Bicycles and left-right tours in locally finite graphs

Henning Bruhn

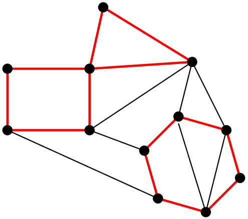
Universität
Hamburg

Infinite Graphs Workshop 2007

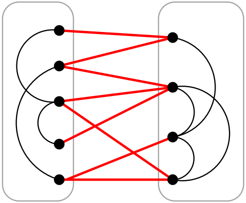
joint with S. Kosuch & M. Win Myint

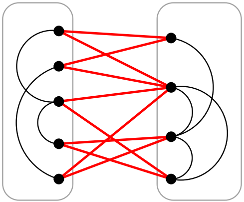
Bicycles in finite graphs

cycle space $\mathcal{C}(G)$



cut space $\mathcal{C}^*(G)$





bicycle space
$\mathcal{B}(G) = \mathcal{C}(G) \cap \mathcal{C}^*(G)$

Tripartition theorem

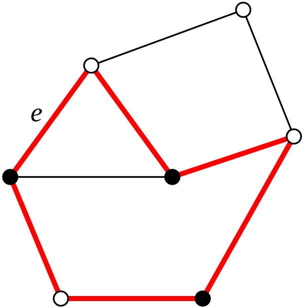
Theorem (Read & Rosenstiehl)

Let G finite, e edge of G. Then exactly one of the following holds:

(i) $\exists B \in \mathcal{B}(G)$ *with* $e \in B$

(ii) $\exists Y \in \mathcal{C}(G)$ *with* $e \in Y$ *and* $Y + e \in \mathcal{C}^*(G)$

(iii) $\exists Z \in \mathcal{C}(G)$ *with* $e \notin Z$ *and* $Z + e \in \mathcal{C}^*(G)$.



Tripartition theorem

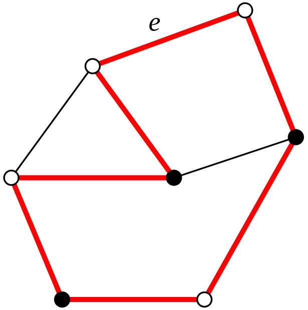
Theorem (Read & Rosenstiehl)

Let G finite, e edge of G. Then exactly one of the following holds:

(i) $\exists\, B \in \mathcal{B}(G)$ *with* $e \in B$

(ii) $\exists\, Y \in \mathcal{C}(G)$ *with* $e \in Y$ *and* $Y + e \in \mathcal{C}^*(G)$

(iii) $\exists\, Z \in \mathcal{C}(G)$ *with* $e \notin Z$ *and* $Z + e \in \mathcal{C}^*(G)$.



Tripartition theorem

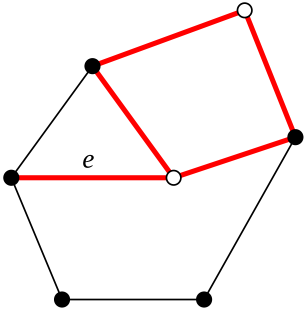
Theorem (Read & Rosenstiehl)

Let G finite, e edge of G. Then exactly one of the following holds:

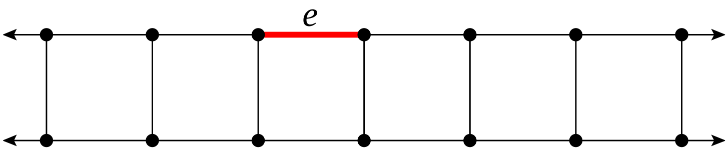
(i) $\exists\, B \in \mathcal{B}(G)$ with $e \in B$

(ii) $\exists\, Y \in \mathcal{C}(G)$ with $e \in Y$ and $Y + e \in \mathcal{C}^*(G)$

(iii) $\exists\, Z \in \mathcal{C}(G)$ with $e \notin Z$ and $Z + e \in \mathcal{C}^*(G)$.

