List edge-colouring and total colouring in graphs of low treewidth

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Abstract

We prove that the list chromatic index of a graph of maximum degree Δ and treewidth $\leq \sqrt{2\Delta} - 3$ is Δ; and that the total chromatic number of a graph of maximum degree Δ and treewidth $\leq \Delta/3 + 1$ is $\Delta + 1$. This improves results by Meeks and Scott.

1 Introduction

We treat two common generalisations of graph colouring: list colouring and total colouring. In analogy to the chromatic number, the *list chromatic number* $\mathrm{ch}(G)$ of a graph G is the smallest integer k so that for each choice of k legal colours at every vertex, there is a proper colouring that picks a legal colour at every vertex. In a similar way, the *list chromatic index* $\mathrm{ch}'(G)$ generalises the chromatic index.

While the list chromatic number and chromatic number may differ widely, the same is not true for the list chromatic index and the chromatic index. No example is known where these invariants differ. Whether this is a general truth is one of the central open questions in the field of list colouring:

List edge-colouring conjecture. *Equality* $\mathrm{ch}'(G) = \chi'(G)$ *holds for all graphs* G.

The conjecture appeared for the first time in print in 1985 in [3]. But, according to Alon [1], Woodall [15] and Jensen and Toft [8], the conjecture was suggested independently by Vizing, Albertson, Collins, Erdős, Tucker and Gupta in the late seventies. The conjecture was verified for bipartite graphs by Galvin [6].

While list colouring generalises either vertex or edge colouring, total colouring applies to both, vertices and edges. The *total chromatic number* $\chi''(G)$ is the smallest integer k so that there is a proper vertex colouring of the graph G with at most k colours and at the same time a proper edge colouring with the same k colours, so that no edge receives the same colour as any of its end vertices. If the list edge-colouring conjecture is true an easy argument[1] shows that $\chi''(G) \leq \Delta(G) + 3$ for all graphs G. The next conjecture asserts a little more:

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[1] If we colour the vertices of G using the colours $1, \ldots, \Delta(G) + 3$, then for each edge there are still $\Delta(G) + 1$ colours available. We can colour the edges from those sets if the list edge-colouring conjecture holds.

Total colouring conjecture. *$\chi''(G) \leq \Delta(G) + 2$ holds for all graphs G.*

The conjecture has been proposed independently by Behzad [2] and Vizing [14] during the seventies. The conjecture is known to hold up to a maximal degree of 5 (see Kostochka [10]), and Molloy and Reed [13] proved that $\chi''(G) \leq \Delta(G) + 10^{26}$, provided the graph G has large maximal degree.

It is clear that $\mathrm{ch}'(G)$ is bounded from below by $\Delta(G)$, the maximum degree of G. Also, $\chi''(G) \geq \Delta(G) + 1$, since a vertex of maximum degree and its incident edges have to receive distinct colours. We show that these trivial lower bounds are already sufficient for graphs of low treewidth and high maximum degree. (The treewidth of a graph is a way to measure how much the graph resembles a tree, a proper definition is given in Section 2.) In particular, our results imply the list edge-colouring conjecture as well as the total colouring conjecture for these classes of graphs.

Theorem 1. *Let G be graph of treewidth k and maximum degree $\Delta(G) \geq (k+3)^2/2$. Then $\mathrm{ch}'(G) = \Delta(G)$.*

Theorem 2. *Let G be a graph of treewidth $k \geq 3$ and maximum degree $\Delta(G) \geq 3k - 3$. Then $\chi''(G) = \Delta(G) + 1$.*

