Generating Cycles in Graphs With at Most One End

Henning Bruhn*

UNIVERSITÄT HAMBURG, HAMBURG GERMANY

E-mail: hbruhn@gmx.net

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Abstract: Answering a question of Halin, we prove that in a 3-connected graph with at most one end the cycle space is generated by induced non-separating cycles. J Graph Theory 00: 1–8, 2003

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1. INTRODUCTION

The *cycle space* of a graph G is the set of all symmetric differences of the edge-sets of finitely many cycles (2-regular connected subgraphs). With the addition defined through the symmetric difference, the cycle space is a $\mathbb{Z}_2$ vector space.

A classical result of Tutte [9] states that in a finite 3-connected graph the cycle space is generated by induced non-separating cycles. Following Tutte we shall call those cycles *peripheral*. For infinite graphs, however, this needs not to be true. An obvious counterexample is the cartesian product of a cycle C with a *double ray* (a 2-way infinite path). There, every copy of C is separating but not the sum of any set of peripheral cycles.

*Correspondence to: Henning Bruhn; Universität Hamburg, Hamburg, Germany. E-mail: hbruhn@gmx.net

Two aproaches to deal with this counterexample have been pursued. Diestel and Kühn [4,5] developed a notion of the cycle space that is better adapted to infinite graphs, and with which Tutte's theorem can be extended to locally finite graphs. For a brief discussion see the end of the next section.

In contrast, Halin [7] looked for a class of infinite graphs for which Tutte's theorem still holds (with the usual definition of the cycle space). For this, note that the counterexample above has two ends. (An *end* is the equivalence class of *rays* (1-way infinite paths), where two rays are said to be equivalent if they cannot be separated by finitely many vertices.) In fact, all known counterexamples have at least two ends. In addition, Halin observed that in planar one-ended graphs and in rayless graphs (which, thus, do not have ends) Tutte's theorem is valid.

Theorem 1.1 (Halin [7]). *The cycle space of a* 3*-connected rayless graph is generated by peripheral cycles.*

Motivated by these results he raised the following problem:

Problem 1.1 (Halin [7]). *Is the cycle space of every* 3*-connected graph with at most one end generated by peripheral cycles?*

Weakening Problem 1.1, Halin made the following conjecture:

Conjecture 1.1 (Halin [7]). *The cycle space of every* 3*-connected graph that does not contain a double ray is generated by peripheral cycles.*

We will give a positive answer to Problem 1.1, thereby proving Conjecture 1.1. This will be done by an extension of Halin's proof of Theorem 1.1, which in turn employs a powerful tool for rayless graphs developed by Schmidt [8].

2. GENERATING THE CYCLE SPACE

In general our notation and terminology follows Diestel [3]. We recall a standard concept that naturally arises when dealing with cycles. Here, we have taken the definition from Bondy and Murty [1].

Definition 2.1. *Let H be a subgraph of a graph G. We define an equivalence relation on $E(G)\setminus E(H)$ by $e \sim f$ if there is a walk W such that:*

(i) *the first edge of W is e and the last is f; and*
(ii) *W meets H at most at its ends.*

A connected non-trivial subgraph B of $G-E(H)$ whose edge set is closed under this equivalence relation is called a bridge of H. The vertices of B on H are called the vertices of attachment of B.

One sees easily that a bridge is either a chord of H or a subgraph of G consisting of a component K of $G-H$ with the edges $E(K,H)$ added. Note, that a cycle is peripheral, if and only if, it has at most one bridge.

The key to the solution of Problem 1.1 is the following observation. Let a cycle C in a 3-connected graph G with at most one end be given. As C is finite it cannot separate any two rays of G. Consequently, C has at most one bridge containing rays. Substituting that bridge through a suitable rayless one we obtain a rayless graph G' whose cycle space differs not too much from that of G, but is, by Theorem 1.1, generated by peripheral cycles.

The following simple lemma shows how we can find an appropriate substitute bridge for the unwanted bridge.

