The graph formulation of the union-closed sets conjecture

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Abstract

In 1979 Frankl conjectured that in a finite non-trivial union-closed collection of sets there has to be an element that belongs to at least half the sets. We show that this is equivalent to the conjecture that in a finite non-trivial graph there are two adjacent vertices each belonging to at most half of the maximal stable sets. In this graph formulation other special cases become natural. The conjecture is trivially true for non-bipartite graphs and we show that it holds also for the classes of chordal bipartite graphs, subcubic bipartite graphs, bipartite series-parallel graphs and bipartitioned circular interval graphs.

1 Introduction

A set $\mathcal{X}$ of sets is *union-closed* if $X, Y \in \mathcal{X}$ implies $X \cup Y \in \mathcal{X}$. The following conjecture was formulated by Peter Frankl in 1979 [8].

Union-closed sets conjecture. *Let $\mathcal{X}$ be a finite union-closed set of sets with $\mathcal{X} \neq \{\emptyset\}$. Then there is a $x \in \bigcup_{X \in \mathcal{X}} X$ that lies in at least half of the members of $\mathcal{X}$.*

In spite of a great number of papers, see e.g. the good bibliography of Marković [16] for papers up to 2007, this conjecture is still wide open. Several special cases are known to hold, for example when $|\bigcup_{X \in \mathcal{X}} X|$ is upper bounded, with current best being 11 by Bošnjak and Marković [1], or when $|\mathcal{X}|$ is upper bounded, with current best being 46. This follows from a lemma by Lo Faro [7], and independently by Roberts and Simpson [22]. The conjecture also holds when certain sets are present in $\mathcal{X}$, such as a set of size 2 as shown by Sarvate and Renaud [24]. Possibly as a reflection of its general difficulty, Gowers [10] suggested that work on this conjecture could fruitfully be done as a collaborative Polymath project. See [2] for a survey of the literature on the union-closed sets conjecture.

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Various equivalent formulations have been discovered. We mention in particular Poonen [18] who translates the conjecture into the language of lattice theory. Several subsequent results together with their proofs belong to lattice theory, for example Reinhold [21] who proves this conjecture for lower semimodular lattices. A version of the conjecture is also known for hypergraphs; see El-Zahar [6].

In this paper we give a formulation of the conjecture in the language of graph theory. A set of vertices in a graph is *stable* if no two vertices of the set are adjacent. A stable set is *maximal* if it is maximal under inclusion, that is, every vertex outside has a neighbour in the stable set.

Conjecture 1. *Let G be a finite graph with at least one edge. Then there will be two adjacent vertices each belonging to at most half of the maximal stable sets.*

Note that Conjecture 1 is true for non-bipartite graphs. Indeed, if vertices u and v are adjacent there is no stable set containing them both and so one of them must belong to at most half of the maximal stable sets. An odd cycle will therefore imply the existence of two adjacent vertices each belonging to at most half of the maximal stable sets. The conjecture is for this reason open only for bipartite graphs. Moreover, in a connected bipartite graph, for any two vertices u and v in different bipartition classes we have a path from u to v containing an odd number of edges, so that if u and v each belongs to at most half the maximal stable sets there will be two adjacent vertices each belonging to at most half the maximal stable sets. Conjecture 1 is therefore equivalent to the following.

Conjecture 2. *Let G be a finite bipartite graph with at least one edge. Then each of the two bipartition classes contains a vertex belonging to at most half of the maximal stable sets.*

In this paper we show that Conjectures 1 and 2 are equivalent to the union-closed sets conjecture. The merit of this graph formulation is that other special cases become natural, in particular subclasses of bipartite graphs. We show that the conjecture holds for the classes of chordal bipartite graphs and bipartitioned circular interval graphs, and for subcubic and series-parallel bipartite graphs. Moreover, the reformulation allows to test Frankl's conjecture in a probabilistic sense: In [3] it is shown that almost every random bipartite graph satisfies Conjecture 2 up to any given $\delta > 0$, that is, almost every such graph contains in each bipartition class a vertex for which the number of maximal stable sets containing it is at most $\frac{1}{2} + \delta$ times the total number of maximal stable sets.