Our proofs rely on the fact that graphs with low treewidth and a high maximum degree contain substructures that are suitable for classical colouring arguments. This method has been used before: Zhou, Nakano and Nishizeki [17] show that $\chi'(G) = \Delta(G)$ if the graph G has treewidth $\leq \frac{1}{2}\Delta(G)$; Juvan, Mohar and Thomas [9] prove that the edges of any graph of treewidth 2 can be coloured from lists of size Δ; and in [11] the latter results are extended to graphs of treewidth 3 and maximum degree ≥ 7. Finally, this approach has also been employed by Meeks and Scott [12], who prove that determining the list chromatic index as well as the list total chromatic number is fixed parameter tractable, when parameterised by treewidth. As a by-product they obtain that $\chi''(G) = \Delta(G) + 1$ (and $\mathrm{ch}'(G) = \Delta(G)$) for all graphs G of treewidth k and maximum degree $\geq (k+2)2^{k+2}$. In Theorem 2, we improve their exponential bound on the maximum degree to a linear bound.

Our other result, Theorem 1, is only a slight improvement of an earlier bound that follows from results of Borodin, Kostochka and Wodall [4] (see also Woodall [16]). As graphs with treewidth k have maximum average degree at most $2k$, the results from [4] imply that $\mathrm{ch}'(G) = \Delta(G)$ for any graph of treewidth k and maximum degree $\Delta(G) \geq 2k^2$.

We mention, moreover, that while the list edge-colouring conjecture is usually formulated so as to cover multigraphs as well, our methods will fail if parallel edges are allowed.

The rest of the article is organised as follows. In the next section we will prove a lemma that provides a useful substructure, if applied to a graph of low treewidth and high maximum degree. This lemma will be used for the proofs of both our main results. The last two sections are independent of each other. In Section 3 we give a proof of Theorem 1 and in Section 4 we show Theorem 2. We remark that if we replace the bound $\Delta(G) \geq 3k - 3$ in Theorem 2, with the bound $\Delta(G) \geq 3k - 1$, then Theorem 2 becomes substantially easier to prove: all after Remark 10 will be unnecessary.

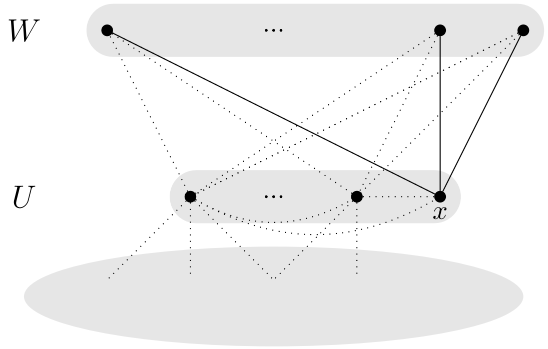

Figure 1: A useful substructure.

2 A structural lemma

We follow the notation of Diestel [5]. Let us recall the definition of a tree-decomposition and of treewidth. For a graph G a *tree decomposition* $(T, \mathcal{V})$ consists of a tree T and a collection $\mathcal{V} = \{V_t : t \in V(T)\}$ of *bags* $V_t \subseteq V(G)$ such that

- $V(G) = \bigcup_{t \in V(T)} V_t$,
- for each $vw \in E(G)$ there exists a $t \in V(T)$ such that v, $w \in V_t$ and
- if $v \in V_{t_1} \cap V_{t_2}$ then $v \in V_t$ for all vertices t that lie on the path connecting t_1 and t_2 in T.

A tree decomposition $(T, \mathcal{V})$ of a graph G has *width* k if all bags have size at most $k+1$. Note that in this case, if t is a leaf in T, then the degree of the vertices in $V_t \setminus \bigcup_{t' \neq t} V_{t'}$ is bounded by k. The *treewidth* of G is the smallest number k for which there exists a width k tree decomposition of G.

Given a tree decomposition $(T, \mathcal{V})$ of G, where T is rooted in some vertex $r \in V(T)$, we define the *height* $h(t)$ of any vertex $t \in V(T)$ to be the distance from r to t. For $v \in V(G)$ we define t_v as the (unique) vertex of minimum height in T for which $v \in V_{t_v}$. In particular, if $v \in V_r$, then $t_v = r$.

