Supplementary material

Proof of main result from NW's Hall-theorem

For a graph $G = (V, E)$ and a subset $U \subseteq V$, denote by $\mathcal{E}_G(U)$ (or simply by $\mathcal{E}(U)$, if G is clear from the context) the set of edges with at least one endvertex in U. For a single vertex v, we abbreviate $\mathcal{E}(\{v\})$ to $\mathcal{E}(v)$. Let H be a bipartite graph with partition classes M and W, and define for $X \subseteq W$ the demand-set $D_H(X)$ to be $\{m \in M : N(m) \subseteq X\}$. We will often simply write $D(X)$ if it is obvious which is the underlying graph H.

Theorem 1 (Hall [2])**.** *Let H be a bipartite graph with partition classes M and W. Assume that every vertex in M has only finitely many neighbours. Then there is a matching of M if and only if $|D(X)| \leq |X|$ for every finite set $X \subseteq W$.*

Let $\mathcal{W}_\theta = (W_\lambda)_{\lambda \leq \theta}$ be a queue in W. Set $q(\mathcal{W}_0) = -|D(W_0)| = -|D(\emptyset)|$ and define

(i) $q(\mathcal{W}_\lambda) := q(\mathcal{W}_\kappa) + |W_\lambda \setminus W_\kappa| - |D(W_\lambda) \setminus D(W_\kappa)|$ if $\lambda = \kappa + 1$ is a successor ordinal; and

(ii) $q(\mathcal{W}_\lambda) := \liminf_{\mu<\lambda} q(\mathcal{W}_\mu) - |D(W_\lambda) \setminus \bigcup_{\mu<\lambda} D(W_\mu)|$ otherwise.

If confusion may arise we will write q_H to indicate the graph H in which we are measuring q.

Theorem 2 (Nash-Williams [3])**.** *Let H be a countable bipartite graph with partition classes M and W. Then there is a matching of M if and only if $q(\mathcal{W}) \geq 0$ for each queue $\mathcal{W}$ in W.*

We deduce our main theorem from Nash-Williams' theorem. For this, let a countable graph $G = (V, E)$ and bounds l and u be given. Assume that there are no deficient and no faulty sets.

For each $v \in V$, set $X_v := \{(v, i) : i = 1, \ldots, u(v)\}$ (if $u(v) = \infty$ we choose countably infinitely many copies of v) and $V_u := \bigcup_{v \in V} X_v$. So V_u consists of $u(v)$ copies of each $v \in V$. Define a bipartite graph H with partition classes V_u and E. Let $v' \in X_v$ and $e \in E$ be adjacent in H if and only if v is incident with e in G. For each $v \in V$ pick $l(v)$ of its copies in X_v, denote the set of those by Y_v and set $V_l := \bigcup_{v \in V} Y_v$.

We will find a matching M_l of V_l and a matching M_u of E in H. From these two we shall construct a common matching M of V_l and E. Once we have done this, we orient an edge $e \in E$ towards its endvertex $v \in V$ if e is matched with some $v' \in X_v$. This yields the desired orientation.

So, let us first find the matching M_l, for which we work within the graph $H' := H[V_l \cup E]$. Considering an arbitrary queue $\mathcal{W} = (W_\lambda)_{\lambda \leq \theta}$ in E, we want to show that $q_{H'}(\mathcal{W}) \geq 0$.

Put $U_\lambda := \{v \in V : \mathcal{E}_{H'}(v) \subseteq W_\lambda\}$ (observe that $U_0 = \emptyset$, since there are no deficient sets). Now, at this stage we would like to see that the sets U_λ form a queue $\mathcal{U}$ in the graph G and that $q_{H'}(\mathcal{W}) \geq \eta_G(\mathcal{U}, l)$. Unfortunately, this is only almost true. However, the only reason this fails is a small technical detail, namely that we had required in our definition for a queue in the context of degree constrained orientations that $U_\lambda = \bigcup_{\mu<\lambda} U_\mu$ for any limit ordinal λ. So, we will turn the chain $\mathcal{U}$ into a queue $\tilde{\mathcal{U}}$ by padding it, that is, for every limit ordinal λ we will insert the set $\bigcup_{\mu<\lambda} U_\mu$ into the chain. For this, we introduce a funtion σ on the ordinals that will provide the needed space, so that we can insert the new sets.