Stable sets are also called independent sets, with the maximal stable sets being exactly the independent dominating sets. A stable set of a graph is a clique of the complement graph and the graph formulation of the conjecture can also be stated in terms of maximal cliques, instead of maximal stable sets. The set of all maximal stable sets of a bipartite graph, or rather maximal complete bipartite cliques (bicliques) of the bipartite complement graph, was studied by Prisner [19] who gave upper bounds on the size of this set, also when excluding certain subgraphs. More recently, Duffus, Frankl and Rödl [5] and Ilinca and Kahn [12] investigate the number of maximal stable sets in certain regular and biregular bipartite graphs. In work related to the graph parameter boolean-width, Rabinovich, Vatshelle and Telle [20] study balanced bipartitions of a graph that bound the number of maximal stable sets. However, we have not found in the graph theory literature any previous work focusing on the number of maximal stable sets that vertices belong to.

2 Equivalence of the conjectures

For a subset S of vertices of a graph we denote by $N(S)$ the set of vertices adjacent to a vertex in S. All our graphs will be finite, and whenever we consider a union-closed set $\mathcal{X}$ of sets, it will be a finite set, all of whose member-sets will be finite as well. As Poonen [18] observed the latter assumption does not restrict generality, while the conjecture becomes false if $\mathcal{X}$ is allowed to have infinitely many sets.

We need two easy lemmas. The proof of the first is trivial.

Lemma 3. *Let G be a bipartite graph with bipartition U, W, and let S be a maximal stable set. Then $S = (U \cap S) \cup (W \setminus N(U \cap S))$.*

Lemma 4. *Let G be a bipartite graph with bipartition U, W, and let S and T be maximal stable sets. Then $(U \cap S \cap T) \cup (W \setminus N(S \cap T))$ is a maximal stable set.*

Proof. Clearly, $R = (U \cap S \cap T) \cup (W \setminus N(S \cap T))$ is stable. Trivially, any vertex in $W \setminus R$ has a neighbour in R. A vertex u in $U \setminus R$ does not lie in S or not in T (perhaps, it is not contained in either), let us say that $u \notin T$. As T is maximal, u has a neighbour $w \in W \cap T$. This neighbour w cannot be adjacent to any vertex in $U \cap S \cap T$ as T is stable. So, w belongs to R as well, which shows that R is a maximal stable set. □

For a fixed graph G let us denote by $\mathcal{A}$ the set of all maximal stable sets, and for any vertex v let us write $\mathcal{A}_v$ for the sets of $\mathcal{A}$ that contain v and $\mathcal{A}_{\overline{v}}$ for the sets of $\mathcal{A}$ that do not contain v. Let us call a vertex v *rare* if $|\mathcal{A}_v| \leq \frac{1}{2}|\mathcal{A}|$.

Theorem 5. *Conjecture 2 is equivalent to the union-closed sets conjecture.*

Proof. Let us consider first a union-closed set $\mathcal{X} \neq \{\emptyset\}$, which, without restricting generality, we may assume to include $\emptyset$ as a member. We put $U = \bigcup_{X \in \mathcal{X}} X$ and we define a bipartite graph G with vertex set $U \cup \mathcal{X}$, where we make $X \in \mathcal{X}$ adjacent with all $u \in X$.

Now we claim that $\tau : S \mapsto U \setminus S$ is a bijection between $\mathcal{A}$ and $\mathcal{X}$. First note that indeed $\tau(S) \in \mathcal{X}$ for every maximal stable set: Set $A = U \cap S$ and $\mathcal{B} = \mathcal{X} \cap S$. If $U \subseteq S$ then $U \setminus S = \emptyset \in \mathcal{X}$, by assumption. So, assume $U \nsubseteq S$, which implies $\mathcal{B} \neq \emptyset$. As S is a maximal stable set, it follows that $U \setminus S = U \setminus A = N(\mathcal{B})$. On the other hand, $N(\mathcal{B})$ is just the union of the $X \in S \cap \mathcal{X} = \mathcal{B}$, which is by the union-closed property equal to a set X' in $\mathcal{X}$. To see that τ is injective note that, by Lemma 3, S is determined by $U \cap S$, which in turn determines $U \setminus S$. For surjectivity, consider $X \in \mathcal{X}$. We set $A = U \setminus N(X)$ and observe that $S = A \cup (\mathcal{X} \setminus N(A))$ is a stable set. Moreover, as $X \in \mathcal{X} \setminus N(A)$ every vertex in $U \setminus A$ is a neighbour of $X \in S$, which means that S is maximal.