The proof of the following lemma can be extracted from [12]. For the sake of completeness we include a proof here.

Lemma 3 (Meeks and Scott [12])**.** *For Δ_0, $k \in \mathbb{N}$ with $\Delta_0 \geq 2k-1$, let G be a non-empty graph of treewidth at most k and*

$$\deg(v) + \deg(w) \geq \Delta_0 + 2$$

for each edge $vw \in E(G)$. Then there are disjoint vertex sets $U, W \subseteq V(G)$ and a vertex $x \in U$, such that

(a) *W is stable with $N(W) \subseteq U$;*

(b) *$\deg(w) \leq k$ for every $w \in W$;*

(c) *$W \subseteq N(x) \subseteq W \cup U$; and*

(d) $|U| \leq k+1$ *and* $|W| \geq \Delta_0 + 2 - 2k$.

Proof. By the assumptions of the lemma we have

$$\deg(v) + \deg(w) \geq \Delta_0 + 2 \geq 2k+1 \text{ for any edge } vw. \tag{1}$$

In particular, of any two adjacent vertices, at least one has degree at least $k+1$ (and G has at least one vertex of degree at least $k+1$). We define $B \subseteq V(G)$ to be the (non-empty) set of vertices of degree at least $k+1$. Then $S := V(G) \setminus B$ is stable.

Fix a width k tree decomposition $(T, \mathcal{V})$ of G and root the associated tree T in an arbitrary vertex $r \in V(T)$. Let $x \in B$ such that $h(t_x) = \max_{v \in B} h(t_v)$. Define T' as the subtree of T rooted at t_x, that is, the subgraph of T induced by all vertices $t \in V(T)$ where the path from t to the root r contains t_x.

Set $U := V_{t_x}$ and $X := \bigcup_{t \in V(T')} V_t$. Note that $|U| \leq k+1$. We have $B \cap X \subseteq U$, since any $v \in (B \cap X) \setminus U$ would have $h(t_v) > h(t_x)$, contrary to the choice of x. Consequently

$$X \setminus U \subseteq S. \tag{2}$$

By definition of the tree decomposition, no element of $X \setminus U$ can appear in a bag indexed by a vertex $t \in V(T - T')$. Since S is stable this gives

$$N(X \setminus U) \subseteq U. \tag{3}$$

By definition of t_x, also x does not appear in any bag V_t of a vertex $t \in T - T'$. So, $N(x) \subseteq X$.

Set $W := N(x) \setminus U$, and observe that W is non-empty as the at least $k+1$ neighbours of x do not all fit in $U \setminus \{x\}$, which has cardinality at most k. Note, moreover, that $W \subseteq X \setminus U$. So by (2), we can guarantee (b), and by (3), we have (a). Also, assertion (c) and the first part of (d) hold.

Using the assumptions of the lemma and (b), we get

$$\deg(x) \geq \Delta_0 + 2 - \deg(w) \geq \Delta_0 + 2 - k,$$

where w is any vertex in W. Since $N(x) \subseteq U \cup W$ we obtain

$$|W| \geq |N(x) \setminus (U \setminus \{x\})| \geq \Delta_0 + 2 - 2k,$$

which is as desired for the second part of (d). $\square$

3 List edge-colouring

We define an *assignment of lists* for a graph G as a function $L : E(G) \to \mathcal{P}(\mathbb{N})$ that maps the edges of G to *lists of colours* $L(v)$. A function $\gamma : E(G) \to \mathbb{N}$ is called an *L-edge-colouring* of G, if $\gamma(e) \in L(e)$ for each $e \in E(G)$ and if no two edges with a common endvertex receive the same colour. The *list chromatic index* $\mathrm{ch}'(G)$ is the smallest integer k such that for each assignment of lists L to G, where all lists have size k, there is an L-edge-colouring of G.

For the remainder of this section we suppose all bipartite graphs to have bipartition classes U and W, unless stated otherwise.