More precisely, we define inductively sets $\tilde{U}_\lambda$ and a function σ on the ordinals. Start with $\tilde{U}_0 = \emptyset$ and $\sigma(0) = 0$. For a successor ordinal $\lambda = \kappa + 1$ put $\sigma(\lambda) = \sigma(\kappa) + 1$. If λ is a limit ordinal set $\nu = \bigcup_{\mu<\lambda} \sigma(\mu)$, $\tilde{U}_\nu = \bigcup_{\mu<\nu} \tilde{U}_\mu$ and $\sigma(\lambda) = \nu + 1$. In any case we define $\tilde{U}_{\sigma(\lambda)} = U_\lambda$. The resulting $\tilde{\mathcal{U}} = (\tilde{U}_\lambda)_{\lambda \leq \sigma(\theta)}$ is indeed a queue.

Claim. *We claim that for all λ it holds that*

$$q_{H'}(\mathcal{W}_\lambda) \geq \eta_G(\tilde{\mathcal{U}}_{\sigma(\lambda)}, l) + |W_\lambda \setminus \mathcal{E}_G(\tilde{U}_{\sigma(\lambda)})|. \tag{1}$$

Proof. In the proof of the claim, q and D are always with respect to H', while $\mathcal{E}$ and η are always measured in G, so we will omit these subscripts. As every vertex $v \in V_l$ has a neighbour in H' we obtain $q(\mathcal{W}_0) = 0$. Since we also have that $W_0 = \emptyset$ and $\eta(\tilde{U}_0, l) = 0$, (1) holds for $\lambda = 0$.

So, let $\lambda > 0$ and assume first $\lambda = \kappa + 1$ to be a successor ordinal. We observe that

$$|D(W_\lambda) \setminus D(W_\kappa)| = \left| \bigcup_{v \in U_\lambda \setminus U_\kappa} Y_v \right| = \sum_{v \in U_\lambda \setminus U_\kappa} l(v) = l(U_\lambda \setminus U_\kappa) = l(\tilde{U}_{\sigma(\lambda)} \setminus \tilde{U}_{\sigma(\kappa)}).$$

Next, since $W_\lambda \supseteq W_\kappa \supseteq \mathcal{E}(\tilde{U}_{\sigma(\kappa)})$ and $W_\lambda \supseteq \mathcal{E}(\tilde{U}_{\sigma(\lambda)}) \supseteq \mathcal{E}(\tilde{U}_{\sigma(\kappa)})$, we get that

$$\begin{aligned} |W_\lambda \setminus W_\kappa| + |W_\kappa \setminus \mathcal{E}(\tilde{U}_{\sigma(\kappa)})| &= |W_\lambda \setminus \mathcal{E}(\tilde{U}_{\sigma(\kappa)})| \\ &= |W_\lambda \setminus \mathcal{E}(\tilde{U}_{\sigma(\lambda)})| + |\mathcal{E}(\tilde{U}_{\sigma(\lambda)}) \setminus \mathcal{E}(\tilde{U}_{\sigma(\kappa)})| \end{aligned}$$

Thus, by induction hypothesis, we get

$$\begin{aligned}
q(\mathcal{W}_\lambda) &= q(\mathcal{W}_\kappa) + |W_\lambda \setminus W_\kappa| - |D(W_\lambda) \setminus D(W_\kappa)| \\
&\geq \eta(\tilde{\mathcal{U}}_{\sigma(\kappa)}, l) + |W_\kappa \setminus \mathcal{E}(\tilde{U}_{\sigma(\kappa)})| + |W_\lambda \setminus W_\kappa| - |D(W_\lambda) \setminus D(W_\kappa)| \\
&= \eta(\tilde{\mathcal{U}}_{\sigma(\kappa)}, l) + |W_\lambda \setminus \mathcal{E}(\tilde{U}_{\sigma(\lambda)})| + |\mathcal{E}(\tilde{U}_{\sigma(\lambda)}) \setminus \mathcal{E}(\tilde{U}_{\sigma(\kappa)})| - l(\tilde{U}_{\sigma(\lambda)} \setminus \tilde{U}_{\sigma(\kappa)}) \\
&= \eta(\tilde{\mathcal{U}}_{\sigma(\lambda)}, l) + |W_\lambda \setminus \mathcal{E}(\tilde{U}_{\sigma(\lambda)})|,
\end{aligned}$$

which is (1).