Now, assuming that Conjecture 2 is true, there is an rare $u \in U$, that is, it holds that $|\mathcal{A}_u| \leq \frac{1}{2}|\mathcal{A}|$. Clearly $\mathcal{A}$ is the disjoint union of $\mathcal{A}_u$ and of $\mathcal{A}_{\overline{u}}$, so that

$$|\tau(\mathcal{A}_{\overline{u}})| = |\mathcal{A}_{\overline{u}}| \geq \frac{1}{2}|\mathcal{A}| = \frac{1}{2}|\mathcal{X}|.$$

As $u \in \tau(S) \in \mathcal{X}$ for every $S \in \mathcal{A}_{\overline{u}}$, the union-closed sets conjecture follows.

For the other direction, consider a bipartite graph with bipartition U, W and at least one edge. Define $\mathcal{X} := \{U \setminus S : S \in \mathcal{A}\}$, and note that $\mathcal{X} \neq \{\emptyset\}$ as G has at least two distinct maximal stable sets. By Lemma 3, there is a bijection between $\mathcal{X}$ and $\mathcal{A}$. Moreover, it is a direct consequence of Lemma 4 that $\mathcal{X}$ is union-closed. From this, it is straightforward that Conjecture 2 follows from the union-closed sets conjecture. □

3 Application to four graph classes

For a set X of vertices we define $\mathcal{A}_X$ to be the set of maximal stable sets containing all of X. As before, we abbreviate $\mathcal{A}_{\{x\}}$ to $\mathcal{A}_x$.

Lemma 6. *Let x be a vertex of a bipartite graph G. Then there is an injection $\mathcal{A}_{N(x)} \to \mathcal{A}_x$.*

Proof. We define

$$i : \mathcal{A}_{N(x)} \to \mathcal{A}_x,\ S \mapsto S \setminus L_1 \cup \{x\} \cup (L_2 \setminus N(S \cap L_3)),$$

where L_i denotes the set of vertices at distance i to x. That $i(S)$ is stable and maximal is a direct consequence of the definition. Moreover, $i(S) = i(T)$ for $S, T \in \mathcal{A}_{N(x)}$ implies that S and T are identical outside $L_1 \cup L_2$. Moreover, S and T are also identical on $L_1 \cup L_2$: First, $L_1 = N(x)$ shows that L_1 lies in both S and T. Second, since every vertex in L_2 is a neighbour of one in $L_1 \subseteq S \cap T$, no vertex of L_2 can lie in either of S or T. Thus, $S = T$, and we see that i is an injection. □

We denote by $N^2(x) = N(N(x))$ the second neighbourhood of a vertex x. The following lemma generalises the observation that if a union-closed set contains a singleton then it satisfies the union-closed sets conjecture:

Lemma 7. *Let x, y be two adjacent vertices in a bipartite graph G with $N^2(x) \subseteq N(y)$. Then y is rare.*

Proof. From $N^2(x) \subseteq N(y)$ it follows that every maximal stable set containing y must contain all of $N(x)$. Thus, $\mathcal{A}_y = \mathcal{A}_{N(x)}$, which means by Lemma 6 that $|\mathcal{A}_y| \leq |\mathcal{A}_x|$ and as $|\mathcal{A}_y| + |\mathcal{A}_x| \leq |\mathcal{A}|$ the lemma is proved. □

We now apply the lemma to the class of *chordal bipartite* graphs. This is the class of bipartite graphs in which every cycle with length at least six has a chord.

This graph class was originally defined in 1978 by Golumbic and Gross [9]. It is also known as the class of bipartite weakly chordal graphs.

A vertex v in a bipartite graph is *weakly simplicial* if the neighbourhoods of its neighbours form a chain under inclusion. Hammer, Maffray and Preissmann [11], and also Pelsmajer, Tokaz and West [17] prove the following:

Theorem 8. *A bipartite graph with at least one edge is chordal bipartite if and only if every induced subgraph has a weakly simplicial vertex. Moreover, such a vertex can be found in each of the two bipartition classes.*

Let us say that a bipartite graph *satisfies Frankl's conjecture* if each of its bipartition classes contains a rare vertex. In order to avoid repeating the trivial condition that the graph has to contain at least one edge, we will also consider edgeless graphs to satisfy Frankl's conjecture.