Let G be a graph with an assignment of lists $L : E(G) \to \mathcal{P}(\mathbb{N})$ to the edges of G. Suppose that for some stable subset $W' \subseteq V(G)$ we can find an L-edge-colouring of $G - W'$. In order to extend this to an L-edge-colouring of G we have to colour the edges of the bipartite graph H induced by the edges incident with W'. Note that in the colouring problem we now have for H, the list of each edge vw with $w \in W'$ has size of at least $\Delta - \deg_{G-H}(v) \geq \deg_H(v)$.

This motivates the following notion. For a bipartite graph G, we call a non-empty subset $C \subseteq W$ *choosable*, if for any assignment of lists L to the edges of the induced graph $H = G[C \cup N(C)]$ with $|L(vw)| \geq d_H(v)$ for each edge vw with $w \in C$ and $v \in N(C)$, there is an L-edge-colouring of H.

Lemma 4. *Let G be a (non-empty) bipartite graph with $2|W| > |U|(|U| - 1)$. Then W contains a choosable subset.*

To prove this we will use the following refined version of Galvin's theorem:

Theorem 5 (Borodin, Kostochka and Woodall [4])**.** *Let G be a bipartite graph with an assignment of lists L to the edges of G such that such that $|L(vw)| \geq \max\{\deg(v), \deg(w)\}$ for each edge $vw \in E(G)$. Then G has an L-edge-colouring.*

Corollary 6. *Let G be a bipartite graph with $\deg(v) \geq \deg(w)$ for each edge $vw \in E(G)$ with $w \in W$. Then W is choosable.*

Proof of Lemma 4. We proceed by induction on $k = |U|$. If $|U| = 1$, then for any vertex $w \in W$ the set $\{w\}$ is choosable. Given a graph G that satisfies the assumptions of the lemma and for which $|U| = k + 1$, we can assume that there is a vertex $v \in U$ of degree at most k. Otherwise W itself is choosable by Corollary 6: Indeed, we have $\deg(w) \leq k + 1$ for every $w \in W$ as w has all its neighbours in U, which is of size $k + 1$.

Let $W' := W \setminus N(v)$ and $U' := U \setminus \{v\}$. As $|U'| = k$ and

$$2|W'| = 2|W| - 2|N(v)| > (k + 1)k - 2k = k(k - 1)$$

the graph $G' = G[U' \cup W']$ fulfils the assumptions of the lemma. By the induction assumption W' contains a subset of vertices that is choosable with respect to G' and hence also choosable with respect to G. $\square$

Proof of Theorem 1. We prove the following assertion.

> *Let G be a graph of treewidth at most k with an assignment of lists L to the edges of G, such that each list $L(vw)$ has size $\max\{(k + 3)^2, \Delta(G)\}$. Then G has an L-edge-colouring.*

Set $\Delta := \max\left(\frac{(k+3)^2}{2}, \Delta(G)\right)$ and let G be a counterexample to the claim with $|V(G)|+|E(G)|$ minimal. So there are lists $L(vw)$ of size Δ for each $vw \in E(G)$, such that there is no L-edge-colouring of G. Clearly, G is connected and non-empty. Moreover, for every edge $vw \in E(G)$ we have

$$\deg(v) + \deg(w) \geq \Delta + 2.$$

Otherwise choose an L-edge-colouring of $G - vw$ by minimality and observe that $L(vw)$ retains at least one available colour, which can be used to colour vw. By

Lemma 3 (with $\Delta_0 = \Delta$), we know that G has subsets $U, W \subseteq V(G)$, such that $|U| \leq k+1$ and

$$|W| \geq \Delta + 2 - 2k \geq \frac{(k+3)^2}{2} + 2 - 2k > \frac{(k+1)k}{2}.$$

Let H be the bipartite graph induced by the edges between U and W. Then Lemma 4 provides a subset $C \subseteq W$ that is choosable with respect to H. By minimality there is an L-edge-colouring γ of the graph $G - C$. Since C is choosable, we can extend γ to an L-edge-colouring of G. This gives the desired contradiction. $\square$

Theorem 1 is almost certainly not best possible. In the introduction we mentioned the result of Zhou et al [17] that $\chi'(G) = \Delta(G)$ whenever $\Delta(G)$ is at least twice the treewidth. If one believes the list edge-colouring conjecture then this indicates that in Theorem 1 a maximum degree that is linear in k is already sufficient to guarantee the assertion.