So, assume λ to be a limit ordinal. Then, by definition of σ, $\sigma(\lambda) = \nu + 1$ where $\nu = \bigcup_{\mu<\lambda} \sigma(\mu)$. We get

$$|D(W_\lambda) \setminus (\bigcup_{\mu<\lambda} D(W_\mu))| = |\bigcup_{v \in U_\lambda \setminus (\bigcup_{\mu<\lambda} U_\mu)} Y_v| = |\bigcup_{v \in \tilde{U}_{\nu+1} \setminus \tilde{U}_\nu} Y_v| = l(\tilde{U}_{\nu+1} \setminus \tilde{U}_\nu),$$

and thus

$$\begin{aligned}
q(\mathcal{W}_\lambda) &= \liminf_{\mu<\lambda} q(W_\mu) - |D(W_\lambda) \setminus (\bigcup_{\mu<\lambda} D(W_\mu))| \\
&\geq \liminf_{\mu<\lambda} (\eta(\tilde{\mathcal{U}}_{\sigma(\mu)}, l) + |W_\mu \setminus \mathcal{E}(\tilde{U}_{\sigma(\mu)})|) - l(\tilde{U}_{\nu+1} \setminus \tilde{U}_\nu) \\
&\geq \eta(\tilde{\mathcal{U}}_\nu, l) + \liminf_{\mu<\lambda} |W_\mu \setminus \mathcal{E}(\tilde{U}_{\sigma(\mu)})| - l(\tilde{U}_{\nu+1} \setminus \tilde{U}_\nu).
\end{aligned}$$

Now, $\liminf_{\mu<\lambda} |W_\mu \setminus \mathcal{E}(\tilde{U}_{\sigma(\mu)})| \geq \liminf_{\mu<\lambda} |W_\mu \setminus \mathcal{E}(\tilde{U}_\nu)|$, and since $W_\lambda = \bigcup_{\mu<\lambda} W_\mu$ it follows that

$$\liminf_{\mu<\lambda} |W_\mu \setminus \mathcal{E}(\tilde{U}_{\sigma(\mu)})| \geq |W_\lambda \setminus \mathcal{E}(\tilde{U}_\nu)| = |W_\lambda \setminus \mathcal{E}(\tilde{U}_{\nu+1})| + |\mathcal{E}(\tilde{U}_{\nu+1}) \setminus \mathcal{E}(\tilde{U}_\nu)|.$$

(Note that $\mathcal{E}(\tilde{U}_{\nu+1}) \subseteq W_\lambda$.) Substituting in the above estimation for $q(\mathcal{W}_\lambda)$ we obtain

$$\begin{aligned}
q(\mathcal{W}_\lambda) &\geq \eta(\tilde{\mathcal{U}}_\nu, l) + |W_\lambda \setminus \mathcal{E}(\tilde{U}_{\nu+1})| + |\mathcal{E}(\tilde{U}_{\nu+1}) \setminus \mathcal{E}(\tilde{U}_\nu)| - l(\tilde{U}_{\nu+1} \setminus \tilde{U}_\nu) \\
&= \eta(\tilde{\mathcal{U}}_{\nu+1}, l) + |W_\lambda \setminus \mathcal{E}(\tilde{U}_{\nu+1})|.
\end{aligned}$$

Since $\nu + 1 = \sigma(\lambda)$ this shows (1) when λ is a limit ordinal. □

Having proved Claim (1), we see that $q_{H'}(\mathcal{W}) \geq 0$ as there are no deficient sets in G. Therefore, we can apply Theorem 2 and obtain a matching M_l of V_l in H.

Next, we find a matching M_u of E in H. For this, pick a vertex v of $V(G)$ with $u(v) = \infty$. Hence, its set X_v of copies in H is infinite and we can easily match all incident edges to a separate copy of v in H. Delete X_v and all those already matched edges from H and pick the next v' with $|X_{v'}| = \infty$. Continuing in this manner, we arrive at a subgraph H'' of H in which all the sets X_v are finite.

