Clique or hole in claw-free graphs

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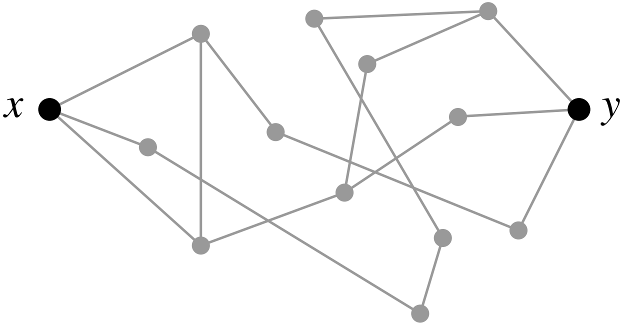
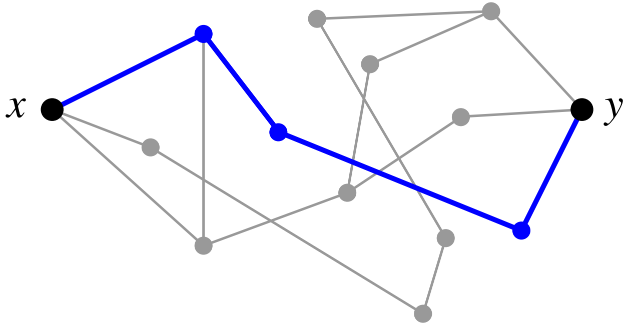
Reachability

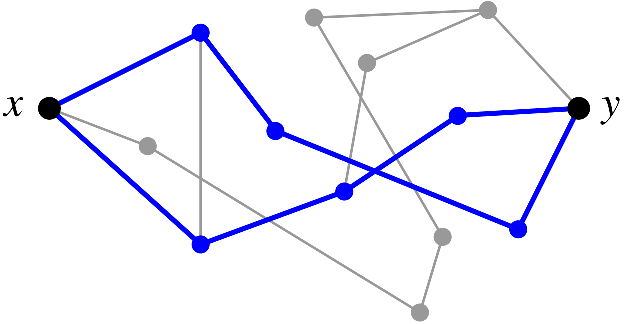


Reachability



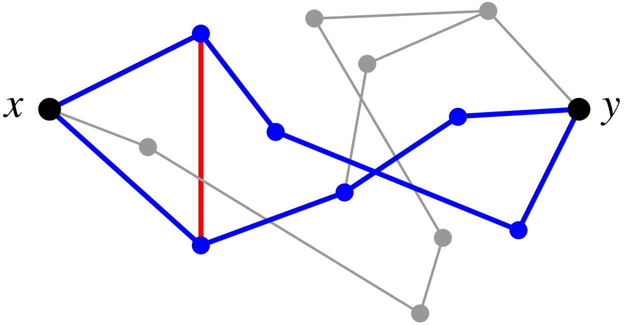
x–*y* path

Reachability



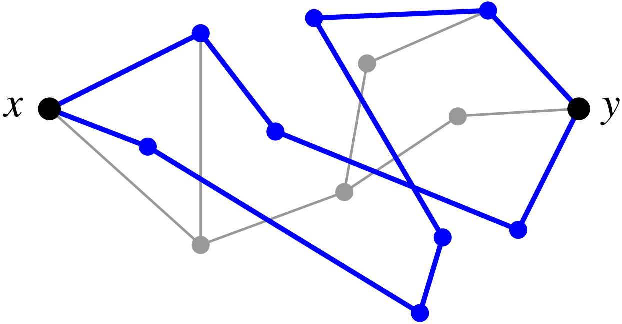
two disjoint *x*–*y* paths

Reachability



interference

Reachability

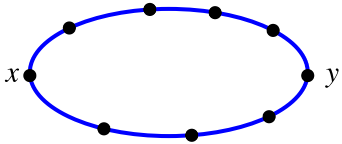


two disjoint x–y paths without interference

x–y holes

x–y HOLE

Given: G, x, y
Is there a hole through x and y?