One obvious way to improve the theorem would be to improve the bound on the size of W in Lemma 4. That bound, however, is the best we can obtain by our simple use of Theorem 5 and its corollary. An illustration is given in the following example.

Consider the family of bipartite graphs G_i, which is constructed as follows. Let G_1 be the complete bipartite graph with two vertices in partition class U_1, and one vertex in the other class, W_1. We obtain G_{i+1} from G_i by adding one vertex to U_i, and i vertices to W_i, thus obtaining U_{i+1} and W_{i+1}. The vertices in $W_{i+1} \setminus W_i$ are made adjacent to all vertices in U_{i+1}. (Thus, the vertex in $U_{i+1} \setminus U_i$ is only adjacent to $W_{i+1} \setminus W_i$.)

From the construction it is clear that $|W_i| = \sum_{j=1}^{i} j$ and $|U_i| = i+1$. So for each $\in \mathbb{N}$, we have

$$2|W_i| = 2\sum_{j=1}^{i} j = (i+1)i = |U|(|U|-1).$$

Moreover, we can not apply Corollary 6 to any induced bipartite subgraph $H = G[C \cup N(C)]$ with $C \subseteq W_i$ for some i. To see this, let C be any subset of W_i. Choose $\ell \leq i$ maximal such that there exists $w \in C \cap W_\ell \setminus W_{\ell-1}$. By construction of G_i, the vertex w has degree $|U_\ell| = \ell + 1$ in H, but any neighbour of w in $U_\ell \setminus U_{\ell-1}$ has degree $|W_\ell \setminus W_{\ell-1}| = \ell$ in H, by the maximality of ℓ. Thus Corollary 6 does not apply to $(C, N(C))$.

However, there is another version of Galvin's theorem, which can be used to show that for any $i \geq 3$, the set W_i itself is choosable in G_i:

Theorem 7 (Borodin, Kostochka and Woodall [4]). *Let G be a bipartite graph. Then W is choosable if and only if G has an L-edge-colouring from the lists $L^*(uw) = \{1, \ldots, \deg(u)\}$ for $u \in U$.*

Let us show by induction that the graphs G_i are colourable from the lists L^*, for $i \geq 3$. It is not hard to see that the graph G_3 (which equals $K_{3,3} - e$) can be coloured from the lists L^*. For the graph G_{i+1}, consider the lists L^* as in the above theorem. By induction, colour the edges of G_i from the smaller lists, and colour the edges adjacent to $U_{i+1} \setminus U_i$ with $1, \ldots, i$. The remaining edges lie between $W_{i+1} \setminus W_i$ and U_i, spanning a complete bipartite $(i+1)$-regular graph H. Their lists retain a set C_{i+1} of $i+1$ colours that are unused so far. So we may apply Corollary 6 to see that $W_{i+1} \setminus W_i$ is choosable in H. Thus by Theorem 7, we can colour the $E(H)$ with $i+1$ colours. Substitute these colours with the ones from C_{i+1}, and we are done.

This suggests that the bound on the size of $|W|$ in Lemma 4 might not be optimal. Perhaps Theorem 7 could be used in general to decrease the bound on the maximum degree.

4 Total colouring

Most of this section is devoted to the proof of Theorem 2. The same theorem with the slightly stronger bound $\Delta(G) \geq 3k-1$ can be shown with less effort: the reader interested in this variant may read our proof up to Remark 10 and skip everything afterwards.

