Infinite circuits in infinite graphs

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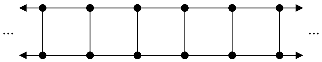
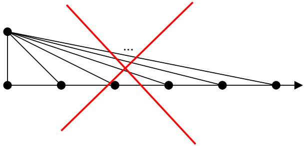
Locally Finite Graphs

- (Almost) every graph in this talk will be locally finite

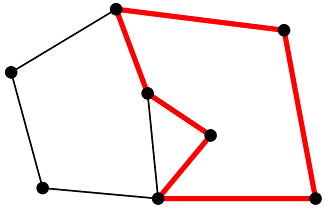




The Cycle Space of a Finite Graph

Cycle Space $\mathcal{C}(G)$ for finite G:

- set of all symmetric differences of circuits
- $\mathbb{Z}_2$-vector space
- 1. homology group



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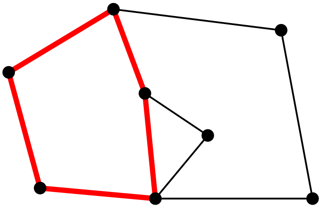

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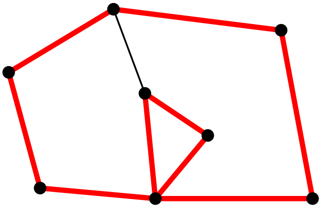
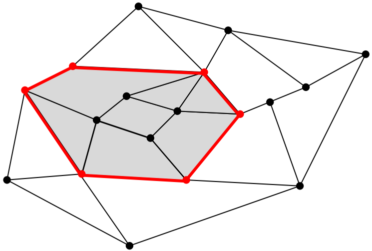


MacLane's Planarity Criterion

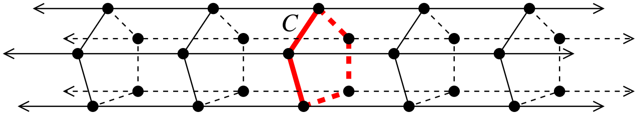
Theorem (MacLane)

If G is finite then G is planar iff $\mathcal{C}(G)$ has simple generating set.



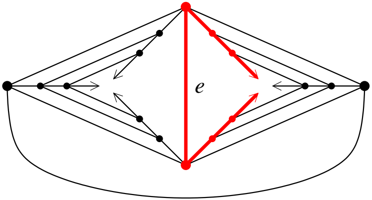
- simple: no edge in two members
- face boundaries generate $\mathcal{C}(G)$

MacLane Fails in Infinite Graphs



$\Rightarrow$ need infinite sums

Too few circuits



- face boundaries containing *e* infinite
- cannot generate circuits containing *e*

$\Rightarrow$ need more circuits

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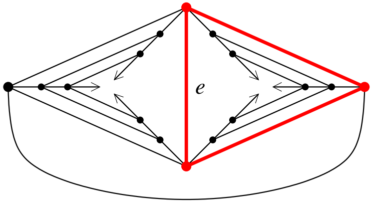

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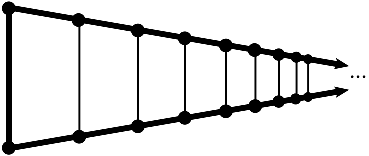
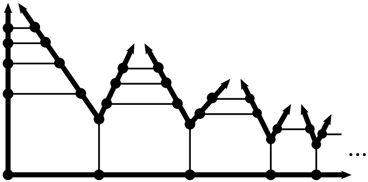
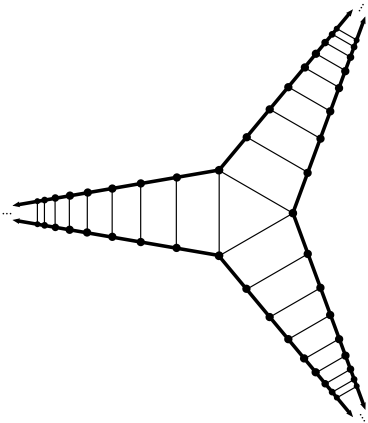
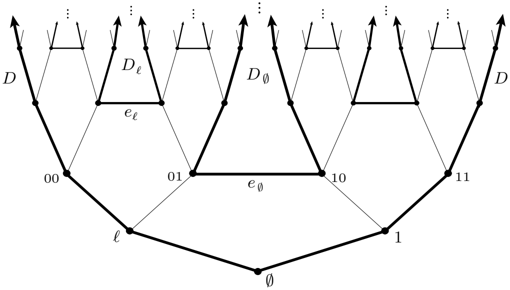
Infinite Face Boundaries