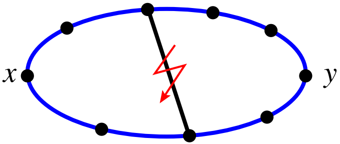
Theorem (Bienstock '91)

x–y HOLE is NP-complete.

x–y holes

x–y HOLE

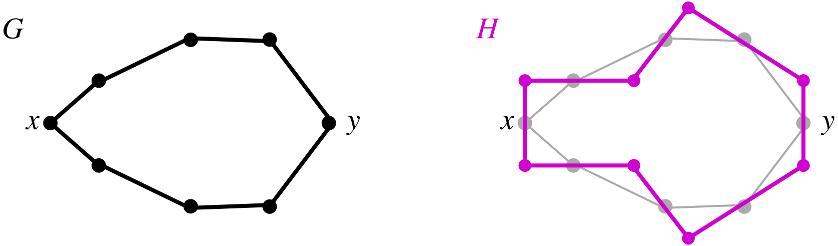
Given: G, x, y
Is there a hole through x and y?



Theorem (Bienstock '91)

x–y HOLE is NP-complete.

Easy for line graphs $G = L(H)$

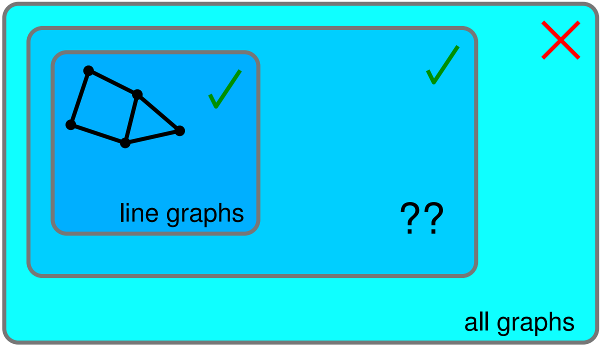


hole through x and y — cycle through x and y

- just need to check for two disjoint paths
- simple polynomial time algorithm

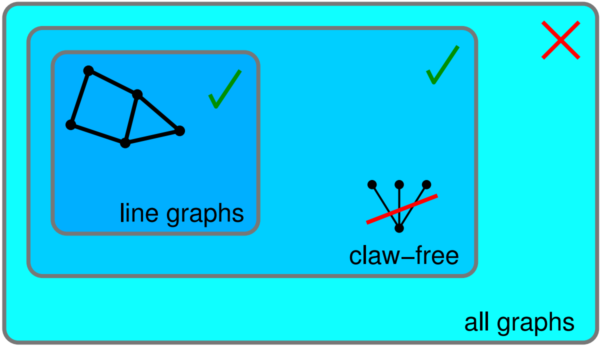
Polynomial time algorithm

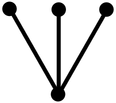
Existence of polynomial time algorithm...



Polynomial time algorithm

Existence of polynomial time algorithm...



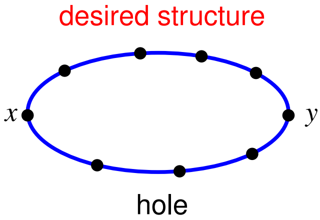


Lévêque, Lin, Maffray and Trotignon:

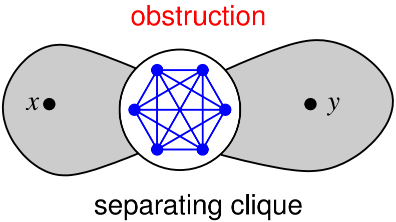
- $O(n^4)$-algorithm for G (n vertices)
- algorithm not very practical

Hole or clique

Given: non-adjacent vertices *x* and *y*



or



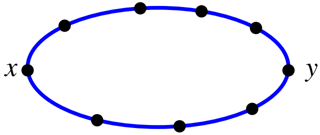
Naive conjecture

No clique that separates *x* from *y*
⇒ *x*–*y* hole

Hole or clique

Given: non-adjacent vertices *x* and *y*

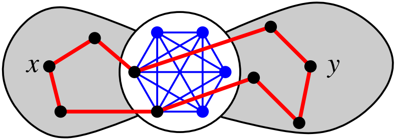
desired structure



hole

or

obstruction

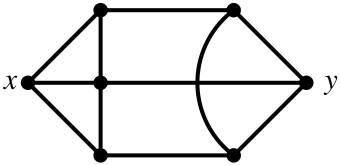


separating clique

Naive conjecture

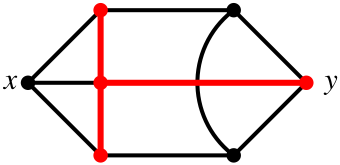
No clique that separates *x* from *y*
⇒ *x*–*y* hole