Lemma 2.1. *Let G be a 3-connected graph and let C be a cycle in G with a bridge B that is not a chord. Denote by G' the graph obtained from G by contracting the bridge B to a vertex v_B (with any loops and multi-edges deleted). Then G' is 3-connected. Moreover, every peripheral cycle D in G' that avoids v_B is a peripheral cycle in G as well.*

Proof. It is straightforward to see that G' is 3-connected, so let us verify the second assertion. Clearly, a cycle D in G' that avoids v_B may be viewed as a cycle in G. We claim that D has exactly one bridge in G. So, consider an edge $e \in E(B)$. We show that every edge $f \in E(G) \setminus E(D)$ is equivalent to e. Should f be another edge of B this is obvious, so assume $f \notin E(B)$. In particular, f is then an edge of G' as well. Let e' be an edge of G' incidident with v_B. D avoids v_B and has only a single bridge in G'. Consequently, there is a walk $W' = f \cdots e'$ in G' internally avoiding D. Denote by v the predecessor (of the first occurrence) of v_B on W'. Note that $v \in V(C \cap B)$. Hence, there is a path $Q = v \cdots e \subseteq B$ internally disjoint to C (and, thus, to D as well). Observe, that v lies not in D; otherwise W' could not be internally disjoint to D. With this, we obtain the walk $W = W'vQ$ from f to e which meets D at most in its endvertices. ■

We know now how to obtain a rayless graph G' from our original graph G suitable to our problem. In that graph Theorem 1.1 delivers generating cycles for an arbitrary cycle; we only have to ensure that these avoid our substitute bridge. This amounts to the following slightly strengthened version of Halin's theorem.

Lemma 2.2. *Let G be 3-connected and rayless. Let C be a cycle in G with a finite bridge B. Then C is the sum of peripheral cycles each meeting B at most in C.*

We start with the finite version of Lemma 2.2. For that, we need some further terminology. Let C be a cycle in a finite graph G. Two bridges B, B' of C are called *overlapping* if there is no path $P \subseteq C$ such that all vertices of attachment of B belong to P, but no inner vertex of P is a vertex of attachment of B'. Two bridges B, B' are called *skew* if there are vertices u, u', v, v' appearing in that circular order on C such that $u, v \in V(B)$ and $u', v' \in V(B')$. It is easy to see that in a 3-connected graph two overlapping bridges are skew or have three vertices in common.

Lemma 2.3 (Tutte [9])**.** *Let G be a finite and 3-connected graph. Let C be a cycle in G with a bridge B. Then either C is peripheral or there is another bridge B' of C overlapping B.*

Lemma 2.4, the finite version of Lemma 2.2, is essentially Tutte's [9] original theorem. As the proof is rather short, we include it here for the convenience of the reader.

Lemma 2.4. *Let G be a finite and 3-connected graph. Let C be a cycle in G with a bridge B. Then C is the sum of peripheral cycles each meeting B at most in C.*

Proof. Let U be the set of sums of peripheral cycles that meet B at most in C. If $C \in U$ we are done, so assume otherwise. Then there is a cycle $D \notin U$ with:

(i) D has a bridge $B' \supseteq B$; and
(ii) no cycle $D' \notin U$ has a bridge that properly contains B'.

By Lemma 2.3 there is another bridge $\tilde{B}$ that overlaps B'.

First, let B' and $\tilde{B}$ be skew. Thus, there are vertices u, x, v, y that appear in that circular order on D, such that $u, v \in V(B')$ and $x, y \in V(\tilde{B})$. Denote by $P = x \cdots y \subseteq \tilde{B}$ a path internally disjoint to D. Then $C_1 := xCyP$ and $C_2 := yCxP$ are cycles each having a bridge that properly contains B'. Thus, we have $C_1, C_2 \in U$, in contradiction to $C_1 + C_2 = D \notin U$.

Finally, let B' and $\tilde{B}$ have three vertices x, y, z in common. There is a $b \in V(\tilde{B})$ and three paths $P_x = x \cdots b$, $P_y = y \cdots b$, and $P_z = z \cdots b$ in $\tilde{B}$ meeting only in b and that are each internally disjoint to D. With these we obtain three cycles: $C_{xy} := xCyP_yP_x$, $C_{yz} := yCzP_zP_y$ and $C_{zx} := zCxP_xP_z$. Every one of these has a bridge properly containing B'. Thus, we have again the contradiction of $C_{xy}, C_{yz}, C_{zx} \in U$ but $D = C_{xy} + C_{yz} + C_{zx}$. ■

Lemma 2.2 can be proved by closely following Halin's proof of Theorem 1.1. Our sole contribution is to include the bridge B into the reasoning, which is an easy task as B is assumed to be finite. But first, we need a result by Schmidt [8] (see Halin [6] for an exposition in English):

For every ordinal μ we define a set $A(\mu)$ of graphs. Let $A(0)$ be the set of all finite graphs. If $\mu > 0$ is such that $A(\lambda)$ is defined for all $\lambda < \mu$, then we put a graph G in $A(\mu)$ if there is a finite set $F \subseteq V(G)$ such that every component of $G - F$ is contained in $A(\lambda)$ for some $\lambda < \mu$. Let A be the union of all the $A(\mu)$ and define $o(G) := \min\{\mu \mid G \in A(\mu)\}$ for all $G \in A$.