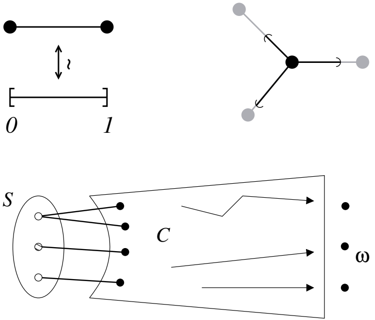
The Monster Circuit



- combinatorial description seems hopeless

Topology on *G*+ends

Define topology on *G*+ends...

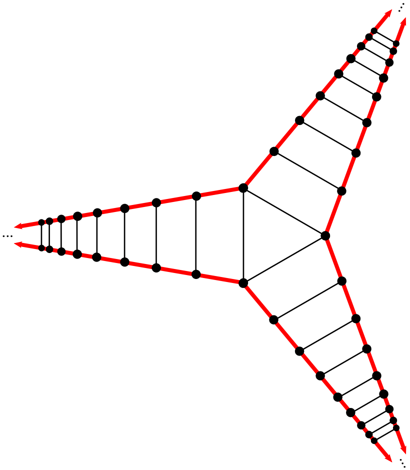


- on *G*: topology of 1-complex
- Freudenthal compactification
- Hausdorff

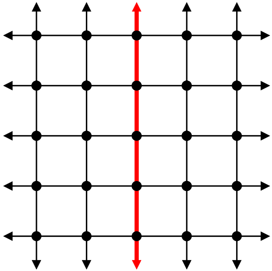
Circles

For locally finite G...

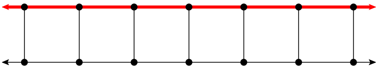
- Circle homeomorphic image of S^1 in G+ends
- Circuit edge set of a circle

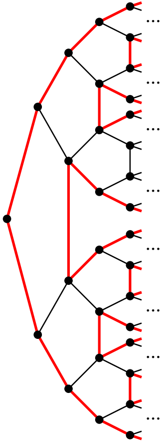


Infinite circuits: examples



no cycle:

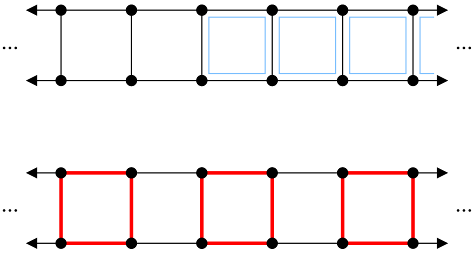




Topological Cycle Space

Topological
cycle space $\mathcal{C}(G)$
set of thin sums
of circuits

(Diestel&Kühn)



- we allow infinite sums!
- $\mathcal{C}(G) \neq$ 1. homology group
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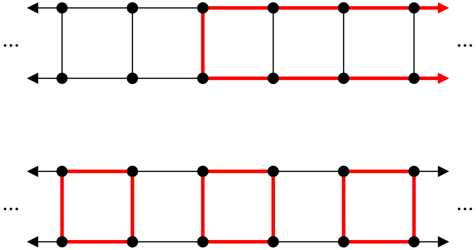

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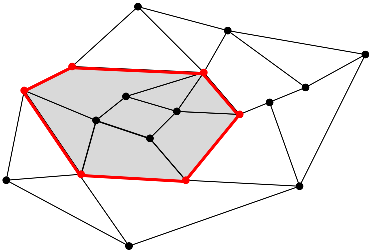
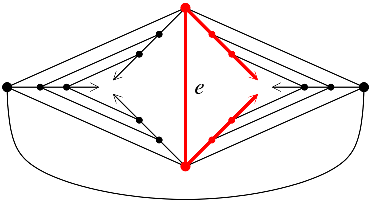
Theorem (Bruhn&Stein)

If G is locally finite then G is planar iff $\mathcal{C}(G)$ has simple generating set.