Naive conjecture



- no x–y hole...
- no clique that separates x from y

Naive conjecture

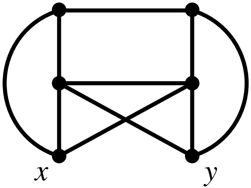


- no x–y hole...
- no clique that separates x from y
- but: claw

Structure theorem

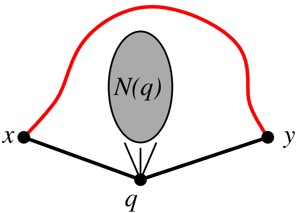
Theorem

Let G be claw-free and x, y non-adjacent and without common neighbours. Then there exists a hole through x and y iff there is no clique separating x and y.



- false with common neighbours

With common neighbours



- a hole through a common neighbour

Theorem

Let G be claw-free and x, y non-adjacent. Then:

- *There exists a hole through x and y; or*
- *there exists a clique separating x and y in $G - N(x) \cap N(y)$, and for every $q \in N(x) \cap N(y)$: $q \cup N(q) \setminus \{x, y\}$ separates x and y in G.*

Algorithmic consequences

x–y HOLE

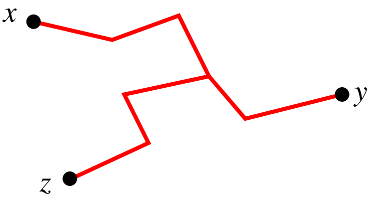
Given G, x, y check whether G contains a hole through x and y.

$\exists$ hole through x and y iff $\nexists$ x–y clique separator

For claw-free G...

- Use Tarjan's clique decomposition algorithm
- x–y HOLE can be solved in $O(|E| \cdot |V|)$-time
  previously $O(|V|^4)$ (Lévêque et al)

Three-in-a-Tree



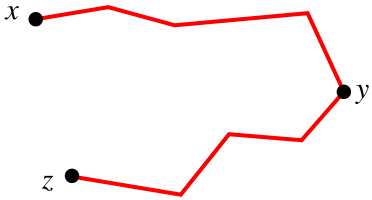
THREE-IN-A-TREE

Given G, x, y, z check whether G contains an induced tree containing x, y and z.

Theorem (Chudnovsky & Seymour)

THREE-IN-A-TREE *can be solved in* $O(|V|^4)$*-time.*

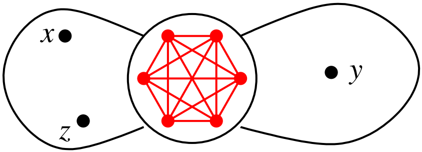
Three-in-a-Path

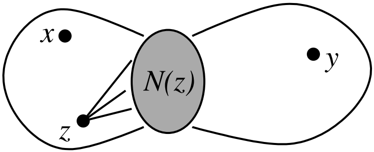


G claw-free →
tree becomes induced path

When is there an induced *x*–*z* path through *y*?

- obstructions:





Induced x–z path through y

Theorem

Let G be claw-free and x, y, z non-adjacent. Then:

- *There exists an induced x–z path through y; or*
- *there exists a clique separating $\{x, z\}$ and y, or $N(x)$ separates z from y, or $N(z)$ separates x from y.*

- follows from x–y hole theorem
- allows for $O(|E| \cdot |V|)$-algorithm

Extensions?

Given k vertices X in claw-free G, when is there a hole through all of X?

$\rightarrow$ in line graphs: search for cycle through k edges
$\rightarrow$ too ambitious

Lovász-Woodall Conjecture

Let H be k-connected, F set of k independent edges, and if k is odd, assume $G - F$ to be connected. Then there is a cycle through F.

- true for $k = 3, 4$
- full proof announced by Kawarabayashi

Open problem

- When is there a hole through x, y, z in a claw-free graph?
- complexity?