subset \Omega_1 \times \Omega_2$ and $x \in \Omega_1$, $y \in \Omega_2$, we define the **cuts** $[Q]_x$ and $[Q]_y$ by

$$[Q]_x := \{y \in \Omega_2 : (x,y) \in Q\} \quad \text{and} \quad [Q]_y := \{x \in \Omega_1 : (x,y) \in Q\}.$$

Lemma 3.89. *For $x \in \Omega_1$, $y \in \Omega_2$ and $Q \in \Sigma_1 \otimes \Sigma_2$ we have $[Q]_x \in \Sigma_2$ and $[Q]_y \in \Sigma_1$.*

Proof. We put $\mathscr{G} := \{Q \in \Sigma_1 \otimes \Sigma_2 : [Q]_x \in \Sigma_2\}$. We claim that $\mathscr{G}$ is a σ-algebra on $\Omega_1 \times \Omega_2$. Clearly (S1) holds, since $[\Omega_1 \times \Omega_2]_x = \Omega_2$. (S2) and (S3) follow from the identities

$$[Q^{\mathsf{c}}]_x = ([Q]_x)^{\mathsf{c}} \quad \text{and} \quad \Big[\bigcup_{n\in\mathbb{N}} Q_n\Big]_x = \bigcup_{n\in\mathbb{N}} [Q_n]_x,$$

which hold for every $Q \in \Sigma_1 \otimes \Sigma_2$ and every sequence (Q_n) in $\Sigma_1 \otimes \Sigma_2$, respectively.

To finish the proof, it suffices to observe that for $A \in \Sigma_1$ and $B \in \Sigma_2$ we have $[A \times B]_x = B$ or $[A \times B]_x = \emptyset$ depending on whether $x \in A$ or $x \notin A$. Since $B, \emptyset \in \Sigma_2$, it follows that every rectangle $A \times B$ belongs to $\mathscr{G}$. As these rectangles generate the product σ-algebra, one obtains $\mathscr{G} = \Sigma_1 \otimes \Sigma_2$.

The proof for the cuts $[Q]_y$ is completely analogous. $\square$

By Lemma 3.89, it makes sense to consider $\mu_2([Q]_x)$ and $\mu_1([Q]_y)$ for $Q \in \Sigma_1 \otimes \Sigma_2$, $x \in \Omega_1$ and $y \in \Omega_2$.

Lemma 3.90. *Let $Q \in \Sigma_1 \otimes \Sigma_2$. If μ_1 and μ_2 are σ-finite, then the maps*

$$x \mapsto \mu_2([Q]_x) \quad \textit{and} \quad y \mapsto \mu_1([Q]_y)$$

are well-defined and Σ_1-measurable, respectively Σ_2-measurable.

Proof. We put $\varphi_Q(x) := \mu_2([Q]_x)$ and prove that φ_Q is Σ_1-measurable for all $Q \in \Sigma_1 \otimes \Sigma_2$. The proof for the second map is analogous.

Let us first assume that $\mu_2(\Omega_2) < \infty$. Then

$$\mathscr{D} := \{Q \in \Sigma_1 \otimes \Sigma_2 : \varphi_Q \text{ is } \Sigma_1\text{-measurable}\}$$

is a Dynkin system. Indeed, $\varphi_{\Omega_1 \times \Omega_2}(x) \equiv \mu_2(\Omega_2)$ is constant, hence measurable. So (D1) holds as $\Omega_1 \times \Omega_2 \in \mathscr{D}$. For (D2) note that if φ_Q is measurable then $\varphi_{Q^{\mathsf{c}}} = \varphi_{\Omega_1 \times \Omega_2} - \varphi_Q$ is measurable as difference of measurable functions. Finally, if (Q_n) is a sequence of pairwise disjoint sets in $\mathscr{D}$, then $\varphi_{\bigcup_n Q_n} = \sum_{n=1}^{\infty} \varphi_{Q_n}$ is measurable by Proposition 3.50.

Next observe that if $A \in \Sigma_1$ and $B \in \Sigma_2$, then $\varphi_{A\times B} = \mu_2(B)\mathbb{1}_A$ is measurable. Hence $A \times B \in \mathscr{D}$. Since these rectangles generate $\Sigma_1 \otimes \Sigma_2$ and are stable under intersections, it follows that $\mathscr{D} = \Sigma_1 \otimes \Sigma_2$.

Now assume that μ_2 is merely σ-finite. Then there exists a sequence (B_n) with $\mu_2(B_n) < \infty$ and $B_n \uparrow \Omega_2$. In this case, $\mu_2^{(n)} \colon B \mapsto \mu_2(B \cap B_n)$ is a finite measure on Σ_2, whence, by the above, $\varphi_Q^{(n)} \colon x \mapsto \mu_2^{(n)}([Q]_x)$ is measurable for all $Q \in \Sigma_1 \otimes \Sigma_2$. Since $\mu_2^{(n)}([Q]_x) \uparrow \mu_2([Q]_x)$ for all $x \in \Omega_1$, it follows that $\varphi_Q = \sup_n \varphi_Q^{(n)}$ is measurable as the pointwise limit of measurable functions. $\square$

We can now prove the existence of the product measure.