We show the following assertion, which clearly implies Theorem 2:

$$\chi''(G) \leq \max\{\Delta(G), 3k-3, 2k\} + 1 \textit{ for any graph } G \textit{ of treewidth} \leq k.$$

Suppose this is not true, and let G be an edge-minimal counterexample. Put $\Delta := \max\{\Delta(G), 3k-3, 2k\}$. (Thus we assume G cannot be totally coloured with $\Delta+1$ colours, but $G-e$ can, for any edge e.)

Claim 8. *We have* $\deg(u) + \deg(v) \geq \Delta + 1$ *for each edge* $uv \in E(G)$.

Proof. Suppose G contains an edge uv for which the degree sum is at most Δ, where we assume that $\deg(u) \geq \deg(v)$. Let $G - uv$ be totally coloured with at most $\Delta + 1$ colours.

Now, if u and v receive the same colour, we recolour v: Note that v has $\deg(v)$ coloured neighbours and is incident with $\deg(v) - 1$ coloured edges. As

$$2\deg(v) - 1 \leq \deg(u) + \deg(v) - 1 \leq \Delta - 1,$$

there is a colour among the $\Delta + 1$ colours available that can be given to v.

Finally, we observe that the edge uv is incident with two coloured vertices and adjacent to $\deg(u) + \deg(v) - 2$ coloured edges. That means there are at most $\deg(u) + \deg(v) \leq \Delta$ different colours that cannot be chosen for uv – but we have $\Delta + 1$ colours at our disposal. Thus, G can be totally coloured with $\Delta + 1$ colours. □

By Claim 8 we may apply Lemma 3 with parameters $\Delta_0 = \Delta - 1$ and k; let U, W, x as obtained by the lemma. We choose a neighbour $w^* \in W$ of x and totally colour $G - w^*x$ with at most $\Delta + 1$ colours. Further, we uncolour every vertex in W. Observe that it will not be a problem to colour W once all the rest of $V(G) \cup E(G)$ has been coloured: The vertices in W have degree at most k each, so there will be at most $2k \leq \Delta$ forbidden colours at each $w \in W$.

We will say that a colour γ is *missing at a vertex* v, if neither v nor any edge incident with v is coloured with γ (neighbours of v, though, are allowed to have colour γ). Let $M(v)$ be the set of all colours missing at v.

As x is incident with at most $\Delta - 1$ coloured edges, there is a colour α missing at x. Call an edge coloured α an *α-edge.* Note that

$$\alpha \notin M(w^*). \tag{4}$$

Indeed, otherwise we could colour w^*x with α, then colour W as described above, and thus get a $(\Delta + 1)$-colouring of G, which by assumption does not exist.

Let F be the set of colours on edges between x and U together with the colour of x itself. Note that, since $|U| \leq k + 1$, we have that

$$|F| \leq k + 1. \tag{5}$$

Colours that are not in F, but missing at w^* are useful to us, because they could be used to colour xw^* (after possibly recolouring some edges in $E(U, W)$). Let us make this more precise:

Claim 9. *For every colour $\beta \in M(w^*) \setminus F$ there is a vertex $v_\beta \in W$ so that xv_β has colour β. Furthermore, there is an α-edge incident with v_β.*

Proof. If there is no $v_\beta \in W$ with xv_β coloured β, then, since $\beta \notin F$, the colour β is also missing at x, and we may use it for the edge xw^*. This proves the first part of the claim.

Next, if α is missing at v_β, we can colour xv_β with α and xw^* with β. Colouring W as described above, this gives a $(\Delta+1)$-colouring of G, a contradiction. Thus, we may assume that α is not missing at v_β, which, as the vertices of W are uncoloured, means that there is an α-edge at v_β. ☐

Denote by n_α the number of α-edges between U and W. Using Claim 9 and the fact that there is an α-edge at w^* by (4), we see that

$$n_\alpha \geq |M(w^*) \setminus F| + 1. \tag{6}$$

Let us now estimate how many colours are missing at w^*. Of the $\Delta + 1$ colours available, at most $\deg(w^*) - 1 \leq k - 1$ are used for incident edges, and none on w^*.