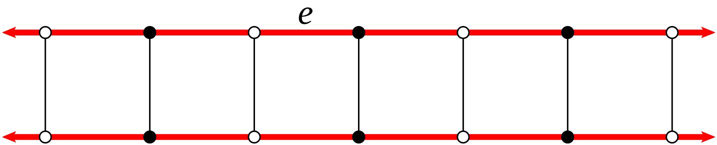


Tripartition in ∞ graphs?



- no finite bicycle $\ni e$
- no finite $Z \in \mathcal{C}(G)$ with $Z + e \in \mathcal{C}^*(G)$

Tripartition in ∞ graphs?



- no finite bicycle $\ni e$
- no finite $Z \in \mathcal{C}(G)$ with $Z + e \in \mathcal{C}^*(G)$

$\Rightarrow$ infinite bicycle $\ni e$

Tripartition in ∞ graphs!

Hamburg version

Theorem

Let G locally finite, e edge of G. Then exactly one of the following holds:

(i) $\exists\, B \in \mathcal{B}(G)$ with $e \in B$

(ii) $\exists$ finite $Y \in \mathcal{C}(G)$ with $e \in Y$ and $Y + e \in \mathcal{C}^*(G)$

(iii) $\exists$ finite $Z \in \mathcal{C}(G)$ with $e \notin Z$ and $Z + e \in \mathcal{C}^*(G)$.

- real tripartition

Waterloo version

Theorem (Casteels&Richter)

Let G locally finite, e edge of G. Then exactly one of the following holds:

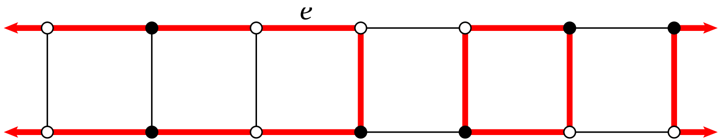
(i) $\exists$ finite $B \in \mathcal{B}(G)$ with $e \in B$

(ii) $\exists\, X \in \mathcal{C}(G)$ with $X + e \in \mathcal{C}^*(G)$

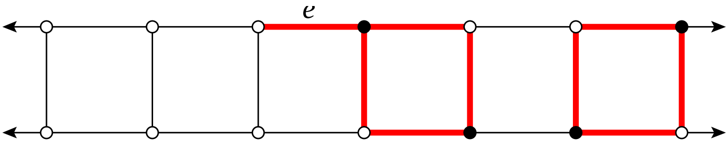
- consequence of more general theorem

Ambiguous edges

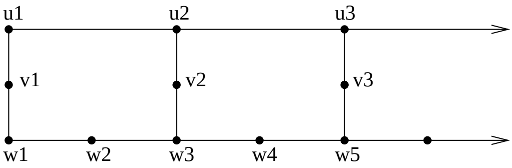
- $\exists\, Y \in \mathcal{C}(G)$ with $e \in Y$ and $Y + e \in \mathcal{C}^*(G)$



- $\exists\, Z \in \mathcal{C}(G)$ with $e \notin Z$ and $Z + e \in \mathcal{C}^*(G)$



Pedestrian graphs



G pedestrian
$:\Leftrightarrow \mathcal{B}(G) = \{\emptyset\}$

Property

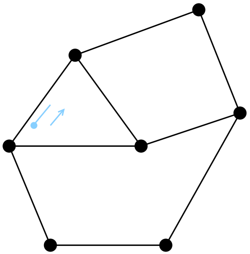
Locally finite G is pedestrian iff for all $e \in E(G)$ there exist finite $Z \in \mathcal{C}(G)$ with $Z + e \in \mathcal{C}^*(G)$.

Theorem (Read & Rosenstiehl)

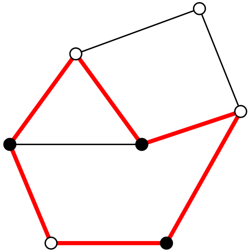
G finite and connected. Then G pedestrian iff $\#$ of spanning trees $=$ odd.

Question: when is an ∞ graph pedestrian?

Left-right tours



A left-right tour...