Theorem 3.91. *Let $(\Omega_i, \Sigma_i, \mu_i)$ be σ-finite measure spaces for $i = 1, 2$. Then there exists a unique measure π on $(\Omega_1 \times \Omega_2, \Sigma_1 \otimes \Sigma_2)$ such that $\pi(A \times B) = \mu_1(A)\mu_2(B)$ for all $A \in \Sigma_1$ and $B \in \Sigma_2$. Moreover, for all $Q \in \Sigma_1 \otimes \Sigma_2$ we have*

$$\pi(Q) = \int_{\Omega_1} \mu_2([Q]_x)\, \mathrm{d}\mu_1(x) = \int_{\Omega_2} \mu_1([Q]_y)\, \mathrm{d}\mu_2(y). \tag{3.2}$$

Proof. We define $\pi(Q) := \int_{\Omega_1} \mu_2([Q]_x)\, \mathrm{d}\mu_1(x)$ for all $Q \in \Sigma_1 \otimes \Sigma_2$. This is well-defined by Lemma 3.90. Then π is a measure. Indeed, $\pi(\emptyset) = 0$ and, if (Q_n) is a sequence of disjoint sets in $\Sigma_1 \otimes \Sigma_2$, then $[Q_n]_x$ is a sequence of disjoint sets in Σ_2. Thus

$$\begin{aligned}\pi\Big(\bigcup_{n\in\mathbb{N}} Q_n\Big) &= \int_{\Omega_1} \mu_2\Big(\bigcup_{n\in\mathbb{N}} [Q_n]_x\Big)\, \mathrm{d}\mu_1(x) = \int_{\Omega_1} \sum_{n=1}^{\infty} \mu_2([Q_n]_x)\, \mathrm{d}\mu_1(x)\\ &= \sum_{n=1}^{\infty} \int_{\Omega_1} \mu_2([Q_n]_x)\, \mathrm{d}\mu_1(x) = \sum_{n=1}^{\infty} \pi(Q_n),\end{aligned}$$

where we have used monotone convergence in the third step. Since

$$\pi(A \times B) = \int_{\Omega_1} \mu_2([A \times B]_x)\, \mathrm{d}\mu_1(x) = \int_{\Omega_1} \mu_2(B)\mathbb{1}_A\, \mathrm{d}\mu_1 = \mu_1(A)\mu_2(B),$$

there exists a measure with the required properties. Its uniqueness follows from Corollary 3.38. Now define $\tilde{\pi}(Q) := \int_{Q_2} \mu_1([Q]_y)\, \mathrm{d}\mu_2(y)$. Repeating the above computations, we see that $\tilde{\pi}$ is also a measure with with $\tilde{\pi}(A \times B) = \mu_1(A)\mu_2(B)$. Consequently, by uniqueness, $\pi = \tilde{\pi}$ and thus (3.2) holds. □

Here is a neat application that allows to evaluate an integral by integrating the measures of the super-level sets.

Proposition 3.92. *Let (Ω, Σ, μ) be a σ-finite measure space and $f \colon \Omega \to [0, \infty]$ be measurable. Then*

$$\int_{\Omega} f\, \mathrm{d}\mu = \int_0^{\infty} \mu(\{f \geq t\})\, \mathrm{d}t.$$

This result holds without the assumption that the measure space is σ-finite. Moreover, the integral on the right hand side can be understood as an improper Riemann integral. So this identity describes the Lebesgue integral in terms of a Riemann integral.

Proof. We may assume that $f(x) < \infty$ for all $x \in \Omega$. Consider the set $G := \{(x, t) \in \Omega \times [0, \infty) : f(x) \geq t\}$. Then $G \in \Sigma \otimes \mathscr{B}([0, \infty])$. Indeed, the maps $\Phi, \Psi \colon \Omega \times [0, \infty)$, given by $\Phi(x, t) = f(x)$ and $\Psi(x, t) = t$, are clearly $\Sigma \otimes \mathscr{B}([0, \infty])$-measurable. Hence so is $\Phi - \Psi$. But then $G = (\Phi - \Psi)^{-1}[[0, \infty]]$ is measurable.

By Theorem 3.91, we have, on the one hand,

$$(\mu \otimes \lambda)(G) = \int_0^{\infty} \mu([G]_t)\, \mathrm{d}t = \int_0^{\infty} \mu(\{f \geq t\})\, \mathrm{d}t$$

and, on the other hand,

$$(\mu \otimes \lambda)(G) = \int_\Omega \lambda([G]_x)\,\mathrm{d}\mu(x) = \int_\Omega f(x)\,\mathrm{d}\mu(x). \qquad \square$$

We next turn to the task to determine when a map $f\colon \Omega_1 \times \Omega_2 \to \mathbb{K}$ is integrable with respect to $\mu_1 \otimes \mu_2$ and how to compute the integral. We start with a lemma about measurability.

Lemma 3.93. *Let (Ω_i, Σ_i) for $i = 1,2,3$ be measurable space. If $f\colon \Omega_1 \times \Omega_2 \to \Omega_3$ is $(\Sigma_1 \otimes \Sigma_2)/\Sigma_3$-measurable, then $f(x,\cdot)$ is Σ_2/Σ_3-measurable for all $x \in \Omega_1$ and $f(\cdot,y)$ is Σ_1/Σ_3-measurable for all $y \in \Omega_2$.*