Thus,

$$|M(w^*)| \geq \Delta + 1 - (\deg(w^*) - 1) \geq 2k - 1. \tag{7}$$

Remark 10. *Our argumentation so far is enough to prove that any graph of treewidth k and maximum degree $\Delta(G) \geq 3k - 1$ satisfies $\chi''(G) = \Delta(G) + 1$. Indeed, note that with the assumption $\Delta(G) \geq 3k-1$, we obtain $|M(w^*)| \geq 2k+1$ in (7). Plugging this into (6), and using (5), we get $n_\alpha \geq k + 1$. On the other hand, the α-edges form a matching, which means there can be at most k, as α is missing at x and as $|U| \leq k + 1$.*

Let ρ_x be the colour of x.

Claim 11. *We have $F - \rho_x \subseteq M(w^*)$. Moreover, $\rho_x \in M(w^*)$ if and only if there is a vertex in U that is coloured α.*

Proof. Let u_α be the number of vertices of U coloured α. No vertex in U may be incident with two of the α-edges counted by n_α. As, moreover, α is missing at x, we get that

$$n_\alpha \leq |U| - u_\alpha - 1 \leq k - u_\alpha. \tag{8}$$

On the other hand,

$$|M(w^*) \setminus F| - |F \setminus M(w^*)| = |M(w^*)| - |F| \overset{(5),(7)}{\geq} k-2. \tag{9}$$

Putting (6), (8) and (9) together, we get

$$k - u_\alpha \geq |F \setminus M(w^*)| + k - 1.$$

In other words,

$$|F \setminus M(w^*)| + u_\alpha \leq 1.$$

In the case $u_\alpha > 0$, this proves the claim. So suppose $u_\alpha = 0$. If $\rho_x \in M(w^*)$, we can recolour x with α, colour the edge xw^* with ρ_x and colour W as above. Therefore, $\rho_x \notin M(w^*)$, and the claim follows. ☐

Claim 12. *We have* $|F| = k+1$ *and* $\deg(w^*) = k$.

Note that the claim, in particular, implies that x is adjacent to every vertex in U, as $|U| \leq k+1$.

Proof. Suppose either of the two equalities does not hold. Then the estimate in (9) is never tight, and we deduce

$$|M(w^*) \setminus F| - |F \setminus M(w^*)| \geq k-1.$$

This leads to

$$|F \setminus M(w^*)| + u_\alpha \leq 0.$$

Thus both $u_\alpha = 0$ and $\rho_x \in M(w^*)$, contradicting Claim 11. ☐

We next investigate which colours are missing at the vertices v_β from Claim 9.

Claim 13. $M(v_\beta) \subseteq M(w^*)$ *for every colour* $\beta \in M(w^*) \setminus F$.

Proof. First, note that $\rho_x \notin M(v_\beta) \setminus M(w^*)$. Indeed, otherwise $\rho_x \notin M(w^*)$ and therefore, by Claim 11, no vertex in U is coloured with α. Thus we can recolour xv_β with ρ_x, colour xw^* with β, recolour x with α and finish by colouring W.

Now, for contradiction suppose there is a colour $\beta^* \in M(v_\beta) \setminus M(w^*)$. By the previous paragraph, $\beta^* \neq \rho_x$. Hence, by Claim 11, $\beta^* \notin F$.

Then, there must be a vertex $y \in W$ so that xy has colour β^*, as otherwise we can colour the edge xw^* with colour β, and the edge xv_β with colour β^*, colour W, and are done. Moreover, y is incident with an α-edge. Indeed, otherwise we can colour the edge xy with α, the edge xw^* with β, and the edge xv_β with β^*, colour W, and are done.