Prior work due to Thomassen ('80), also Bonnington&Richter ('03)

$\mathcal{C}(G)$ is the Right Concept

A host of results:

- orthogonality of circuits and cuts [Diestel & Kühn]
- cycle space elements are disjoint union of circuits [DK]
- Tutte's generating theorem [Bruhn]
- Tutte's planarity criterion [B&Stein]
- Gallai's cycle-cocycle partition [BDS]
- Whitney's planarity criterion [BD]
- duality of spanning trees [BD]
- characterisation by degrees [BS]
- tree packing [S]
- Fleischner's theorem [Georgakopoulos]
- $\mathcal{C}(G)$ is generated by geodesic circuits [G&Sprüssel]

Also: work by Richter&Vella in more general context

Tutte's Planarity Criterion

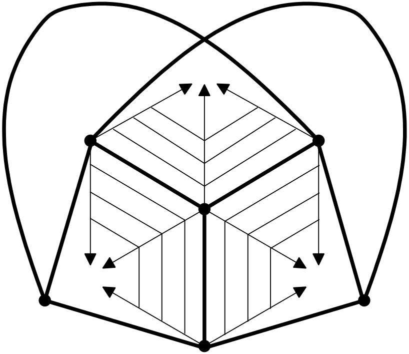
Peripheral circuit: non-separating and without chords

Theorem (Tutte)

If G is finite and 3-connected then G is planar iff every edge lies in at most two peripheral circuits.

Theorem (Bruhn&Stein)

If G is locally finite and 3-connected then G is planar iff every edge lies in at most two peripheral circuits.



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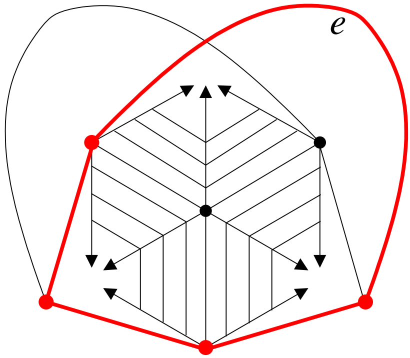

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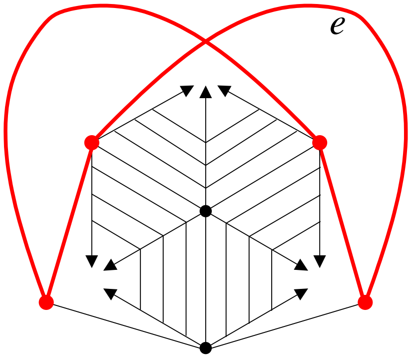

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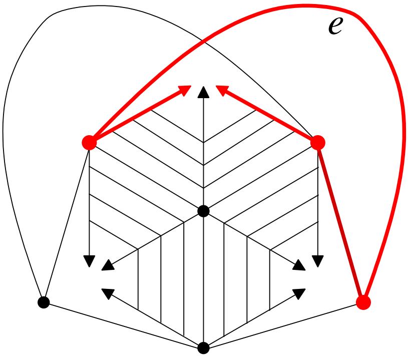
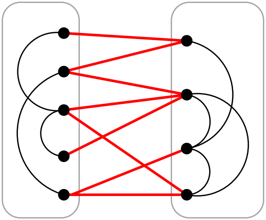
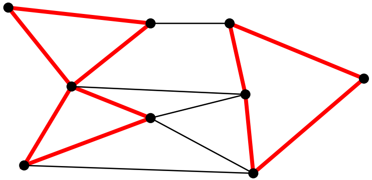


Gallai's edge partition

- a cut

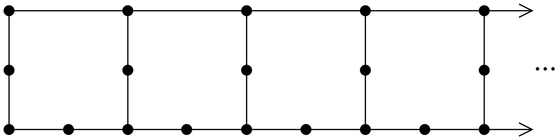




Theorem (Gallai)

For finite G, there is a partition $Z \cup F = E(G)$, so that F is a cut and $Z \in \mathcal{C}(G)$.

Gallai's edge partition



- unique partition
- need infinite circuits

Theorem (Bruhn, Diestel & Stein)

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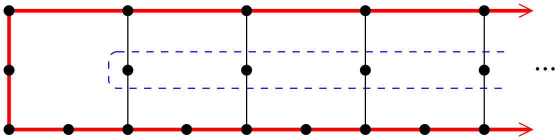
Gallai's edge partition



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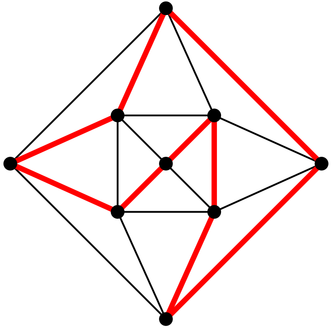
Hamilton Circuits in Finite Graphs



Theorem (Tutte)

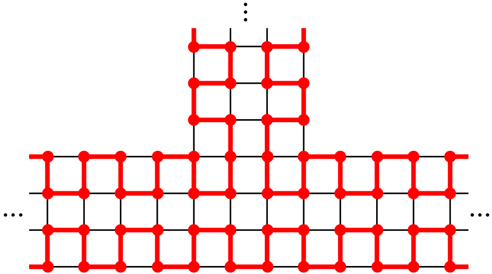
If G is finite planar and 4-connected then it has a Hamilton circuit.

