Periodical states and marching groups in a closed owari

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Abstract

Owari is an old African game that consists of cyclically ordered pits that are filled with pebbles. In a sowing move all the pebbles are taken out of one pit and distributed one by one in subsequent pits. Repeated sowing will give rise to recurrent states of the owari. Bouchet studied such periodical states in an idealised setup, where there are infinitely many pits. We characterise periodical states in owaris with finitely many pits. Our result implies Bouchet's result.

1 Introduction

Owari is an over 1000 years old game of African origin that is now played worldwide. It and its variants are known under many names, some of which are oware, awalé, warri and awari. Its gameboard consists of a number of cyclically arranged pits or "houses", typically twelve of them, each of which is filled with 4 seeds or pebbles at the beginning of the game. In a turn, one of the two players takes all pebbles out of one pit and distributes them one by one in subsequent pits in counterclockwise direction. This is called *sowing*. After having sown the player *captures* certain pebbles. The player who captures more than half of all pebbles wins. For precise rules see [1].

Owari is not only a very popular game in certain parts of Africa but has also attracted some interest in the scientific community. The strategical aspect was studied by Bal and Romein [2], who have determined by use of computer that the game is always a draw if both players play optimally. Erickson [7] considered similar games from a more combinatorial perspective. In contrast, Eglash [5, 6] and Bouchet [3, 4] focussed on the sowing operation and, in particular, on recurrent constellations of pebbles under repeated sowing. This will be our main interest too.

Let us make this more precise. A *state* of the owari is described by the number h of the pit from which we will sow next, this is called the *active* pit, and a tuple with the numbers of pebbles in the respective pits. In the sowing move, we *sow from* the active pit, which means that we remove all pebbles from pit h and distribute them one by one in subsequent pits. Then we advance to pit $h+1$, i.e. pit $h+1$ becomes active (here, we take the index $h+1$ modulo the number of pits). In the next sowing move, we sow from pit $h+1$ and so on. We call a state *periodical* if after p sowing moves the number of pebbles in pit $i+p$ equals the number of pebbles in pit i of the starting state, for all i.

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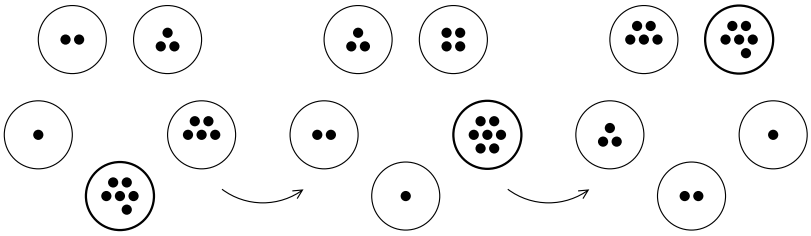

Figure 1: A period of length 2; the active pit is drawn in bold

A periodical state of period length 2 is illustrated in Figure 1. We start with the pit to the bottom of the left owari, sow from it and advance one pit in counterclockwise direction. After two steps we have regained our starting state (modulo a rotation of the owari).

In Figure 1 we can witness another phenomenon. Since the first pit contains at least as many pebbles as there are pits, we make a complete tour when sowing from it and thus put a pebble in it.[1] Whenever this happens we say that the sowing *overlaps*.

In order to avoid overlapping Bouchet considered open owaris. While a *closed owari* has finitely many pits arranged in a cycle as described in the previous paragraphs, an *open owari* comprises infinitely many pits, layed out so that each pit has a successor. An open owari may be seen as the limit of closed owaris.

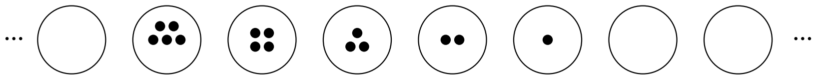

Figure 2: A marching group in an open owari

Sowing is done as in closed owaris. An example of a periodical state in an open owari is given in Figure 2. In contrast, adding two more pebbles to the first (non-empty) pit would yield a non-periodical state. The constellation as in the figure is of special interest as it is invariant to sowing; such a state is called a *marching group* (of an open owari). It is easy to see that marching groups always have the following form: the first non-empty pit is active and contains $r \in \mathbb{N}$ pebbles, the pit following it contains $r-1$ pebbles, and so on, to the last non-empty pit containing exactly one pebble. The integer r is called the *order* of the marching group.

We define an *augmented marching group* as follows: starting from a marching group we can decide for every non-empty pit independently whether to add a pebble or not. In addition, we are allowed (but do not need to do so) to put one pebble in the first empty pit following the marching group. As an example, consider Figure 3, where the marching group of order 3 is augmented by putting

[1] We should point out that the standard rules are slightly different. When sowing from pit h it is not allowed to put a pebble in pit h. Rather, if necessary, pit h is skipped and the pebble put in the next pit. Our rule bending, however, is not substantial. It is not hard to see that there is a natural bijection between periodical states under standard rules and periodical states under our simplified rules.

one extra pebble in the first non-empty pit, and one pebble in the empty pit succeeding the marching group.