Proof. For $A \in \Sigma_3$, we have

$$f(x,\cdot)^{-1}[A] = \{y : f(x,y) \in A\} = [f^{-1}[A]]_x$$

and

$$f(\cdot,y)^{-1}[A] = \{x : f(x,y) \in A\} = [f^{-1}[A]]_y.$$

Hence the claim follows from Lemma 3.89. $\square$

Theorem 3.94 (Tonelli)**.** *Let $(\Omega_i, \Sigma_i, \mu_i)$ be σ-finite measure spaces for $i = 1,2$. Moreover, let $f\colon \Omega_1 \times \Omega_2 \to [0,\infty]$ be $\Sigma_1 \otimes \Sigma_2/\mathscr{B}([0,\infty])$-measurable. Then the maps*

$$y \mapsto \int_{\Omega_1} f(x,y)\,\mathrm{d}\mu_1(x) \quad \text{and} \quad x \mapsto \int_{\Omega_2} f(x,y)\,\mathrm{d}\mu_2(y)$$

are measurable and

$$\begin{aligned} \int_{\Omega_1\times\Omega_2} f\,\mathrm{d}\mu_1 \otimes \mu_2 &= \int_{\Omega_2}\int_{\Omega_1} f(x,y)\,\mathrm{d}\mu_1(x)\,\mathrm{d}\mu_2(y) \\ &= \int_{\Omega_1}\int_{\Omega_2} f(x,y)\,\mathrm{d}\mu_2(x)\,\mathrm{d}\mu_1(y). \end{aligned} \tag{3.3}$$

In particular, if one of the iterated integrals is finite, then f is integrable with respect to $\mu_1 \otimes \mu_2$.

Proof. Set $\Omega := \Omega_1 \times \Omega_2$, $\Sigma := \Sigma_1 \times \Sigma_2$ and $\pi := \mu_1 \otimes \mu_2$. First, let f be a simple function, say $f = \sum_{k=1}^n \alpha_k \mathbb{1}_{Q_k}$. Then $f(x,\cdot) = \sum_{k=1}^n \alpha_k \mathbb{1}_{Q_k}(x,\cdot) = \sum_{k=1}^n \alpha_k \mathbb{1}_{[Q_k]_x}(\cdot)$. Thus

$$\int_{\Omega_2} f(x,y)\,\mathrm{d}\mu_2(y) = \sum_{k=1}^n \alpha_k \mu_2([Q_k]_x).$$

Taking Lemma 3.90 into account, it follows in particular that the map

$x \mapsto \int_{\Omega_2} f(x,y)\,\mathrm{d}\mu_2(y)$ is measurable. Moreover, by (3.2), one has

$$\int_{\Omega_1}\int_{\Omega_2} f(x,y)\,\mathrm{d}\mu_2(y)\,\mathrm{d}\mu_1(y) = \sum_{k=1}^{n} \alpha_k \int_{\Omega_1} \mu_2([Q_k]_x)\,\mathrm{d}\mu_1(x)$$
$$= \sum_{k=1}^{n} \alpha_k \pi(Q_k) = \int_{\Omega} f\,\mathrm{d}\pi.$$

Now let f be an arbitrary nonnegative, measurable function and (f_n) be a sequence of simple functions increasing to f. Such a sequence exists by Proposition 3.53. By the above, $(f_n(x,\cdot))$ is a sequence of simple functions which increases to $f(x,\cdot)$. Thus, by monotone convergence,

$$\varphi(x) := \int_{\Omega_2} f(x,y)\,\mathrm{d}\mu_2(y) = \sup_{n\in\mathbb{N}} \int_{\Omega_2} f_n(x,y)\,\mathrm{d}\mu_2(y) =: \sup_{n\in\mathbb{N}} \varphi_n(x)$$

for all $x \in \Omega_1$. In particular, φ is measurable as the pointwise limit of simple functions. It now follows that

$$\int_{\Omega} f\,\mathrm{d}\pi = \sup_{n\in\mathbb{N}} \int_{\Omega} f_n\,\mathrm{d}\pi = \sup_{n\in\mathbb{N}} \int_{\Omega_1} \varphi_n\,\mathrm{d}\mu_1 = \int_{\Omega_1} \varphi\,\mathrm{d}\mu_1.$$

Here, we have used monotone convergence on (Ω, Σ, π), then the above result for simple functions and finally monotone convergence on $(\Omega_1, \Sigma_1, \mu_1)$. This proves the first equality in (3.3). The rest follows by interchanging the roles of x and y. ☐

The following example shows that one can not do without the assumption that the measure spaces are σ-finite in Tonelli's theorem.

Example 3.95. Let $(\Omega_1, \Sigma_1, \mu_1) = ([0,1], \mathscr{B}([0,1]), \lambda)$ and $(\Omega_2, \Sigma_2, \mu_2) = ([0,1], \mathscr{P}([0,1]), \zeta)$, where ζ is the counting measure. Note that μ_2 is not σ-finite. Now consider the function $f\colon \Omega_1 \times \Omega_2 \to [0,1]$ given by $f(x,y) = 0$ if $x \neq y$ and $f(x,x) = 1$ for all $x, y \in [0,1]$. It is easily seen that f is $(\Sigma_1 \otimes \Sigma_2)$-measurable. However,

$$\int_{\Omega_1}\int_{\Omega_2} f(x,y)\,\mathrm{d}\mu_2(y)\,\mathrm{d}\mu_1(x) = 1 \neq 0 = \int_{\Omega_2}\int_{\Omega_1} f(x,y)\,\mathrm{d}\mu_1(x)\,\mathrm{d}\mu_2(y).$$

Theorem 3.96 (Fubini). *Let $(\Omega_i, \Sigma_i, \mu_i)$ be σ-finite measure spaces for $i = 1, 2$. If $f\colon \Omega_1 \times \Omega_2 \to \mathbb{K}$ is $\Sigma_1 \otimes \Sigma_2$-measurable and integrable with respect to $\mu_1 \otimes \mu_2$, then $f(x,\cdot)$ is integrable with respect to μ_2 for μ_1-a.e. $x \in \Omega_1$ and $f(\cdot, y)$ is integrable with respect to μ_1 for μ_2-a.e. $y \in \Omega_2$. Moreover,*

$$\begin{aligned}\int_{\Omega_1\times\Omega_2} f\,\mathrm{d}\mu_1\otimes\mu_2 &= \int_{\Omega_2}\int_{\Omega_1} f(x,y)\,\mathrm{d}\mu_1(x)\,\mathrm{d}\mu_2(y) \\ &= \int_{\Omega_1}\int_{\Omega_2} f(x,y)\,\mathrm{d}\mu_2(x)\,\mathrm{d}\mu_1(y).\end{aligned} \tag{3.4}$$