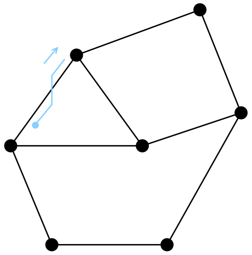


...and its residue

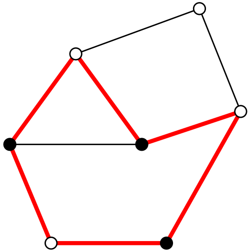
Theorem (Shank)

The residue of a left-right tour in a finite plane graph is a bicycle.

Left-right tours



A left-right tour...

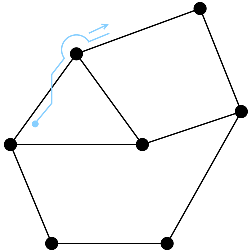


...and its residue

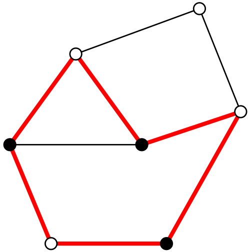
Theorem (Shank)

The residue of a left-right tour in a finite plane graph is a bicycle.

Left-right tours



A left-right tour...

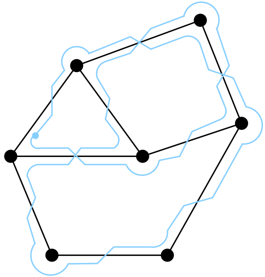


...and its residue

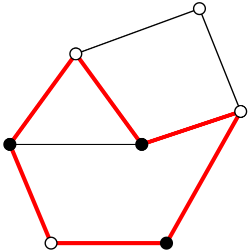
Theorem (Shank)

The residue of a left-right tour in a finite plane graph is a bicycle.

Left-right tours



A left-right tour...

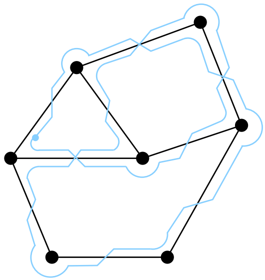


...and its residue

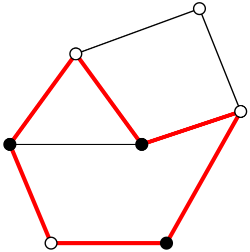
Theorem (Shank)

The residue of a left-right tour in a finite plane graph is a bicycle.

Left-right tours



A left-right tour...



...and its residue

Theorem (Shank)

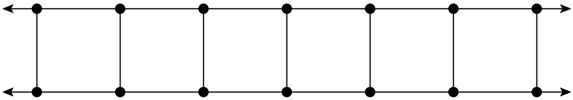
The residue of a left-right tour in a finite plane graph is a bicycle.

Theorem (Horton, Shank)

The residues of left-right tours generate $\mathcal{B}(G)$ in finite plane G.

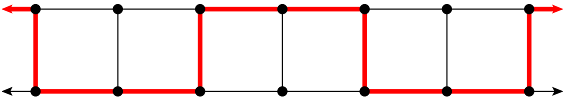
How to define LRT in ∞ graphs?

Left-right tour should be...



How to define LRT in ∞ graphs?

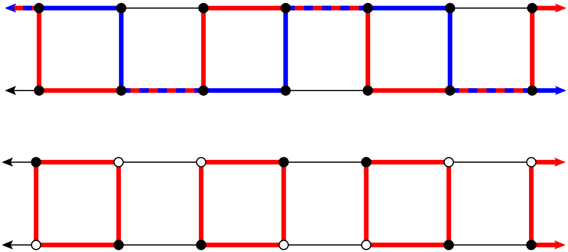
Left-right tour should be...



- ...left-right

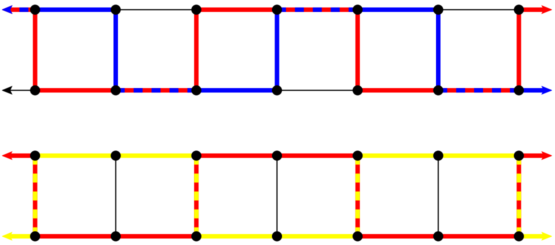
How to define LRT in ∞ graphs?

Left-right tour should be...