Proposition 2.1 (Schmidt [8])**.** *A is the set of all rayless graphs. In particular, the function o is defined for all rayless graphs G and has the following properties:*

(i) *G is finite, if and only if, $o(G) = 0$; and*
(ii) *if $o(G) > 0$, there is a finite set $F \subseteq V(G)$ such that for every component K of $G - F$ we have $o(K) < o(G)$.*

We call $o(G)$ the order of G.

To illustrate this further, consider the graph G in Figure 1, which has order 1. Indeed, for $F := \{u, v, w\}$ all the components of $G - F$ are single vertices, and thus have order 0.

We will need the following properties below. Firstly, for every infinite rayless graph G there is a unique minimal vertex set satisfying (ii). This set is called the *kernel* of G and denoted by $K(G)$.

Secondly, we need the notion of a *subkernel*. For a vertex set $S \subseteq K(G)$, we denote by $\mathcal{C}_S$ the set of all components K of $G - K(G)$ with neighbourhood $N(K) = S$. Then S is called a subkernel of G if the graph $G[S \cup \bigcup \mathcal{C}_S]$ has the same order as G. In Figure 1, the graph has kernel $K(G) = \{u, v, w\}$ while the subkernels are precisely $\{u\}$ and $\{v, w\}$ (the kernel is always the union of the subkernels). In particular, $\{w\}$ is not a subkernel as there is only one component, a single vertex, with a neighbourhood that equals $\{w\}$.

Thirdly, for finitely many rayless graphs $G_1, \ldots, G_n$, the order of the union is the maximum of the orders of the constituent graphs: $o(G_1 \cup \cdots \cup G_n) = \max_{i=1\cdots n} o(G_i)$.

Finally, for a subkernel S the set $\mathcal{C}_S$ is necessarily infinite; otherwise $G[S \cup \bigcup \mathcal{C}_S]$ would be the finite union of graphs each having a strictly smaller order than G, resulting in $o(G[S \cup \bigcup \mathcal{C}_S]) < o(G)$, which is impossible for a subkernel.

Proof of Lemma 2.2. We perform a transfinite induction on $o(G)$. For $o(G) = 0$, G is finite. Hence, Lemma 2.4 ensures the induction start.

So, assume $o(G) \geq 1$ and let the assertion be true for smaller orders. There is a set $\mathcal{S}$ of components of $G - K(G)$ with the following properties:

(i) $C \cup B$ is contained in $G' := G[K(G) \cup \bigcup \mathcal{S}]$;
(ii) for every subkernel S of G the set of $K \in \mathcal{S}$ with $N(K) = S$ is finite but has at least $|K(G)| + 2$ elements; and
(iii) $\mathcal{S}$ contains all the components K of $G - K(G)$ whose neighbourhood $N(K)$ is not a subkernel.

In fact, we may choose $\mathcal{S}$ as follows. Let $\mathcal{S}_1$ be the finitely many of the components of $G - K(G)$ that meet $C \cup B$ (which is a finite set). With this we

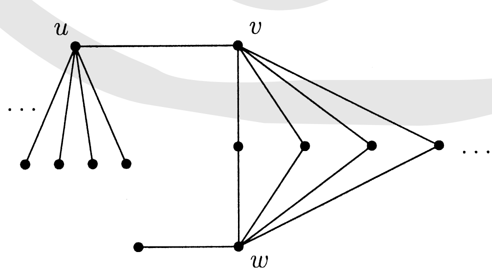

FIGURE 1. A graph with order $o(G) = 1$ and kernel $K(G) = \{$[illegible]$w\}$.

have ensured (i). Now consider a subkernel S for which the set $\mathcal{S}_1$ does not satisfy (ii). By definition of a subkernel there are infinitely many components of $G - K(G)$ whose neighbourhood is S. Thus, by adding to $\mathcal{S}_1$ for each such S the finitely many missing components we arrive at a finite set $\mathcal{S}_2$ that satisfies (i) and (ii). All that is left to deal with is (iii), which is easily satisfied by including the by (iii) required components into $\mathcal{S}_2$. Note that doing this violates neither (i) nor (ii).

Having established the existence of such a set $\mathcal{S}$ we claim that G' is 3-connected and its order $o(G')$ is strictly smaller than $o(G)$.