Theorem 9. *Chordal bipartite graphs satisfy Frankl's conjecture.*

Proof. For a given bipartition class, let x be a weakly simplicial vertex in it. Among the neighbours of x denote by y the one whose neighbourhood includes the neighbourhoods of all other neighbours of x. Then y is rare, by Lemma 7. $\square$

Going beyond chordal bipartite graphs, we quickly encounter graphs that cannot be handled anymore by Lemma 7: No vertex in an even cycle of length at least six can be proved to be rare by applying Lemma 7. We will, therefore, strengthen the lemma to at least cover all even cycles.

For this, let us extend our notation a bit. For two vertices u, v let us denote by $\mathcal{A}_{uv}$ the set of $S \in \mathcal{A}$ containing both of u and v, by $\mathcal{A}_{u\overline{v}}$ the set of $S \in \mathcal{A}$ containing u and but not v, and by $\mathcal{A}_{\overline{uv}}$ the set of $S \in \mathcal{A}$ containing neither of u and v.

Lemma 10. *Let G be a bipartite graph. Let y and z be two neighbours of a vertex x so that $N^2(x) \subseteq N(y) \cup N(z)$. Then one of y and z is rare.*

Proof. We may assume that $|\mathcal{A}_{y\overline{z}}| \leq |\mathcal{A}_{\overline{y}z}|$. Now, from $N^2(x) \subseteq N(y) \cup N(z)$ we deduce that $\mathcal{A}_{yz} = \mathcal{A}_{N(x)}$. Thus, by Lemma 6, we obtain $|\mathcal{A}_{yz}| \leq |\mathcal{A}_x|$. Since $\mathcal{A}_x \subseteq \mathcal{A}_{\overline{yz}}$ it follows that $|\mathcal{A}_y| = |\mathcal{A}_{y\overline{z}}| + |\mathcal{A}_{yz}| \leq |\mathcal{A}_{\overline{y}z}| + |\mathcal{A}_{\overline{yz}}| = |\mathcal{A}_{\overline{y}}|$. As $|\mathcal{A}| = |\mathcal{A}_y| + |\mathcal{A}_{\overline{y}}|$, we see that y is rare. $\square$

Again, the lemma generalises a fact that is well known for the set formulation of the union-closed sets conjecture: If one of the sets in the union-closed set $\mathcal{X}$ contains exactly two elements then one of the two elements will lie in at least half of the members of $\mathcal{X}$; see Sarvate and Renaud [24].

Next we give an application of Lemma 10 to a class of graphs derived from *circular interval graphs*. The class of circular interval graphs plays a fundamental role in the structure theorem of claw-free graphs of Chudnovsky and Seymour [4]. Circular interval graphs are defined as follows: Let a finite subset of a circle be the vertex set, and for a given set of subintervals of the circle consider two vertices to be adjacent if there is an interval containing them both. This class is equivalent to what is known as the proper circular arc graphs.

Circular interval graphs are not normally bipartite. The only exceptions are even cycles and disjoint unions of paths. Nevertheless, we may obtain a rich class of bipartite graphs from circular interval graphs: For any circular interval graph, partition its vertex set and delete every edge with both its endvertices in the same class. We call any graph arising in this manner a *bipartitioned circular interval graph*.

Theorem 11. *Bipartitioned circular interval graph satisfy Frankl's conjecture.*

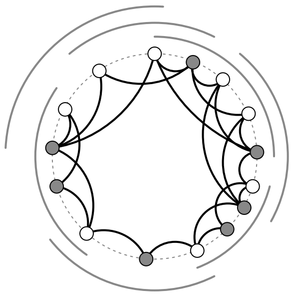

Figure 1: A bipartitioned circular interval graph

Proof. Consider a bipartitioned circular interval graph defined by intervals $\mathcal{I}$, and let x be a non-isolated vertex of the graph.

For every neighbour u of x we choose an interval $I_u \in \mathcal{I}$ containing both x and u. If $\bigcup_{v \in N(x)} I_v$ covers the whole circle, then there are already two such intervals I_y and I_z that cover the circle. Clearly, every vertex not in the same bipartition class as y and z is adjacent to at least one of them. In particular, $N^2(x) \subseteq N(y) \cup N(z)$.