Proof. First let $\mathbb{K} = \mathbb{R}$. By Tonelli,

$$\begin{aligned}\int_{\Omega_2}\int_{\Omega_1} |f(x,y)| \,\mathrm{d}\mu_1(x)\,\mathrm{d}\mu_2(y) &= \int_{\Omega_1}\int_{\Omega_2} |f(x,y)| \,\mathrm{d}\mu_2(x)\,\mathrm{d}\mu_1(y)\\ &= \int_{\Omega_1\times\Omega_2} |f| \,\mathrm{d}\mu_1\otimes\mu_2 < \infty\end{aligned}$$

by assumption. Hence $x \mapsto \int_{\Omega_1} |f(x,y)|\,\mathrm{d}\mu_2(y)$ is μ_1-integrable and thus, in particular, μ_1-a.e. finite. This proves that $f(x,\cdot)$ is μ_2-integrable for μ_1-a.e. x. By the definition of the integral and Tonelli,

$$\begin{aligned}\int_{\Omega_1\times\Omega_2} f \,\mathrm{d}\mu_1\otimes\mu_2 &= \int_{\Omega_1\times\Omega_2} f^+ \,\mathrm{d}\mu_1\otimes\mu_2 - \int_{\Omega_1\times\Omega_2} f^- \,\mathrm{d}\mu_1\otimes\mu_2\\ &= \int_{\Omega_1}\int_{\Omega_2} f^+ \,\mathrm{d}\mu_1\,\mathrm{d}\mu_2 - \int_{\Omega_1}\int_{\Omega_2} f^- \,\mathrm{d}\mu_1\,\mathrm{d}\mu_2\\ &= \int_{\Omega_1}\int_{\Omega_2} f \,\mathrm{d}\mu_1\,\mathrm{d}\mu_2.\end{aligned}$$

This proves one equality in (3.4). The other one follows by interchanging the roles of x and y. If $\mathbb{K} = \mathbb{C}$, consider $\operatorname{Re} f$ and $\operatorname{Im} f$ separately. □

Example 3.97. Consider $(\Omega_i, \Sigma_i, \mu_i) = ((0,\infty), \mathscr{B}(0,\infty), \lambda)$ for $i = 1,2$. Then the product space is $((0,\infty)^2, \mathscr{B}((0,\infty)^2), \lambda_2)$. Consider $f\colon (0,\infty)^2 \to \mathbb{R}$, given by $f(x,y) = ye^{-(1+x^2)y^2}$. Since f is positive and continuous, we may evaluate the one-dimensional integrals as improper Riemann integrals. We obtain

$$\int_0^\infty |f(x,y)|\,\mathrm{d}y = \Big[-\frac{1}{2}\frac{1}{1+x^2}e^{-(1+x^2)y^2}\Big|_0^\infty = \frac{1}{2}\frac{1}{1+x^2}.$$

Hence

$$\int_0^\infty\int_0^\infty |f(x,y)|\,\mathrm{d}y\,\mathrm{d}x = \int_0^\infty \frac{1}{2}\frac{1}{1+x^2}\,\mathrm{d}x = \frac{1}{2}\arctan x\Big|_0^\infty = \frac{\pi}{4}.$$

It follows from Tonelli's theorem that f is integrable with respect to λ_2. Moreover, the integral is given by $\int_{(0,\infty)^2} f\,\mathrm{d}\lambda_2 = \frac{\pi}{4}$. On the other hand, interchanging the order of integration (which is possible by Tonelli's theorem), we see that

$$\begin{aligned}\frac{\pi}{4} &= \int_0^\infty\int_0^\infty ye^{-(1+x^2)y^2}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_0^\infty e^{-y^2}\int_0^\infty ye^{x^2y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty e^{-y^2}\,\mathrm{d}y\int_0^\infty e^{-z^2}\,\mathrm{d}z,\end{aligned}$$

where we have used the substitution $z = xy$ in the last step. It follows that

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

So by symmetry,

$$\int_{\mathbb{R}} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}.$$

By substituting $x = \frac{t}{\sqrt{2}}$, it follows that

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{t^2}{2}}\,\mathrm{d}x = 1.$$

Remark 3.98. It is possible to generalize the results of this section also to finite products of σ-finite measure spaces. The main task is to prove that the product σ-algebra $\Sigma_1 \otimes \cdots \otimes \Sigma_n$ is the product of the two σ-algebras Σ_1 and $\Sigma_2 \otimes \cdots \otimes \Sigma_n$. One can then proceed by induction. We leave the details to the reader.

In fact, there exists a suitable product measure also for the countable (actually even arbitrary) product of probability spaces. This is convenient to construct an abstract probability space that supports a sequence of suitably distributed independent random variables.

3.11 The L^p-spaces

In this section we introduce and study the Banach spaces of p-integrable 'functions' on a general measure space (Ω, Σ, μ). These spaces are very important for applications and generalise the sequence spaces ℓ^p. In the case $p = 2$, for example, one obtains a Banach space with particularly nice geometric properties that plays a central role in quantum mechanics, thermodynamics and signal processing.

