Proof of Hu's theorem

This short note contains the details of the proof of Theorem 24 in [1].

Theorem 1 (Hu [2])**.** *Suppose there is a $c > 2$ so that any separating union-closed family $\mathcal{A}'$ with $|\mathcal{A}'| \leq c|U(\mathcal{A}')|$ satisfies the union-closed sets conjecture. Then, in every union-closed family $\mathcal{A}$, there is an element that appears in at least $\frac{c-2}{2(c-1)}|\mathcal{A}|$ member-sets of $\mathcal{A}$.*

Proof. Let $\mathcal{A}$ be a union-closed family. Clearly, we may assume $\mathcal{A}$ to be separating. Moreover, we can suppose that $n := |\mathcal{A}| > cm$, where $m := |U(\mathcal{A})|$ is the size of the universe; otherwise there is even, by assumption, an element in $U(\mathcal{A})$ that appears in at least half of the member-sets.

Put

$$p := \left\lceil \frac{n - cm}{c - 1} \right\rceil \leq \frac{n - cm}{c - 1} + 1 \tag{1}$$

Pick some element $x_0 \in U(\mathcal{A})$, and introduce p new elements $X := \{x_1, \ldots, x_p\}$, disjoint from $U(\mathcal{A})$. We define a family

$$\begin{aligned} \mathcal{A}' :=& \{A \cup X : A \in \mathcal{A},\, x_0 \in A\} \cup \{A \in A : x_0 \notin A\} \\ & \cup \{U(\mathcal{A}) \cup (X - x_i) : i = 1, \ldots, p\} \end{aligned}$$

The family is obviously union-closed, and moreover, it is separating. Indeed, any elements of $U(\mathcal{A})$ can still be separated as $\mathcal{A}$ is separating, while the elements $x_1, \ldots, x_p$ are separated by the sets $\{U(\mathcal{A}) \cup (X - x_i) : i = 1, \ldots, p\}$.

The number of sets in $\mathcal{A}'$ is $n+p$, while the universe has grown to $m+p$. We can easily check that $n + p \leq c(m + p)$, so that we can use the assumption that there is an element u^* in $U(\mathcal{A}')$ that appears in at least $\frac{n+p}{2}$ member-sets. Note that x_0 appears more often than any of $x_1, \ldots, x_p$, so that we may assume that $u^* \in U(\mathcal{A})$. Then, however, we see that u^* appears in at least $\frac{n+p}{2} - p = \frac{n-p}{2}$ of the member-sets of $\mathcal{A}$.

We compute with (1) that

$$\begin{aligned} \frac{n-p}{2} &\geq \frac{1}{2}\left(n - \frac{n - cm}{c - 1} - 1\right) = \frac{1}{2}n\left(\frac{c-2}{c-1} + \frac{1}{n}\left(\frac{cm}{c-1} - 1\right)\right) \\ &> \frac{1}{2}n \cdot \frac{c-2}{c-1}, \end{aligned}$$

as $c > 2$ entails that $cm > c - 1$. Thus, there is an element that appears in at least $\frac{c-2}{2(c-1)}n$ of members-sets of $\mathcal{A}$. □

References

[1] H. Bruhn and O. Schaudt, *The journey of the union-closed sets conjecture*, to appear in *Graphs and Combinatorics*.

[2] Y. Hu, Master's thesis, in preparation.