So, let us assume that there is a point p on the circle that is not covered by any I_v, $v \in N(x)$. We choose y as the first neighbour of x from p in clockwise direction, and z as the first neighbour of x from p in counterclockwise direction. Then y, v, z appear in clockwise order for every $v \in N(x)$ and $v' \in I_y \cup I_z$ for every vertex v' so that y, v', z appear in clockwise order.

Let us show that again $N^2(x) \subseteq N(y) \cup N(z)$. For this consider a $u \in N^2(x)$, and a neighbour w of x that is adjacent to u. Thus, there is a $J \in \mathcal{I}$ containing both u and w. If y, u, z appear in clockwise order, then $u \in I_y \cup I_z$, which implies $u \in N(y) \cup N(z)$. If not, then J meets one of y or z as y, w, z appear in clockwise order. Thus, by virtue of J, the vertex u is adjacent to at least one of y and z.

In both cases, we apply Lemma 10 in order to see that one of y and z is rare. As the choice of x was arbitrary, we find rare vertices in both bipartition classes. □

Let us now turn to *subcubic* bipartite graphs: Bipartite graphs in which no vertex has a degree greater than 3.

Theorem 12. *Subcubic bipartite graphs satisfy Frankl's conjecture.*

Our proof of Theorem 12 needs some preparation. Let us call a graph G *reduced* if there is no vertex v whose neighbourhood is equal to the union of neighbourhoods of some other vertices. In particular, reduced graphs are *twin-free*, that is, no two vertices have identical neighbourhoods. The following lemma tells us that we may restrict our attention to reduced bipartite graphs.

Lemma 13. *For any bipartite graph G there is a reduced induced subgraph G' so that G satisfies Frankl's conjecture if G' satisfies it.*

Proof. Assume there are pairwise distinct vertices $u, v_1, v_2, \ldots, v_k$ such that $N(u) = \bigcup_{i=1}^{k} N(v_i)$. Then $\mathcal{A}_u = \mathcal{A}_{\{v_1,v_2,\ldots,v_k\}}$. Thus, if A is a maximal stable set of G, then $A - u$ is one of $G - u$, and conversely, any maximal stable set

A' of $G-u$ is already maximally stable in G if $\{v_1, v_2, \dots, v_k\} \not\subseteq A'$; otherwise $A'+u$ is a maximal stable set of G. Hence, a rare vertex of $G-u$ is also rare in G. The assertion is now obtained by iteratively deleting vertices such as u from G. □

Unlike the two classes above, subcubic graphs do not have an easily exploitable local structure. In particular, Lemmas 7 and 10 will have only limited use. Nevertheless, we can verify Frankl's conjecture by adapting two results on the set formulation of the union-closed sets conjecture into the graph setting. Both results, one of Vaughan and the other of Knill, have surprisingly involved proofs. For a union-closed set $\mathcal{X}$, we say that an element of $\bigcup \mathcal{X}$ is *abundant* if the element appears in at least half of the member-sets of $\mathcal{X}$.

Theorem 14 (Vaughan [25])**.** *Let $\mathcal{X}$ be a union-closed set containing three distinct sets of size 3 all of which have one element in common. Then there is an abundant element in the union of the three sets.*

While Vaughan's theorem gives a local condition, not unlike Lemmas 7 and 10, when a particular union-closed set satisfies the conjecture, the following result of Knill treats a special class of union-closed sets, which he calls *graph-generated families*. In this context, we view edges of a graph H as subsets of $V(H)$ of size two.

Theorem 15 (Knill [14])**.** *Given a graph H with at least one edge, let $\mathcal{B} = \{\bigcup F : F \subseteq E(H)\}$. Then there is an edge $e \in E(H)$ such that $|\{S \in \mathcal{B} : e \subseteq S\}| \leq \frac{|\mathcal{B}|}{2}$.*

Probably unaware of Knill's result, it was restated as a conjecture by El-Zahar [6]. Finally, as a response to El-Zahar's paper, it was reproven by Llano, Montellano-Ballesteros, Rivera-Campo and Strausz [15].