Definition 3.99. Let (Ω, Σ, μ) be a measure space. For $f\colon \Omega \to \mathbb{K}$ measurable and $1 \leq p < \infty$, put

$$\|f\|_p := \Big(\int_\Omega |f|^p \,\mathrm{d}\mu \Big)^{1/p}.$$

We define $\mathscr{L}^p(\Omega, \Sigma, \mu) := \{f\colon \Omega \to \mathbb{K} \text{ measurable} : \|f\|_p < \infty\}$. If it is clear which Σ and μ we use (or we make a statement over generic measure spaces), we will just write $\mathscr{L}^p(\Omega)$.

We now prove that $\mathscr{L}^p(\Omega)$ is a vector space and that $\|\cdot\|_p$ is *nearly* a norm on $\mathscr{L}^p(\Omega)$.

Proposition 3.100. *Let (Ω, Σ, μ) be a measure space and $1 \leq p < \infty$. Then the following statements hold.*

(a) *For all $f \in \mathscr{L}^p(\Omega)$, we have $\|f\|_p \geq 0$ and $\|f\|_p = 0$ if and only if $f = 0$ almost everywhere.*

(b) *For all $f \in \mathscr{L}^p(\Omega)$ and $\lambda \in \mathbb{K}$, we have $\lambda f \in \mathscr{L}^p(\Omega)$ and $\|\lambda f\|_p = |\lambda| \|f\|_p$.*

(c) *For all $f, g \in \mathscr{L}^p(\Omega)$, we have $f + g \in \mathscr{L}^p(\Omega)$ and $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.*

Proof. (a) $\|f\|_p \geq 0$ is obvious and the second assertion follows from Corollary 3.68.

(b) By Corollary 3.64,

$$\|\lambda f\|_p = \Big(\int_\Omega |\lambda f|^p \,\mathrm{d}\mu\Big)^{1/p} = \Big(|\lambda|^p \int_\Omega |f| \,\mathrm{d}\mu\Big)^{1/p} = |\lambda| \|f\|_p.$$

In particular, if $\|f\|_p < \infty$ then $\|\lambda f\|_p < \infty$.

(c) Let us first assume that f and g are simple functions, say $f = \sum_{k=1}^m a_k \mathbb{1}_{A_k}$ and $g = \sum_{k=1}^n b_k \mathbb{1}_{B_k}$. We may assume without loss of generality that $m = n$ and $A_k = B_k$ for all $k = 1, \ldots, m$. We can moreover assume that the A_k are disjoint. Then

$$\begin{aligned}
\|f+g\|_p &= \Big\| \sum_{k=1}^n (a_k + b_k) \mathbb{1}_{A_k} \Big\|_p = \Big(\sum_{k=1}^n |a_k + b_k|^p \mu(A_k) \Big)^{\frac{1}{p}} \\
&\leq \Big(\sum_{k=1}^n (|a_k| \mu(A_k)^{\frac{1}{p}} + |b_k| \mu(A_k)^{\frac{1}{p}})^p \Big)^{\frac{1}{p}} \\
&\leq \Big(\sum_{k=1}^n |a_k|^p \mu(A_k) \Big)^{\frac{1}{p}} + \Big(\sum_{k=1}^n |b_k|^p \mu(A_k) \Big)^{\frac{1}{p}} = \|f\|_p + \|g\|_p,
\end{aligned}$$

where for the first inequality we put $\mu(A_k)^{1/p}$ into the absolute value and used the triangle inequality, and for the second inequality we employed Minkowski's inequality for $\mathbb{R}^n$, Theorem 2.17. This proves (c) for simple functions.

Now let $f, g \in \mathscr{L}^p(\Omega)$. By Proposition 3.53, there exist sequences (f_n) and (g_n) of simple functions converging pointwise to f and g, respectively, with $|f_n| \leq 2|f|$ and $|g_n| \leq 2|g|$. In particular, f_n and g_n belong to $\mathscr{L}^p(\Omega)$ for all $n \in \mathbb{N}$. It also follows that $|f_n + g_n|^p \to |f+g|^p$ pointwise and hence, by Fatou's lemma, Theorem 3.71,

$$\begin{aligned}
\Big(\int_\Omega |f+g|^p \,\mathrm{d}\mu \Big)^{\frac{1}{p}} &= \Big(\int_\Omega \liminf_{n\to\infty} |f_n + g_n|^p \,\mathrm{d}\mu \Big)^{\frac{1}{p}} \\
&\leq \liminf_{n\to\infty} \Big(\int_\Omega |f_n + g_n|^p \,\mathrm{d}\mu \Big)^{\frac{1}{p}} \\
&\leq \liminf_{n\to\infty} (\|f_n\|_p + \|g_n\|_p) \\
&= \lim_{n\to\infty} (\|f_n\|_p + \|g_n\|_p) \\
&= \|f\|_p + \|g\|_p.
\end{aligned}$$

Here we have used the continuity of $t \mapsto |t|^{\frac{1}{p}}$ in the first and second step, Fatou's lemma in the second step, the established inequality for simple functions in the third step and the dominated convergence theorem and the existence of the limits in the last two steps. $\square$

By Proposition 3.100 the map $\|\cdot\|_p$ satisfies all properties of a norm on $\mathscr{L}^p(\Omega)$ except for (N1), i.e., it is possible that $\|f\|_p = 0$ without f being constantly zero. In order to overcome this difficulty, we identify functions which are equal almost everywhere. More precisely, on the space of measurable functions on Ω that we denote by $\mathscr{L}^0(\Omega, \Sigma, \mu)$ and which obviously contains $\mathscr{L}^p(\Omega, \Sigma, \mu)$, we introduce the equivalence relation $\sim$ by defining $f \sim g\colon \Leftrightarrow f = g$ almost everywhere. We then consider the equivalence classes $[f] := \{g : f \sim g\}$ as our primary objects.