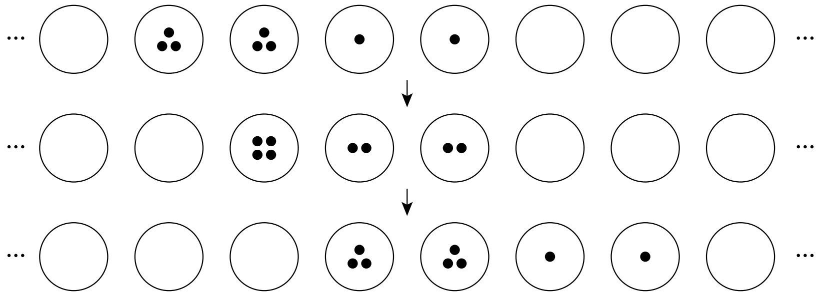

Figure 3: Marching group of order 3, augmented by two pebbles

Bouchet [3] characterises the periodical states in an open owari:

Theorem 1 (Bouchet [3])**.** *A state in an open owari is periodical if and only if it is an augmented marching group.*

Bouchet also determined the period lengths in an open owari:

Theorem 2 (Bouchet [3])**.** *In an open owari, an integer p is the period of a marching group of order r augmented by $m \leq r$ pebbles if and only if $p = \frac{r+1}{d}$ where d is divisor of $r+1$ and of m.*

We prove in the next section an extension of Theorem 1 to closed owaris. Our result will imply Theorem 1. We remark that our proof is quite different from the one given by Bouchet.

2 Closed owaris

We will deal with closed owaris, which consist of a finite number, n say, of cyclically ordered pits. We consider the number n of pits to be fixed. A *state* of the owari will be represented by a tuple $O := [w_0, \ldots, w_{n-1}]$ of non-negative integers, the first of which corresponds to the pebbles in the active pit, i.e. the pit from which we will sow next. Let $w_0 = qn+r$ with $0 \leq r \leq n-1$. The process of sowing can then be modelled by the *sowing operator* $\mathcal{S}$, which we define as follows: $\mathcal{S}O := [w'_0, \ldots, w'_{n-1}]$ where $w'_i = w_{i+1} + q + 1$ for $i = 0, \ldots, r-1$ and $w'_j = w_{j+1} + q$ for $j = r, \ldots, n-1$ (setting $w_n = 0$).

Since we are working in a closed owari rather than an open one, we need to amend the definition of a marching group. For non-negative integers q, r with $r \leq n-1$ we call the state $[m_0, \ldots, m_{n-1}]$ a *marching group* (in a closed owari) and denote it by $M_{q,r}$ if $m_i = (n-i)q + r - i$ for $i = 0, \ldots, r-1$ and $m_j = (n-j)q$ for $j = r, \ldots, n-1$.

It is easy to see, but will formally be proved in the next lemma nevertheless, that, again, a marching group is invariant under sowing. Indeed, $M_{q,r}$ can be seen as the sum of the two marching groups $M_{q,0}$ and $M_{0,r}$. In $M_{q,0}$, sowing simply means putting q pebbles in each pit. On the other hand, as $r \leq n-1$, no overlap occurs when sowing in $M_{0,r}$ and so, $M_{0,r}$ behaves as a marching group in an open owari. See Figure 4 for an example of a marching group in a closed owari.

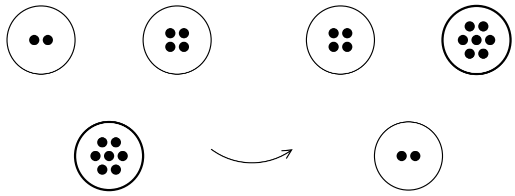

Figure 4: Marching group $M_{2,1}$ in an owari with 3 pits

Lemma 3. *A state in a closed owari is invariant under $\mathcal{S}$ if and only if it is a marching group.*

Proof. Let $W = [w_0, \ldots, w_{n-1}]$ be a state in a closed owari, and put $\mathcal{S}W = [w'_0, \ldots, w'_{n-1}]$.

First, let $W = M_{q,r}$ be a marching group. Then $w'_i = w_{i+1} + q + 1 = (n-(i+1))q + r - (i+1) + q + 1 = (n-i)q + r - i$ for $i = 0, \ldots, r-1$ (note that also $w_i = (n-i)q + r - i$ for $r = i$) and $w'_j = w_{j+1} + q = (n-(j+1))q + q = (n-j)q$ for $j = r, \ldots, n-1$.