In order to use Theorem 1, we consider a finite set $X \subseteq V_u \cap V(H'')$. If there is a $v \in V(G)$ such that X_v meets X but is not completely contained in X then we may delete $X_v \cap X$ from X: Indeed, $|X|$ will get smaller while $D_{H''}(X)$ stays the same, making our task of showing $|D_{H''}(X)| \leq |X|$ only more difficult. Thus, we may assume that for each v either $X_v \subseteq X$ or X_v and X are disjoint. Denoting by V_X the set of vertices in G with $X_v \subseteq X$ we obtain $u(V_X) = |X|$ and $i_G(V_X) = |D_{H''}(X)|$. Since, by assumption, $u(V_X) \geq i_G(V_X)$ we find with Theorem 1 a matching of the remaining edges of G in H'', which together with the already matched edges gives us the desired M_u.

Finally, we construct a common matching M of V_l and E in H. Put $L := (V(H), M_l \cup M_u)$ where we put in a double edge if an edge of H lies in M_l and M_u. Clearly, L has maximum degree 2, and every vertex in $V_l \cup E$ has degree at least one. Thus components of L are cycles, finite or infinite paths. Consider a component P that is a finite path starting in a vertex of V_l. Then the first edge of P is necessarily an edge of M_l. Since we reach every vertex on P in E via an edge in M_l and since each vertex in E is incident with an edge in M_u, P ends in a vertex $w \in V_u$. The last edge of P lies in M_u; therefore, $w \notin V_l$.

Now in every component pick every other edge; if the component is a path (finite or infinite) starting in a vertex v in V_l start picking edges from v. In this way we get a matching M that covers all of $V_l \cup E$.

Proof of Lemma 6

Lemma 6. *Let there be neither deficient sets nor faulty sets in G, and let U be a taut set and L be a tight set. Then $U \setminus L$ is taut and $L \setminus U$ is tight.*

Proof. Let $\mathcal{L} = (L_\lambda)_{\lambda \leq \theta}$ be a queue with $\eta(\mathcal{L}, l) = 0$ and $L_\theta = L$, and define $\mathcal{M} = (L_\lambda \setminus U)_{\lambda \leq \theta}$. By transfinite induction, we show that for any ordinal $\lambda \leq \theta$ it holds that

$$\eta(\mathcal{L}_\lambda, l) \geq \eta(\mathcal{M}_\lambda, l) + i(L_\lambda \cap U) - l(L_\lambda \cap U) + d(L_\lambda \cap U, \overline{L_\lambda}). \tag{2}$$

This is trivially true for $\lambda = 0$. Let λ be such that the induction hypothesis holds for all $\mu < \lambda$. First, assume that λ is a successor ordinal. We use the induction hypothesis for $\lambda - 1$ in what follows:

$$\begin{aligned}
\eta(\mathcal{L}_\lambda, l) &= \eta(\mathcal{L}_{\lambda-1}, l) + i(L'_\lambda) + d(L'_\lambda, \overline{L_\lambda}) - l(L'_\lambda) \\
&\overset{(2)}{\geq} \eta(\mathcal{M}_{\lambda-1}, l) + i(L_{\lambda-1} \cap U) - l(L_{\lambda-1} \cap U) \\
&\quad + d(L_{\lambda-1} \cap U, \overline{L_{\lambda-1}}) + i(L'_\lambda) + d(L'_\lambda, \overline{L_\lambda}) - l(L'_\lambda) \\
&= \eta(\mathcal{M}_{\lambda-1}, l) + i(L_{\lambda-1} \cap U) + d(L_{\lambda-1} \cap U, \overline{L_{\lambda-1}} \setminus M'_\lambda) \\
&\quad d(L_{\lambda-1} \cap U, M'_\lambda) + i(L'_\lambda) + d(L'_\lambda, \overline{L_\lambda}) \\
&\quad - l(L_\lambda \cap U) - l(M'_\lambda)
\end{aligned}$$

With

$$\begin{aligned}
& d(L_{\lambda-1} \cap U, M'_\lambda) + i(L'_\lambda) + d(L'_\lambda, \overline{L_\lambda}) \\
&\qquad = d(L_{\lambda-1} \cap U, M'_\lambda) + i(L'_\lambda \cap U) + d(L'_\lambda \cap U, M'_\lambda) \\
&\qquad\quad + i(M'_\lambda) + d(L'_\lambda \cap U, \overline{L_\lambda}) + d(M'_\lambda, \overline{L_\lambda}) \\
&\qquad = i(M'_\lambda) + d(M'_\lambda, \overline{M_\lambda}) + i(L'_\lambda \cap U) + d(L'_\lambda \cap U, \overline{L_\lambda})
\end{aligned} \tag{3}$$