*An **equivalence relation** is a relation on a set that is reflexive, symmetric and transitive. It partitions the set into classes of 'equivalent' elements.*

Exercise 3.101. Consider the measure space $([0,1], \mathscr{B}([0,1]), \lambda)$. Show that if f, g are continuous functions with $f = g$ almost everywhere, then $f = g$ everywhere. Now endow $([0,1], \mathscr{B}([0,1]))$ with the Dirac measure δ_1. For a measurable function f, determine its equivalence class $[f]$.

Definition 3.102. For a measure space (Ω, Σ, μ), we define

$$L^p(\Omega, \Sigma, \mu) := \{[f] : f \in \mathscr{L}^p(\Omega, \Sigma, \mu)\}.$$

We put $\|[f]\|_p := \|f\|_p$ and define $\lambda[f] := [\lambda f]$ and $[f] + [g] := [f+g]$. In this way, $L^p(\Omega, \Sigma, \mu)$ becomes a normed vector space.

Remark 3.103. That $L^p(\Omega, \Sigma, \mu)$ is a normed vector space is an immediate consequence of Proposition 3.100 and the definition of the norm, scalar multiplication and addition on L^p. However, one needs to check that these maps are **well defined**, i.e., that they do not depend on the choice of the particular representative of $[f]$. For example, we have to show that if $f \sim g$ then $\|f\|_p = \|g\|_p$ (this follows from Corollary 3.68) and $\lambda f = \lambda g$ almost everywhere, etc.

Remark 3.104. As is customary, we will not distinguish between f and $[f]$ and treat elements of $L^p(\Omega, \Sigma, \mu)$ as functions, rather than as equivalence classes, and understand equalities, inequalities, etc. only to hold almost everywhere.

Theorem 3.105. *Let (Ω, Σ, μ) be a measure space and $1 \le p < \infty$. Then $(L^p(\Omega), \|\cdot\|_p)$ is complete.*

The proof of Theorem 3.105 rests on the following Lemma which is also of independent interest.

Lemma 3.106. *Let (Ω, Σ, μ) be a measure space and let $1 \le p < \infty$. If $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(L^p(\Omega), \|\cdot\|_p)$ (in particular, if (f_n) converges in L^p), then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ which converges pointwise almost everywhere to an $f \in L^p(\Omega)$. Moreover, the subsequence can be chosen such that there exists a $g \in L^p(\Omega)$ such that $|f_{n_k}| \le g$ for all $k \in \mathbb{N}$.*

Proof. Since (f_n) is a Cauchy sequence, there exists a subsequence (f_{n_k}) with $\|f_{n_{k+1}} - f_{n_k}\|_p \leq 2^{-k}$. We now define

$$h_k := f_{n_{k+1}} - f_{n_k} \quad \text{and} \quad h := \sum_{k=1}^{\infty} |h_k|.$$

Then, using Proposition 3.100, we see that for every $N \in \mathbb{N}$, we have

$$\Big\| \sum_{k=1}^{N} |h_k| \Big\|_p \leq \sum_{k=1}^{N} \|f_{n_{k+1}} - f_{n_k}\|_p \leq \sum_{k=1}^{N} 2^{-k} \leq 1.$$

It follows from monotone convergence, that $h \in L^p(\Omega)$ with $\|h\|_p \leq 1$. Note that since $h \in L^p(\Omega)$ it follows that $|h| < \infty$ a.e. (Exercise!). In particular, the series $\sum_{k=1}^{\infty} h_k(x)$ converges (absolutely) for almost all $x \in \Omega$. Noting that $\sum_{k=1}^{N} h_k = f_{n_{N+1}} - f_{n_1}$, it follows that (f_{n_k}) converges almost everywhere to $f := f_{n_1} + \sum_{k=1}^{\infty} h_k$. We may suppose that f is defined everywhere and measurable. Then $f \in L^p(\Omega)$ as $\|f\|_p < \infty$. Moreover,

This is a 'teleskopic' sum; all the middle terms cancel.

$$|f_{n_k}| = |h_1 + \cdots + h_{k-1} + f_{n_1}| \leq h + |f_{n_1}| =: g.$$

Since $g \in L^p(\Omega)$, the proof is complete. □

Let us give some examples illustrating Lemma 3.106.

Example 3.107. Consider the measure space $([0,1], \mathscr{B}([0,1]), \lambda)$. Then $f_n(t) := t^n$ converges to $f(t) \equiv 0$ in $L^p([0,1])$ for all $1 \leq p < \infty$. Indeed,

$$\|f_n - f\|_p = \Big(\int_0^1 t^{np} \, \mathrm{d}t \Big)^{\frac{1}{p}} = \Big(\frac{1}{np+1} \Big)^{\frac{1}{p}} \to 0.$$

Moreover, $f_n(t)$ converges to $f(t)$ for all $t \in [0,1)$. Since the singleton $\{1\}$ has Lebesgue measure zero, f_n converges to f almost everywhere, but f_n does not converge to f pointwise.

Example 3.108. Consider the measure space $((0,1], \mathscr{B}((0,1]), \lambda)$. If $m = 2^n + k$ for $n \in \mathbb{N}$ and $0 \leq k \leq 2^n - 1$, put $f_m = \mathbb{1}_{(k2^{-n},(k+1)2^{-n}]}$. Then $f_m \to 0$ in $L^p((0,1])$, since $\|f_{2^n+k}\|_p = 2^{-n/p} \to 0$. By Lemma 3.106, (f_m) has a subsequence converging to 0 almost everywhere (an example being (f_{m_n}), where $m_n = 2^n$ such that $f_{m_n} = \mathbb{1}_{(0,2^{-n}]}$). In fact, every subsequence of (f_m) has a subsequence that converges to 0 almost everywhere, but the whole sequence (f_m) does not converge to 0 almost everywhere. This can be used to show that convergence almost everywhere is not a notion of convergence that comes from a metric.