We first translate Knill's theorem to the graph setting:

Lemma 16. *Let G be a twin-free bipartite graph with bipartition $U \cup W$, where every vertex in U is of degree 2. Then there is a rare vertex in U.*

Proof. Again, let $\mathcal{A}$ be the set of maximal stable sets of G. Observe that G is the subdivision of the graph H on vertex set W, where any two distinct vertices x, y of H are adjacent if and only if they have a common neighbor $u \in U$ in G. As G is twin-free, every edge $e = xy$ of H corresponds to a unique vertex $u_e \in U$ with $N(u_e) = \{x, y\}$.

Let $\mathcal{B} = \{\bigcup F : F \subseteq E(H)\}$, and note that $\mathcal{B} = \{N_G(U') : U' \subseteq U\}$. We will establish a bijection between $\mathcal{B}$ and $\mathcal{A}$. For this, denote by $\mathcal{A}_{\cap W}$ the intersections of maximal stable sets of G with W. Then we define the mapping $\mathcal{B} \to \mathcal{A}_{\cap W}$ by $N_G(U') \mapsto W \setminus N_G(U')$, for $U' \subseteq U$. As $(W \setminus N_G(U')) \cup (U \setminus N_G(W \setminus N_G(U')))$ is a maximal stable set, the mapping is a bijection. Recall that Lemma 3 asserts that every maximal stable set is determined by its intersection with one of the bipartition classes. Thus, the bijection $\mathcal{B} \to \mathcal{A}_{\cap W}$ extends to a bijection $\mathcal{B} \to \mathcal{A}$. In particular, $|\mathcal{A}| = |\mathcal{B}|$.

Now, for any $S \in B$ there exists $U' \subseteq U$ so that $N_G(U') = S$. Any edge $e \in E(H)$ between vertices $x, y \in W$ is contained in S if and only if $x, y \notin W\ N_G(U')$, which means that the unique maximal stable set $A \in \mathcal{A}$ with $A \cap W = W \setminus N_G(U')$ needs to contain u_e, the vertex in U with neighbours x, y.

Therefore, the number of $S \in \mathcal{B}$ with $e \subseteq S$ is equal to the number of maximal stable sets containing u_e.

Applying Theorem 15 we obtain an edge $e = xy \in E(H)$ such that $|\{S \in \mathcal{B} : \{x, y\} \subseteq S\}| \leq \frac{|\mathcal{B}|}{2}$. This then implies that u_e lies in at most $\frac{|\mathcal{B}|}{2} = \frac{|\mathcal{A}|}{2}$ maximal stable sets, which completes the proof. □

Proof of Theorem 12. Let G be a subcubic bipartite graph with bipartition $U \cup W$, and let $\mathcal{A}$ be the set of maximal stable sets of G. By Lemma 13, we may assume that G is reduced and, in particular, twin-free.

Let us prove that there is a rare vertex in U. Then, by symmetry, we know that there must be a rare vertex in W too. If W contains a vertex of degree 1 or 2, we are done by Lemma 10. So, let us assume that every vertex in W has degree 3.

First assume that there is a vertex $u \in U$ of degree 1. Let $x \in W$ be its unique neighbor, and let $y, z \in U$ be the other two neighbors of x. By Lemma 10, y or z is rare and we are done.

Now assume that there is a vertex $u \in U$ of degree 3, say $N(u) = \{x, y, z\}$. Consider the set $\mathcal{B} = \{U \setminus S : S \in \mathcal{A}\}$, which is union-closed by Lemma 4. Then $N(x), N(y), N(z) \in \mathcal{B}$, and $u \in N(x) \cap N(y) \cap N(z)$. Note that $N(x), N(y), N(z)$ are three distinct sets as G is twin-free. From Theorem 14 we know that there is an abundant element of $\mathcal{B}$ in $N(x) \cup N(y) \cup N(z)$, and hence this is a rare vertex in U.

The remaining case, when every vertex in U is of degree 2 is taken care of by Lemma 16. □

Recall that a graph is called *series-parallel* if it does not contain K_4 as a minor. Equivalently, a graph is series-parallel if and only if it is of treewidth at most two. Reusing some of the tools presented above, we can settle Frankl's conjecture for bipartite series-parallel graphs.

Theorem 17. *Bipartite series-parallel graphs satisfy Frankl's conjecture.*

The following lemma gives us enough information on the local structure of a series-parallel graph to prove the theorem with Lemmas 7 and 10.