we get

$$\begin{aligned}
\eta(\mathcal{L}_\lambda, l) &\overset{(3)}{\geq} \eta(\mathcal{M}_{\lambda-1}, l) + i(L_{\lambda-1} \cap U) + d(L_{\lambda-1} \cap U, \overline{L_{\lambda-1}} \setminus M'_\lambda) \\
&\quad i(M'_\lambda) + d(M'_\lambda, \overline{M_\lambda}) + i(L'_\lambda \cap U) + d(L'_\lambda \cap U, \overline{L_\lambda}) \\
&\quad - l(L_\lambda \cap U) - l(M'_\lambda) \\
&= \eta(\mathcal{M}_\lambda, l) + i(L_{\lambda-1} \cap U) + d(L_{\lambda-1} \cap U, \overline{L_{\lambda-1}} \setminus M'_\lambda) \\
&\quad + i(L'_\lambda \cap U) + d(L'_\lambda \cap U, \overline{L_\lambda}) - l(L_\lambda \cap U) \\
&= \eta(\mathcal{M}_\lambda, l) + i(L_{\lambda-1} \cap U) + d(L_{\lambda-1} \cap U, \overline{L_\lambda}) \\
&\quad + d(L_{\lambda-1} \cap U, L'_\lambda \cap U) + i(L'_\lambda \cap U) + d(L'_\lambda \cap U, \overline{L_\lambda}) - l(L_\lambda \cap U) \\
&= \eta(\mathcal{M}_\lambda, l) + i(L_\lambda \cap U) + d(L_{\lambda-1} \cap U, \overline{L_\lambda}) \\
&\quad + d(L'_\lambda \cap U, \overline{L_\lambda}) - l(L_\lambda \cap U) \\
&= \eta(\mathcal{M}_\lambda, l) + i(L_\lambda \cap U) - l(L_\lambda \cap U) + d(L_\lambda \cap U, \overline{L_\lambda})
\end{aligned}$$

So, let λ be a limit ordinal. Then observe that $\liminf_{\mu \leq \lambda} d(L_\mu \cap U, \overline{L_\mu}) = d(L_\lambda \cap U, \overline{L_\lambda})$ as U is finite. Furthermore, $l(L_\mu \cap U)$ is bounded for the same reason. Thus

$$\begin{aligned}
\eta(\mathcal{L}_\lambda, l) &\geq \liminf_{\mu \leq \lambda} \big(\eta(\mathcal{M}_\mu, l) + i(L_\mu \cap U) - l(L_\mu \cap U) + d(L_\mu \cap U, \overline{L_\mu}) \big) \\
&\geq \eta(\mathcal{M}_\lambda, l) + i(L_\lambda \cap U) - l(L_\lambda \cap U) + d(L_\lambda \cap U, \overline{L_\lambda}).
\end{aligned}$$

Now, for $\lambda = \theta$ this yields

$$\begin{aligned}
0 &= \eta(\mathcal{L}, l) + u(U) - i(U) \\
&\geq \eta(\mathcal{M}, l) + i(L \cap U) - l(L \cap U) + u(U) - i(U) + d(L \cap U, \overline{L}) \\
&\geq \eta(\mathcal{M}, l) + u(U \setminus L) - i(U \setminus L) + (u - l)(L \cap U)
\end{aligned}$$

Since $\eta(\mathcal{M}, l) \geq 0$, $u \geq l$ and since $u(U \setminus L) \geq i(U \setminus L)$ it follows that $U \setminus L$ is taut. This then also implies that $\eta(\mathcal{M}, l) = 0$, and hence $L \setminus U$ is tight. □

Wojciechowski's conjecture

Wojciechowski calls a queue $\mathcal{P} := (P_\theta)_{\theta \leq \lambda}$ of partitions P_θ of $V(G)$ (with the obvious order) *proper* if

(i) $P_0 = \{V(G)\}$;

(ii) $P_{\theta+1} = (P_\theta \setminus V_0) \cup \{V'_0, V''_0\}$ where $V_0 \in P_\theta$ and $\{V'_0, V''_0\}$ is a partition of V_0 for all $\theta + 1 < \lambda$; and

(iii) P_γ is the least upper bound of the chain $(P_\theta)_{\theta<\gamma}$.