So in general convergence almost everywhere (almost sure convergence) and convergence in L^p (convergence in the p-th mean) do not imply each other. Under additional assumptions, the dominated convergence theorem can help to show the first implication.

Proof of Theorem 3.105. Let (f_n) be a Cauchy sequence in $(L^p(\Omega), \|\cdot\|_p)$. By Lemma 3.106, there exists a subsequence (f_{n_k}) that converges almost surely to a function $f \in L^p(\Omega)$ and is dominated by a function $g \in L^p(\Omega)$.

By the dominated convergence theorem, $\|f_{n_k} - f\|_p \to 0$, that is, $f_{n_k} \to f$ with respect to $\|\cdot\|_p$. Since a Cauchy sequence converges if and only if it has a convergent subsequence, the whole sequence (f_n) converges to f with respect to $\|\cdot\|_p$. □

We now complement/complete the scale of L^p-spaces by introducing the space $L^\infty(\Omega, \Sigma, \mu)$. As before, we are formally dealing with equivalence classes of functions rather than with functions themselves.

Definition 3.109. Let (Ω, Σ, μ) be a measure space. An equivalence class $[f]$ is said to belong to $L^\infty(\Omega, \Sigma, \mu)$ if there exists a constant $c > 0$ such that $|f| \le c$ almost everywhere. In this case, we set

$$\|[f]\|_\infty := \inf\{c > 0 : |f| \le c \text{ a.e.}\}.$$

In practice, we again do not distinguish between f and $[f]$. As is to be expected, the normed space $(L^\infty(\Omega), \|\cdot\|_\infty)$ turns out to be complete. We formulate this result and leave the proof to the reader.

Proposition 3.110. *Let (Ω, Σ, μ) be a measure space. Then $(L^\infty(\Omega), \|\cdot\|_\infty)$ is a complete normed space.*

Remark 3.111. In what follows, $L^p(\Omega, \Sigma, \mu)$ will always be endowed with the norm $\|\cdot\|_p$. We will therefore frequently drop it from our notation and say, e.g., that $f_n \to f$ in $L^p(\Omega, \Sigma, \mu)$ or that (f_n) is a Cauchy sequence in $L^p(\Omega, \Sigma, \mu)$ with the understanding, that this is to be understood with respect to the norm $\|\cdot\|_p$.

We point out that $L^p((0,1], \mathscr{B}((0,1]), \lambda)$ can be identified with $L^p((0,1), \mathscr{B}((0,1)), \lambda)$. Moreover, the spaces $L^p(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d), \lambda_d)$ and $L^p(\mathbb{R}^d, \mathscr{M}, \lambda_d^*)$ are the same; here $\mathscr{M}$ are the Lebesgue measurable sets and λ_d^* is the completion of λ_d as in Carathéodory's theorem.

Example 3.112. Consider the measure space $(\mathbb{N}, \mathscr{P}(\mathbb{N}), \zeta)$. Then it follows that $L^p(\mathbb{N}, \mathscr{P}(\mathbb{N}), \zeta) = \ell^p$ for $1 \le p \le \infty$.

We now also extend Hölder's inequality to the L^p setting.

Theorem 3.113. *Let (Ω, Σ, μ) be a measure space and $1 \le p, q \le \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^1(\Omega)$ and $\|fg\|_1 \le \|f\|_p \|g\|_q$.*

Proof. The proof is similar to that of Minkowski's inequality in Proposition 3.100. We give a rough sketch.

If f and g are simple functions, then the claim follows from Hölder's inequality for finite sums, Theorem 2.17. The general case follows from an approximation argument using Fatou's lemma and dominated convergence. □

Exercise 3.114. Work out the details of the proof of Theorem 3.113.

Exercise 3.115. Let (Ω, Σ, μ) be a measure space and $1 \leq p, q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Given $g \in L^q(\Omega)$, define $\varphi_g \colon L^p(\Omega) \to \mathbb{K}$ by

$$\varphi_g(f) = \int_\Omega fg \,\mathrm{d}\mu.$$

Show that $\varphi_g \in (L^p(\Omega))'$, the dual space of $L^p(\Omega)$, such that $\|\varphi_g\|_{(L^p(\Omega))'} = \|g\|_q$.

Recall that $\ell^q \subset \ell^p$ for $1 \leq q \leq p \leq \infty$. So for the sequence spaces the inclusions are opposite to the finite measure case. Combining both cases shows that in general one cannot expect the L^p spaces to be ordered by inclusion in either way.

Corollary 3.116. *Let (Ω, Σ, μ) be a* finite *measure space and $1 \leq p \leq q \leq \infty$. Then $L^q(\Omega, \Sigma, \mu) \subset L^p(\Omega, \Sigma, \mu)$. Moreover, if $f_n \to f$ in $(L^q(\Omega, \Sigma, \mu), \|\cdot\|_q)$, then $f_n \to f$ in $(L^p(\Omega, \Sigma, \mu), \|\cdot\|_p)$.*

Proof. Let us first consider the case where $q = \infty$. In this case, if $f \in L^\infty(\Omega)$, then $f \leq \|f\|_\infty \mathbb{1}_\Omega$ almost everywhere. Hence

$$\|f\|_p^p = \int_\Omega |f|^p \,\mathrm{d}\mu \leq \int_\Omega \|f\|_\infty^p \,\mathrm{d}\mu = \mu(\Omega)\|f\|_\infty^p.$$

This proves that $L^\infty(\Omega) \subset L^p(\Omega)$. Now let $f_n \to f$ in $L^\infty(\Omega)$. Then $\|f_n - f\|_p \leq \mu(\Omega)^{1/p}\|f_n - f\|_\infty \to 0$, proving that $f_n \to f$ in $L^p(\Omega)$.