Setting $\delta = 1$ if $\rho_x \in M(w^*)$ and $\delta = 0$ otherwise, we deduce from Claim 9 and (4) that

$$n_\alpha + \delta \geq |M(w^*) \setminus (F \setminus \{\rho_x\})| + 2 \overset{(5),(7)}{\geq} k+1.$$

On the other hand, using the second part of Claim 11, we see that

$$n_\alpha + \delta \leq |U| - 1 \leq k,$$

a contradiction. ☐

Fix $\beta \in M(w^*) \setminus F$. (There is such a β since $M(w^*) \setminus F \neq \emptyset$ by (9) and Claim 13.) The number of colours missing at any vertex v other than x or w^* is equal to $\Delta+1-(\deg(v)+1)$; at x and w^* there is one more colour missing as xw^* is uncoloured. Thus, it follows from $\deg(v_\beta) \leq k = \deg(w^*)$ (by Claim 12) that $|M(v_\beta)| \geq |M(w^*)| - 1$. So, by Claim 13, we get that $M(v_\beta) = M(w^*) \setminus \{\beta\}$. In particular, $F - \rho_x \subseteq M(v_\beta)$.

By Claim 9, there is a vertex $u \in U$ be so that $v_\beta u$ has colour α. The edge ux exists as $|F| = k + 1$ by Claim 12. The colour ρ_{ux} of ux is in $F - \rho_x$, and thus missing at v_β (by Claim 11). So we may swap colours on ux and uv_β. This yields again a total colouring of $(E - xw^*) \cup V \setminus W$. In the new colouring ρ_{ux} is missing at x. As ρ_{ux} is also missing at w^* we may use it to colour xw^*. Finally we fix the colours of the vertices in W in order to obtain a total colouring of G. This finishes the proof of Theorem 2.

We close the article by a short attempt at answering the question: how good is the bound on $\Delta(G)$ in Theorem 2?

Isobe, Zhou and Nishizeki [7] prove, with quite different methods, a very similar result: namely that every k-degenerate graph G with $\Delta(G) \geq 4k + 3$ can be totally coloured with $\Delta(G) + 1$ colours. So, the result of Isobe et al. is at same time stronger and weaker, that is, their result covers more graphs but with a stricter requirement on the maximal degree.

To see which maximal degree is at least necessary to force $\chi''(G) = \Delta(G)+1$ for graphs of treewidth k, let k and b be positive integers so that $k + b$ is even. Take a complete graph K on k vertices and add b new vertices, each complete to K. The resulting graph G then has treewidth k and maximal degree $\Delta(G) = k + b - 1$. Now, define a graph G' by adding a further new vertex b^*, which is adjacent to every vertex in K but to none outside K.

Consider any total colouring γ of G. Define an edge-colouring of G' by keeping all colours $\gamma(e)$ of edges $e \in E(G)$, and by colouring the edges b^*v for every $v \in K$ with the colour $\gamma(v)$ of the vertex v in the total colouring. This shows that

$$\chi''(G) \geq \chi'(G')$$

We lower-bound $\chi'(G)$ as

$$\chi'(G') \geq \frac{|E(G')|}{\lfloor |V(G')|/2 \rfloor} = \frac{\frac{1}{2}\left(k^2 + k + 2kb\right)}{\frac{1}{2}\left(k + b\right)} = k + b + \frac{k - b^2}{k + b}$$

Thus, if $k > b^2$ then $\chi''(G) > \Delta(G) + 1$. This means the bound on $\Delta(G)$ in Theorem 2 cannot be replaced by $\Delta(G) \geq k + \lfloor\sqrt{k}\rfloor - 1$. We have no clear opinion on whether $\sqrt{k}$ should be the right order for the best lower bound on $\Delta(G) - k$ in Theorem 2.

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