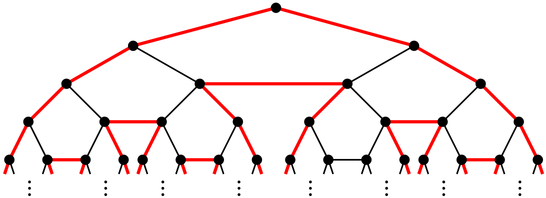
Conjecture (Nash-Williams)

If G is locally finite planar and 4-connected with at most two ends then it has a spanning double ray.

Yu has announced a proof of NW's conjecture

Infinite Hamilton Circuits





Hamilton circuit:
spanning (infinite) circuit

- can go through arbitrarily many ends

Hamilton Circuits in Planar Graphs

Conjecture

If G is locally finite planar and 4-connected then it has an infinite Hamilton circuit.

Two preliminary results...

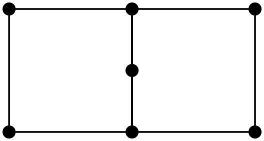
Theorem (Bruhn&Yu)

If G is locally finite, planar, 6-connected and has only finitely many ends then it has a Hamilton circuit.

Theorem (Cui, Wang & Yu)

If G is locally finite, planar, 4-connected and has a VAP-free drawing then it has a Hamilton circuit.

Fleischner's Theorem



Square of G:
put in an edge between
any two vertices of distance 2

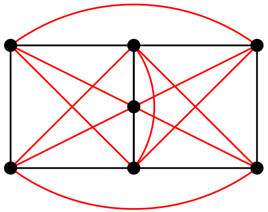
Theorem (Fleischner)

The square of a 2-connected finite graph has a Hamilton circuit.

Theorem (Georgakopoulos)

The square of a 2-connected locally finite graph has a Hamilton circuit.

Fleischner’s Theorem



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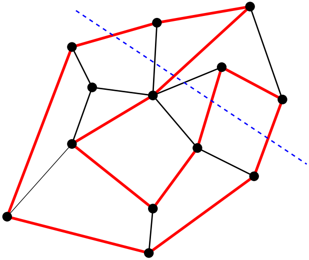
Methods are combinatorial

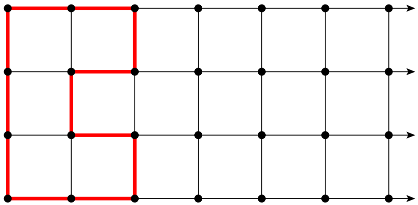
- connected & locally finite
  $\Rightarrow$ countable
- cut criterion



Theorem (Diestel & Kühn)

G locally finite. Then $Z \in \mathcal{C}(G)$ iff $|Z \cap F|$ =even for every finite cut F.

Limits of circuits



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$\Rightarrow$ limit of circuits $\in \mathcal{C}(G)$
$\Rightarrow$ construction from "local to global" possible

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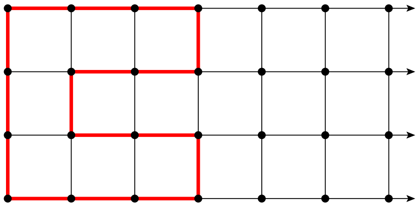

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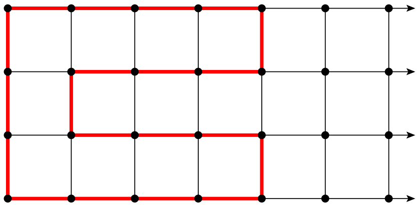

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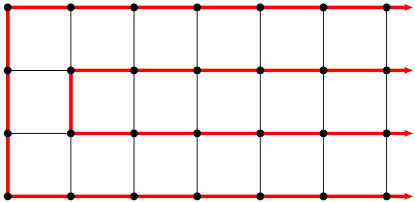

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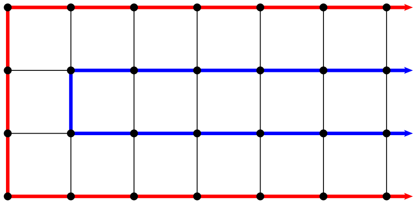