Second, assume $\mathcal{S}W = W$, and let $w_0 = qn + r$ where $0 \le r \le n-1$. Thus, $w_i = w'_i = w_{i+1} + q + 1$ and $w_j = w'_j = w_{j+1} + q$ for $0 \le i \le r-1$ and $r \le j \le n-1$. Starting with $w_{n-1} = q$ we can easily solve this system of equations, whose solution is the marching group $M_{q,r}$. $\square$

We call a state $[m_0 + a_0, \ldots, m_r + a_r, m_{r+1}, \ldots, m_{n-1}]$ an *augmented marching group* (in a closed owari) if $[m_0, \ldots, m_{n-1}] = M_{q,r}$ for some integer $q \ge 0$ and if $a_i \in \{0, 1\}$ for all i. As an example, we remark that Figure 1 shows the marching group $M_{1,1}$ augmented by one pebble in pit 1. Note that the augmentation of the marching group $M_{0,r}$ can be viewed as an augmented marching group in an open owari.

A state W of a closed owari is periodical if there is an integer $p \ge 1$ so that $\mathcal{S}^p W = W$. The smallest such p is the *period* of W. Marching groups are precisely the states that have period 1.

Let us now finally state and prove our main result:

Theorem 4. *A state in a closed owari is periodical if and only if it is an augmented marching group.*

Proof. To check sufficiency, let $W = [m_0 + a_0, \ldots, m_r + a_r, m_{r+1}, \ldots, m_{n-1}]$ be an augmented marching group, i.e. let $M_{q,r} = [m_0, \ldots, m_{n-1}]$ for some q, and let $a_0, \ldots, a_r \in \{0, 1\}$. Then

$$\mathcal{S}W = [m_0 + a_1, \ldots, m_{r-1} + a_r, m_r + a_0, m_{r+1}, \ldots, m_{n-1}],$$

and clearly, $\mathcal{S}^r W = W$.

So, let us prove that every periodical state, $W = [w_0, \ldots, w_{n-1}]$ say, is an augmented marching group. For $j \ge 0$, put $[w^j_0, \ldots, w^j_{n-1}] := \mathcal{S}^j W$. Choose m_0 maximal such that there is an integer $j \ge 0$ and a marching group $M_{q,r} =$

$[m_0, \ldots, m_{n-1}]$ with $m_i \leq w_i^j$ for all i. Put $d_i = w_i - m_i$ for all i. Since W is periodical, and since $w_i^j - m_i \geq 0$ for some j, it follows that $d_i \geq 0$ as well for all i.

We will reduce the sowing operation on W to an operation on the tuple $D = [d_0, \ldots, d_{n-1}]$. More generally, define an operation $\mathcal{S}_r$ on a state $[e_0, \ldots, e_{n-1}]$ as follows:

(I) $[0, e_1, \ldots, e_{n-1}] \to \mathsf{E}$, $e_0 \to \mathsf{pebbles}$ and $r \to \mathsf{position}$

(II) While $\mathsf{pebbles} > 0$ do
    increase E at $\mathsf{position}$ by 1
    decrease $\mathsf{pebbles}$ by 1
    increase $\mathsf{position}$ by 1 (modulo n)

(III) if $\mathsf{E} = [e'_0, \ldots, e'_{n-1}]$ then output $[e'_1, \ldots, e'_{n-1}, e'_0]$

With this operation we get that $\mathcal{S}W = M + \mathcal{S}_r D$, where addition is taken componentwise. Since W is periodical under $\mathcal{S}$ so is, therefore, D under $\mathcal{S}_r$.

Let d_i^j be such that $\mathcal{S}_r^j D = [d_0^j, \ldots, d_{n-1}^j]$. We will prove the theorem in four steps. First, we claim that

$$\textit{for all integers } j \geq 0 \textit{ there is an index } i \in \{0, \ldots, r\} \textit{ such that } d_i^j = 0. \tag{1}$$

Suppose that $d_i^j \geq 1$ for $i = 0, \ldots, r$ for some j. Put $m'_0 = m_0 + 1$, and assume first that $r < n - 1$. Then, $m'_0 = qn + (r + 1)$, and we put $m'_i := (n - i)q + (r + 1) - i = m_i + 1$ for $i = 0, \ldots, r - 1$, $m'_r := (n - r)q + 1 = m_r + 1$ and $m'_i := (n - i)q = m_i$ for $i = r + 1, \ldots, n - 1$. Thus, the m'_i form the marching group $M_{q,r+1}$. Then, $m'_i = m_i + 1 \leq m_i + d_i^j = w_i^j$ for $i = 0, \ldots, r$ and $m'_i = m_i \leq w_i^j$ for $i \geq r + 1$, contradicting our maximal choice of m_0. If $r = n-1$ we see in a similar way that m'_0 and the marching group $M_{q+1,0}$ would have been a better choice. This proves (1).