Lemma 18 (Juvan, Mohar and Thomas [13]). *Every non-empty series-parallel graph G has one of the following:*

(a) *a vertex of degree at most one,*

(b) *two twins of degree two,*

(c) *two distinct vertices u, v and two not necessarily distinct vertices $w, z \in V(G) \setminus \{u, v\}$ such that $N(v) = \{u, w\}$ and $N(u) \subseteq \{v, w, z\}$, or*

(d) *five distinct vertices v_1, v_2, u_1, u_2, w such that $N(w) = \{u_1, u_2, v_1, v_2\}$ and $N(v_i) = \{w, u_i\}$ for $i = 1, 2$.*

Proof of Theorem 17. Let G be a non-empty bipartite series-parallel graph, say with bipartition classes (U, W), and we may assume that G does not contain any isolated vertex. Our argumentation is symmetric, so it suffices to show that there is a rare vertex among the vertices in U. The class of series-parallel graphs is closed under induced subgraphs, and thus by Lemma 13 we may assume that G is reduced.

Let L be the set of *leaves* of G, that is, the set of degree 1 vertices. If there is a leaf in W, we obtain with Lemma 7 a rare vertex in U. So we may assume that $L \subseteq U$. Let $G' = G - L$ be the graph obtained by deleting all leaves. Since $L \subseteq U$, every vertex in $U \cap V(G')$ is of degree at least 2. In particular, G' is not empty.

We claim that in G' there is some vertex $x \in W$ of degree at most 2. If the claim is true then Lemma 10 yields that some $y \in N_{G'}(x) \subseteq U$ is rare in G, since every neighbour of x in $G - G'$ is a leaf.

So it remains to prove the claim. Lemma 18 yields that G' contains one of the configurations in (a), (b), (c), or (d). Clearly, (d) is not possible since G' is bipartite and thus triangle-free.

In case (a), there is a leaf in G', which then needs to be contained in W because every vertex in $U \cap V(G')$ has degree at least 2. In case (b), let u, v be the two twins of degree 2. If $u, v \in U$ then u and v are twins in G as well, which is impossible as G is reduced. Consequently, $u, v \in W$ and the claim is again verified. In the last case (c), there are two distinct vertices u, v and two not necessarily distinct vertices $w, z \in V(G) \setminus \{u, v\}$ such that $N(v) = \{u, w\}$ and $N(u) \subseteq \{v, w, z\}$. But G' is bipartite and so $uw \notin E(G')$. In particular, both u and v are of degree at most two. Since u and v are adjacent, one of them is contained in W. This completes the proof. □

4 Discussion

Lemmas 7 and 10 generalise the cases when there is a vertex x of degree 1 or 2. Then, one of the neighbours of x is rare. In contrast, the subcubic case required a bit of work. This is because none of the neighbours of a vertex of degree at least 3 have to be rare. An example is given in Figure 2 on the left, where no neighbour of the vertex v is rare. Note that both graphs in Figure 2 are subcubic.

Again, this is not new, in the sense that it corresponds directly to an observation of Sarvate and Renaud [23] in the set formulation: A set of size three need not contain any element appearing in at least half of the member sets of the union-closed set.

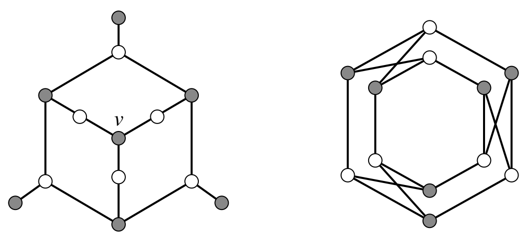

Figure 2: Left: No neighbour of v is rare. Right: Lemmas 7 or 10 not applicable

As chordal bipartite graphs are exactly the $(C_6, C_8, C_{10}, \ldots)$-free graphs one may be tempted to generalise Theorem 9 by allowing one more even cycle, the 6-cycle, as induced subgraph. While Lemma 7 is no longer strong enough even for the C_6, Lemma 10 easily takes care of any graph with a degree 2 vertex in each bipartition class. In general, however, Lemma 10 turns out to be too weak as well to prove the conjecture for $(C_8, C_{10}, C_{12}, \ldots)$-free graphs: The graph on the right in Figure 2 is of that form but has no vertices covered by Lemma 10.