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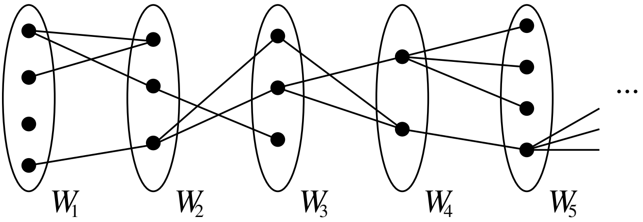
Compactness arguments

Theorem (Kőnig's Infinity Lemma)

Let $W_1, W_2, \ldots$ be finite, non-empty and disjoint, let H graph with $V(H) = \bigcup W_n$. If each vertex in W_{n+1} has neighbour in W_n then there is a ray $w_1 w_2 \ldots$ with $w_n \in W_n$ for all n.

- standard tool
- uncountable $\rightarrow$ Tychonov



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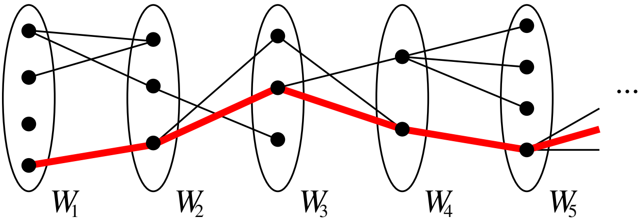
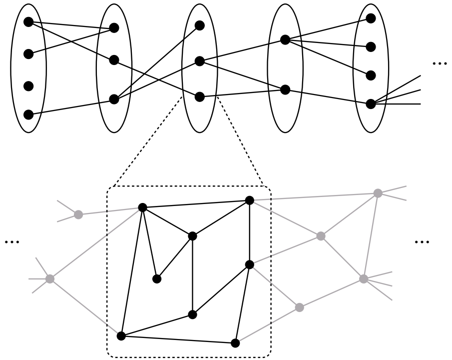


Application of Infinity Lemma



- assume:
  can colour
  finite subgraphs

⇒ colouring for
whole graph

- usually application not this straightforward
- works only in easy cases

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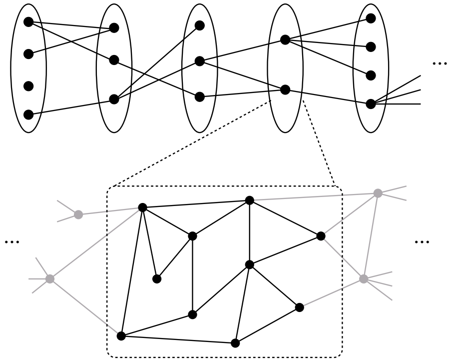

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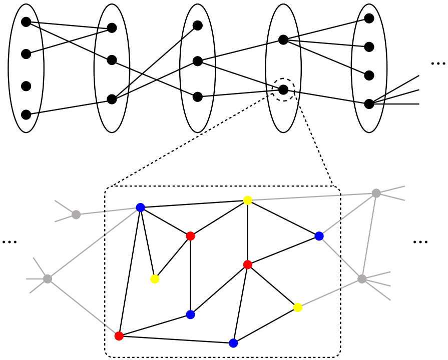

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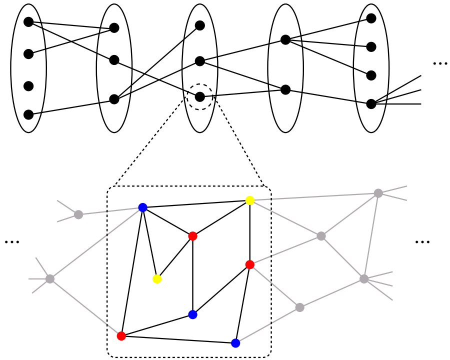

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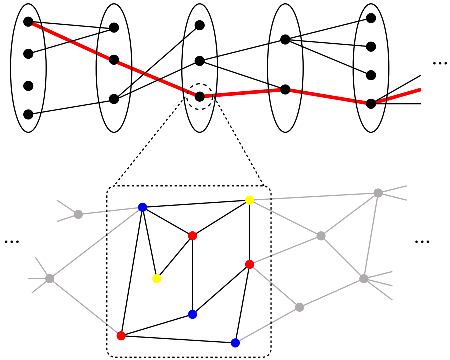

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