Secondly, we show that

$$\textit{there is a constant } c \leq r \textit{ such that for all integers } j \geq 0 \textit{ the number } c^j \textit{ of non-zeros among } d_0^j, \ldots, d_r^j \textit{ is exactly } c. \tag{2}$$

Observe that $c^j \leq c^{j+1}$ for all j. Indeed, sowing or, more precisely, application of $\mathcal{S}_r$ can only introduce a new zero at position r. Since $d_r^{j+1} = d_{r+1}^j + e$ where $e \geq 1$ if $d_0^j \geq 1$, this can only happen if $d_0^j = 0$. But then the gain of the zero at position r is balanced by the loss of a zero at the first position. Thus, $c^j \leq c^{j+1}$. Since D is periodical under $\mathcal{S}_r$ we deduce $c^0 = c^j =: c$ for all j. Claim (1) implies $c \leq r$.

Thirdly, we claim that

$$\textit{for all integers } j \geq 0 \textit{ it holds that } d_i^j \leq 1 \textit{ for } i = 0, \ldots, r. \tag{3}$$

Suppose not. Let s be the smallest positive integer for which there exist an integer j and $k \in \{0, \ldots, r\}$ such that $d_k^j \geq 2$ and $d_{k+s}^j = 0$ where the index is taken mod $(r + 1)$ (such an s exists, by (1)). By applying $\mathcal{S}_r$ k more times, we may assume that $k = 0$. Thus, $\mathcal{S}_r^j D$ starts with an element ≥ 2, which is followed by s entries of 1, which, in turn, precede a 0. Assume first that $r < n-1$. Then, further applications of $\mathcal{S}_r$ yield the following:

$$
\begin{aligned}
[\geq 2, \overbrace{1,\ldots,1}^{s \text{ times}}, 0, *, \ldots, *] &\to [\overbrace{1,\ldots,1}^{s \text{ times}}, 0, *, \ldots, *, \overset{\text{pos } r}{\geq 1}, \geq 1, *, \ldots, *] \\
&\to [\overbrace{1,\ldots,1}^{s-1 \text{ times}}, 0, *, \ldots, *, \geq 1, \overset{\text{pos } r}{\geq 2}, *, \ldots, *] \\
&\to \underset{\cdots}{s-1 \text{ applications of } \mathcal{S}_r} \to \\
&\to [0, *, \ldots, *, \geq 1, \geq 2, \overbrace{\geq 1, \ldots, \geq 1}^{s-2 \text{ times}}, \overset{\text{pos } r}{\geq 1}, *, \ldots, *] \\
&\to [*, \ldots, *, \geq 1, \geq 2, \overbrace{\geq 1, \ldots, \geq 1}^{s-1 \text{ times}}, \overset{\text{pos } r}{\geq 0}, *, \ldots, *].
\end{aligned}
$$

The entry of ≥ 0 at position r in the last state cannot be larger than zero, since in that case c^j would increase, which is impossible, by (2). Then, however, there at most $s-1$ ones between an entry of ≥ 2 and the zero at position r, contradicting our choice of s. For $r = n-1$ the argumentation is similar. This proves (3).

In order to finish the proof of the theorem it suffices, in view of (3), to establish the following claim:

$$d_{r+1} = \ldots = d_{n-1} = 0. \tag{4}$$

Suppose there is an $i \geq r+1$ for which $d_i \geq 1$. Then $d_r^{j+1} \geq d_i + e$ for $j = i - r$ and $e = 1$ if $d_0^j \geq 1$ and $e = 0$ otherwise. From (3) it follows that $e = 0$ as otherwise $d_r^{j+1} \geq 2$. However, $e = 0$ and thus $d_0^j = 0$ entails the introduction of a new non-zero among $d_0^j, \ldots, d_r^j$ in step $j+1$, i.e. $c^j < c^{j+1}$, which contradicts (2). □

We note that Theorem 1 follows easily from our main result. Indeed, let Q be a finite tuple representing consecutive pits of a periodical state in an open owari such that every pit outside this range is empty, and let $n-1$ be the total number of pebbles the state contains. By appending 0-entries we can turn Q into a state W of a closed owari with n pits. Because of the size of the owari an overlap will never occur, and W will be periodical too. Therefore, W is a marching group $M_{q,r}$ augmented by m pebbles. As W contains only n pebbles, it follows that $q = 0$. Since $M_{0,r}$ can be seen as a marching group in an open owari it follows that Q represents an augemented marching group in an open owari.

With the same reduction we can use Theorem 1 to characterise the periods of closed owaris too.

Acknowledgement

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