- ...left-right
- ...a tour
- residue is bicycle

DEF of left-right tours



- parity infomation is lost in ends

DEF A left-right tour is $\tau : S^1 \overset{\text{cont.}}{\longrightarrow} |G|$ that is...

- locally left-right
- locally injective at edges

Results

Lemma

The residue of a left-right tour in a locally finite plane graph is a bicycle.

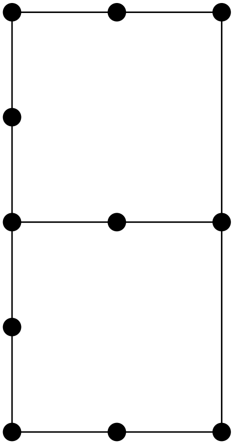
Theorem

The residues of left-right tours generate $\mathcal{B}(G)$ in locally finite plane G.

Problems:

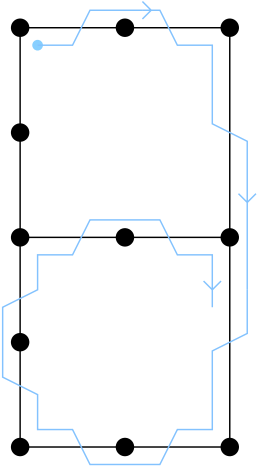
- Finite proof uses plane duals
- Existence of left-right tours?

Unique LRT in pedestrians



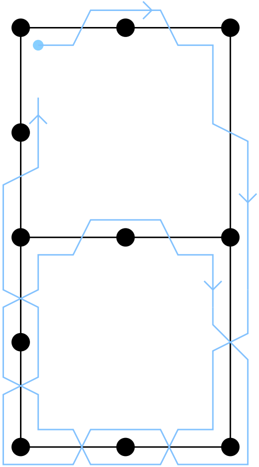
- pedestrian graph has unique LRT
- planarity criterion lists its properties

Unique LRT in pedestrians



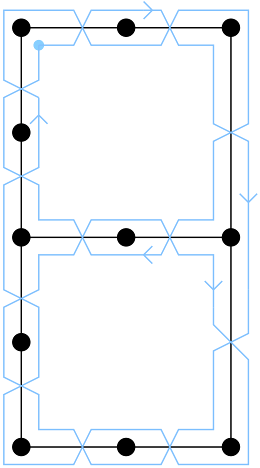
- pedestrian graph has unique LRT
- planarity criterion lists its properties

Unique LRT in pedestrians



- pedestrian graph has unique LRT
- planarity criterion lists its properties

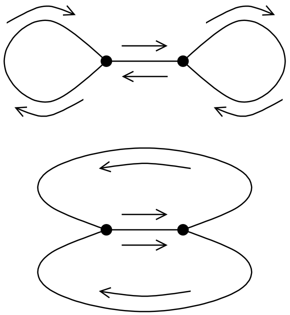
Unique LRT in pedestrians



- pedestrian graph has unique LRT
- planarity criterion lists its properties

Read&Rosenstiehl's planarity criterion

DEF halves



DEF tour W is algebraic diagonal if

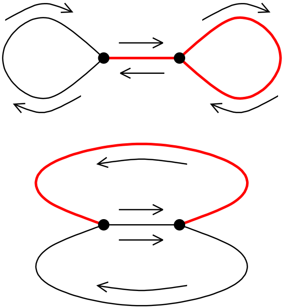
- double cover
- each residue of a half is a cut

Theorem (Read&Rosenstiehl)

Let G be finite and pedestrian. Then G is planar iff it has algebraic diagonal.

→ extends to locally finite
→ for non-pedestrian graphs:
Archdeacon, Bonnington & Little

DEF halves



DEF tour W is algebraic diagonal if

- double cover
- each residue of a half is a cut

Theorem (Read&Rosenstiehl)

Let G be finite and pedestrian. Then G is planar iff it has algebraic diagonal.

$\rightarrow$ extends to locally finite
$\rightarrow$ for non-pedestrian graphs: Archdeacon, Bonnington & Little

Open question

- characterise pedestrian graphs
- left-right tours on other surfaces
- left-right tours for other compactifications