For a partition P of $V(G)$ denote by $E(P)$ the set of cross-edges, i.e. those edges with their endvertices in different partition classes of P. Now, assuming that $\mathcal{P}$ is proper define by transfinite induction for $k \in \mathbb{N}$ the so called *k-margin* $\xi_k(\mathcal{P}_\mu) \in \mathbb{Z} \cup \{-\infty, \infty\}$:

(i) $\xi_k(\mathcal{P}_0) = 0$;

(ii) $\xi_k(\mathcal{P}_\mu) = \xi_k(\mathcal{P}_\theta) + |E(P_\mu) \setminus E(P_\theta)| - k$ if $\mu = \theta + 1$; and

(iii) $\xi_k(\mathcal{P}_\mu) = \liminf_{\theta<\mu} \xi_k(\mathcal{P}_\theta)$ if μ is a limit ordinal.

Motivated by Nash-Williams' version of the Hall theorem Wojciechowki conjectured:

Conjecture 7 (Wojciechowski [4])**.** *Let G be countable, and $k \in \mathbb{N}$. Then G has a spanning tree if and only if for every queue $\mathcal{P}$ of vertex partitions it holds that $\xi_k(\mathcal{P}) \geq 0$.*

It is easy to see that necessity holds.

Using Aharoni and Thomassen's [1] result that for any $k \in \mathbb{N}$ there is a countable $2k$-edge-connected graph without k edge-disjoint spanning trees, the following lemma shows that Wojciechowski's conjecture is false.

Lemma 8. *Let G be a countable $2k$-edge-connected graph. Then for every queue $\mathcal{P}$ of vertex partitions it holds that $\xi_k(\mathcal{P}) \geq 0$.*

Proof. Let $\mathcal{P} = (P_\theta)_{\theta\leq\lambda}$. We define $\nu(\mathcal{P}_\theta) = \sum_{U\in P_\theta}(\frac{1}{2}d(U) - k)$. As $d(U) \geq 2k$ for every nonempty subset $U \subsetneq V(G)$, $\nu(\mathcal{P}_\theta) \in \mathbb{N} \cup \{0, \infty\}$ is well-defined. We claim that

$$\xi_k(\mathcal{P}_\mu) \geq \nu(\mathcal{P}_\mu) \text{ for every } \mu \leq \lambda. \tag{4}$$

Since $\nu(\mathcal{P}_\theta)$ is never negative, the assertion of the lemma follows.

So, let us prove the claim, which we do by transfinite induction. For this, consider first the case when $\mu = \theta + 1$. Let V_0 be the partition class of P_θ that is split up into V_0' and V_0'' in P_μ. Then

$$\begin{aligned}\xi_k(\mathcal{P}_\mu) &= \xi_k(\mathcal{P}_\theta) + d(V_0', V_0'') - k \\ &\geq \nu(\mathcal{P}_\theta) + (d(V_0')/2 - k) + (d(V_0'')/2 - k) - (d(V_0)/2 - k) = \nu(\mathcal{P}_\mu).\end{aligned}$$

Next, let μ be a limit ordinal. It suffices to show that $\nu(\mathcal{P}_\mu) \leq \liminf_{\theta<\mu} \nu(\mathcal{P}_\theta)$. In order to do so, let $K \in \mathbb{N}$ be an integer with $\nu(\mathcal{P}_\mu) \geq K$. Since K is finite, there is a finite subset $Q \subseteq P_\mu$ so that $\sum_{U\in Q}(d(U)/2 - k) \geq K$. For each $U \in Q$ pick $\min\{2(K + k), d(U)\} < \infty$ edges in $D(U, V(G) \setminus U)$, denote the set of these by F_U. Furthermore, since P_μ is the least upper bound, in each P_θ, for $\theta < \mu$, there is a unique set U_θ with $U_\theta \supseteq U$. Choose $\mu_U < \mu$ large enough so that $F_U \subseteq D(U_\theta, V(G) \setminus U_\theta)$ for all ordinals θ with $\mu_U \leq \theta < \mu$. Put $\mu' := \max\{\mu_U : U \in Q\} < \mu$. Then, for any θ with $\mu' \leq \theta < \mu$, it holds that

$$\nu(\mathcal{P}_\theta) = \sum_{U\in P_\theta}(d(U)/2 - k) \geq \sum_{U\in Q}(d(U)/2 - k) \geq K$$

(note that since G is $2k$-edge-connected no cancellation takes place). This shows that $\liminf_{\theta<\mu} \nu(\mathcal{P}_\theta) \geq K$. □

References

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