For the 3-connectivity consider two vertices x and y of G'. For any two other vertices u and v of $G' - \{x, y\}$ there is an $u - v$ path P in $G - \{x, y\}$. The only reason why P may fail to be a path in G' as well is that P meets a component K of $G - K(G)$ with $K \notin \mathcal{S}$. By (iii), the neighbourhood $N(K)$ of K has to be a subkernel. By (ii), there are at least three components of $G - K(G)$ in $\mathcal{S}$ with neighbourhood $N(K)$. At least one of these components is disjoint to $\{x, y\}$ and thus may be used to substitute the part of P that goes through K. By doing this for all such components K we arrive at (a walk that can be pruned to) an $u - v$ path P' in $G' - \{x, y\}$.

To see $o(G') < o(G)$, observe that the set $\mathcal{S}'$ of components in $\mathcal{S}$ whose neighbourhood is a subkernel of G is finite. Indeed, this is because of (ii) and the fact that there are only finitely many subkernels. Hence, we have $o(G[\bigcup \mathcal{S}']) < o(G)$ as each of the components has smaller order than G. For a set $S \subseteq K(G)$ that is not a subkernel, the union of the components $K \in \mathcal{S}$ with S as neighbourhood must have smaller order; otherwise S would be a subkernel. Since the number of subsets of $K(G)$ is finite, there are only finitely many of those unions. Therefore, the order of the union of these unions is smaller than $o(G)$ too. Combined with the preceding observation for subkernels we have established $o(G') < o(G)$.

Next, we show that a peripheral cycle D in G' is still peripheral in G. It is sufficient to assume D to be separating in G as D is induced in G' which, in turn, is an induced subgraph. $G' - D$ is connected and therefore contained in a component K of $G - D$. Suppose there is a different component K' of $G - D$ and denote its set of neighbours $N(K')$ by S. K' being disjoint to G' is therefore certainly disjoint to $K(G)$ as well. But S is contained in $K(G)$ since all the neighbours of $D - K(G)$ are in G'. As a consequence, K' is a component of $G - K(G)$ and by condition (iii), S a subkernel. Condition (ii) asserts that there are $|K(G)| + 2 \geq |S| + 2$ components in $\mathcal{S}$ with neighbourhood S. As K' is separated from K by D, any other component of $G - K(G)$ with neighbourhood S that is not met by D is as well separated from K by D. But D may meet at most $|S|$ of the components in $\mathcal{S}$ with neighbourhood S and, hence, avoids at least two of those components. But then D is already separating in G'—a contradiction.

Since $B \subseteq G'$, B is still a bridge of C in G'. By applying the induction hypothesis to the cycle C with bridge B in G' we obtain C as the sum of peripheral cycles that each meet B at most in C. By the argument above these cycles are peripheral in G as well. ■

As noted, the proof of Halin's problem follows easily.

Theorem 2.1. *The cycle space of a 3-connected graph G with at most one end is generated by peripheral cycles.*

Proof. As every element of the cycle space is the (finite) sum of cycles, it is sufficient to consider a cycle C in G. Since C is finite and G has at most one end there is a bridge B of C containing a tail (a subray) of every ray in G. Hence, the graph G' obtained from G as in Lemma 2.1 is rayless and 3-connected. Lemma 2.2 yields a set of peripheral cycles in G' each avoiding v_B whose sum is C. By Lemma 5 these cycles are peripheral cycles in G too, as required. ■

Because of the obvious counterexample, the cartesian product of a cycle C with a double ray, Halin excluded all graphs with multiple ends in Problem 2.1. However, given the right notion of the cycle space the counterexample ceases to be one. Indeed, if we allow the infinitely many peripheral cycles to the *left* of C, say, to be summed up, we obtain C. Diestel and Kühn [4,5] developed a notion of the cycle space that allows these kind of sums. They obtained this space as an adaptation of the (classical) cycle space to *infinite cycles* involving the ends of the graph. (These infinite cycles are homeomorphic images of the unit circle in the standard topology on the graph together with its ends.)

Building on these results it is shown in [2] that Tutte's theorem generalizes neatly to all locally finite graphs. While this generalization fails for arbitrary infinite graphs, Theorem 2.1 might be an indication that it is still valid for graphs with at most one end. Therefore, it seems worthwhile to explore if the cycle space of (3-connected) graphs with at most one end is generated by peripheral cycles even if infinite cycles and (well-defined) infinite sums are allowed.

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