Next let $1 \leq p < q \neq \infty$ and fix $f \in L^q(\Omega)$. Then $r := \frac{q}{p} \in (1, \infty)$. With $s = \frac{q}{q-p}$, we have $\frac{1}{r} + \frac{1}{s} = 1$. It is clear that $\mathbb{1}_\Omega \in L^s(\Omega)$. Moreover, $|f|^p \in L^r(\Omega)$. By Hölder's inequality, Theorem 3.113, it follows that $|f|^p \mathbb{1}_\Omega \in L^1(\Omega)$ and

$$\int_\Omega |f|^p \,\mathrm{d}\mu \leq \|\mathbb{1}_\Omega\|_s \||f|^p\|_r = \mu(\Omega)^{\frac{1}{s}} \Big(\int_\Omega |f|^q\Big)^{\frac{1}{r}} = \mu(\Omega)^{\frac{q-p}{q}} \|f\|_q^p.$$

This proves that $f \in L^p(\Omega)$. Moreover, taking p-th roots on both sides, $\|f\|_p \leq \mu(\Omega)^{(q-p)/(pq)}\|f\|_q$ follows. As above, this inequality also shows that if $f_n \to f$ in $(L^q(\Omega, \Sigma, \mu), \|\cdot\|_q)$, then $f_n \to f$ in $(L^p(\Omega, \Sigma, \mu), \|\cdot\|_p)$. □

This result also holds for a general metric space (M, d).

Theorem 3.117. *Let X be a normed space and $M \subset X$. Let μ be a finite measure on $(M, \mathscr{B}(M))$. Then $\mathrm{C_b}(M)$ is dense in $L^p(M, \mathscr{B}(M), \mu)$ for all $1 \leq p < \infty$.*

Proof. Let E be the closure of $\mathrm{C_b}(M)$ in $L^p(M, \mathscr{B}(M), \mu)$. Clearly, E is a closed subspace of L^p.

Define $\mathscr{G} := \{A \in \mathscr{B}(M) : \mathbb{1}_A \in E\}$. Then $\mathscr{G}$ is a Dynkin system. Indeed, $\mathbb{1}_\Omega$ is continuous and integrable since $\mu(\Omega) < \infty$. Hence $\Omega \in \mathscr{G}$. If $A \in \mathscr{G}$, then $A^\mathrm{c} \in \mathscr{G}$ since $\mathbb{1}_{A^\mathrm{c}} = \mathbb{1}_\Omega - \mathbb{1}_A$ and E is a vector space. Finally, let (A_k) be a sequence of pairwise disjoint sets in $\mathscr{G}$. Then $\bigcup_{k=1}^n A_k \in \mathscr{G}$ since $\mathbb{1}_{\bigcup_{k=1}^n A_k} = \sum_{k=1}^n \mathbb{1}_{A_k}$ and E is a vector space. Moreover, $\mathbb{1}_{\bigcup_{k=1}^n A_k} \to \mathbb{1}_{\bigcup_{k=1}^\infty A_k}$ pointwise for $n \to \infty$ and all functions are dominated by the integrable function $\mathbb{1}_\Omega$. By dominated convergence and as E is closed it follows that $\bigcup_{k=1}^\infty A_k \in \mathscr{G}$.

Now let $F \subset M$ be relatively closed. With the help of Urysohn's lemma we find, as in the proof of Proposition 3.23, a sequence (f_n) of continuous functions with $0 \le f_n \le 1$ and $f_n \to \mathbb{1}_F$ pointwise. It follows from dominated convergence that $F \in \mathscr{G}$.

Since $\mathscr{C}$, the collection of all relatively closed subsets of M, is a generator of $\mathscr{B}(M)$ that is stable under intersections, Dynkin's π–λ theorem yields $\mathscr{B}(M) = \mathrm{dyn}(\mathscr{C}) \subset \mathscr{G} \subset \mathscr{B}(M)$. Hence $\mathbb{1}_A \in E$ for all $A \in \mathscr{B}(M)$. By linearity, E contains all simple functions. Now an approximation argument shows that $E = L^p(M)$. □

Corollary 3.118. *Let X be a normed space and $K \subset X$ be compact. Let μ be a finite measure on $(K, \mathscr{B}(K))$. Then $L^p(K, \mathscr{B}(K), \mu)$ is separable for all $1 \le p < \infty$.*

We note that other conditions on the underlying measure space are sufficient for the separability of the corresponding L^p spaces for $1 \le p < \infty$. For example, the space $L^p(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d), \lambda_d)$ is separable. The space L^∞ is only 'very rarely' separable. Give an example!

Proof. By Corollary 2.69, there exists a countable set $S \subset \mathrm{C}(K)$ that is dense with respect to the norm $\|\cdot\|_\infty$. Let E be the closure of S in L^p. Then $\mathrm{C}(K) \subset E$. Indeed, given $f \in \mathrm{C}(K)$, there exists a sequence (f_n) in $S \subset E$ such that $f_n \to f$ with respect to $\|\cdot\|_\infty$. By Corollary 3.116, $f_n \to f$ in L^p, hence $f \in E$. Now Theorem 3.117 yields $L^p(K, \mathscr{B}(K), \mu) = E$. □