We contend that the results in the previous section substantiate the usefulness of the graph formulation of the union-closed sets conjecture. Moreover, we believe that a good number of other graph classes should be within reach. Does Frankl's conjecture hold for planar graphs, regular graphs or for graphs of treewidth 3?

References

[1] I. Bošnjak and P. Marković, *The 11-element case of Frankl's conjecture*, Electr. J. Comb. **15** (2008), R88.

[2] H. Bruhn and O. Schaudt, *The journey of the union-closed sets conjecture*, in preparation.

[3] ———, *The union-closed sets conjecture almost holds for almost all random bipartite graphs*, preprint 2013.

[4] M. Chudnovsky and P.D. Seymour, *Claw-free graphs. III. Circular interval graphs*, J. Combin. Theory (Series B) **98** (2008), no. 4, 812–834.

[5] D. Duffus, P. Frankl, and V. Rödl, *Maximal independent sets in bipartite graphs obtained from boolean lattices*, Eur. J. Comb. **32** (2011), no. 1, 1–9.

[6] M. El-Zahar, *A graph-theoretic version of the union-closed sets conjecture*, J. Graph Theory **26** (1997), 155–163.

[7] G. Lo Faro, *Union-closed sets conjecture: Improved bounds*, J. Combin. Math. Combin. Comput. **16** (1994), 97–102.

[8] P. Frankl, *Handbook of combinatorics (vol. 2)*, MIT Press, Cambridge, MA, USA, 1995, pp. 1293–1329.

[9] M.C. Golumbic and C.F. Goss, *Perfect elimination and chordal bipartite graphs*, J. Graph Theory **2** (1978), 155–163.

[10] T. Gowers, *Gowers's weblog: Possible future PolyMath projects*, http://gowers.wordpress.com/2009/09/16/possible-future-polymath-projects, 2009.

[11] P.L. Hammer, F. Maffray, and M. Preissmann, *A characterization of chordal bipartite graphs*, Rutcor research report, Rutgers University, New Brunswick, NJ, 1989.

[12] L. Ilinca and J. Kahn, *Counting maximal antichains and independent sets*, CoRR **abs/1202.4427** (2012).

[13] M. Juvan, B. Mohar, and R. Thomas, *List edge-colorings of series-parallel graphs*, Electron. J. Combin. **6** (1999), 1077–8926.

[14] E. Knill, *Graph generated union-closed families of sets*, arXiv:math/9409215v1 [math.CO], 1994.

[15] B. Llano, J.J. Montellano-Ballesteros, E. Rivera-Campo, and R. Strausz, *On conjectures of frankl and el-zahar*, J. Graph Theory **57** (2008), 344–352.

[16] P. Marković, *An attempt at Frankl's conjecture.*, Publications de l'Institut Mathématique. Nouvelle Série **81(95)** (2007), 29–43.

[17] M.J. Pelsmajer, J. Tokaz, and D.B. West, *New proofs for strongly chordal graphs and chordal bipartite graphs*, preprint 2004.

[18] B. Poonen, *Union-closed families*, J. Combin. Theory (Series A) **59** (1992), 253–268.

[19] E. Prisner, *Bicliques in graphs I: Bounds on their number*, Combinatorica **20** (2000), no. 1, 109–117.

[20] Y. Rabinovich, J.A. Telle, and M. Vatshelle, *Upper bounds on the boolean width of graphs with an application to exact algorithms*, submitted, 2012.

[21] J. Reinhold, *Frankl's conjecture is true for lower semimodular lattices*, Graphs and Combinatorics **16** (2000), no. 1, 115–116.

[22] I. Roberts and J. Simpson, *A note on the union-closed sets conjecture*, Austral. J. Combin. **47** (2010), 265–269.

[23] D.G. Sarvate and J.-C. Renaud, *Improved bounds for the union-closed sets conjecture*, Ars Combin. **29** (1989), 181–185.

[24] ______, *On the union-closed sets conjecture*, Ars Combin. **27** (1989), 149–154.

[25] T.P. Vaughan, *Three-sets in a union-closed family*, J. Combin. Math. Combin. Comput. **49** (2004), 73–84.