Consequence-based Reasoning for Lightweight Description Logics

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Problems with Tableau-based Reasoning

Consequence-based Reasoning for $\mathcal{EL}$

Practical Consequence-based $\mathcal{EL}$ Reasoning

Key Features of Tableau Reasoning

Builds a counter model to test entailments

- Prove $\mathcal{O} \models C \sqsubseteq D$
- Try to build an interpretation of $\mathcal{O}$ that satisfies $C \sqcap \neg D$

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Sound and complete for expressive DLs (up to $\mathcal{SROIQ}$)

Often practical despite the high complexity

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- Example: $\exists R.C \sqsubseteq D$
- General GCI rule: add $\neg\exists R.C \sqcup D$, horrible
- Absorption: rewrite to $\exists R.\top \sqsubseteq D \sqcup \forall R.\neg C$, still bad

Issues Can Be Addressed

Via optimizations

- Smarter consistency algorithms
  (extending absorption, etc.)
- Smarter classification algorithms
  (reduce the number of consistency checks)
- Share information across consistency checks
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Via alternative reasoning approaches

- Consequence-based reasoning

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Brief history

- Long time ago: used in logic programming
- 2005: $\mathcal{EL}^+$ procedure by F. Baader, C. Lutz, and S. Brandt
- 2009: Full GALEN classified in a few seconds (Y. Kazakov)
- 2011: Extended to non-Horn logics

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From its W3C spec:

- suitable for applications employing ontologies that define very large numbers of concepts and/or roles,
- captures the expressivity of many existing ontologies,
- ontology consistency, concept subsumption, and instance checking can be decided in polynomial time

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- General concept inclusions (GCIs):
  $\mathsf{Disease} \sqcap \exists \mathsf{transmittedThrough}.\mathsf{Air} \sqsubseteq \mathsf{AirborneDisease}$
- Concept equivalences:
  $\mathsf{InfectiousDisease} \equiv \mathsf{Disease} \sqcap \exists \mathsf{resultsFrom}.\mathsf{Infection}$

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- Subproperty chains: $\mathsf{hasProperPart} \circ \mathsf{hasPart} \sqsubseteq \mathsf{hasProperPart}$

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Semantics (model theory) is the same as for $\mathcal{ALC}$: $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$

Tractability of $\mathcal{EL}$

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Answer: Complexity is polynomial in the size of $\mathcal{O}$

Each $A \sqsubseteq B$ can be decided in polynomial time

Tractability vs Practicality

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Let's make a simple calculation:

- SNOMED CT contains roughly 300,000 medical terms
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This is NOT practical, need a goal-oriented approach

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$\mathcal{EL}$: Exp $=$ axioms, Closure contains the inferred taxonomy

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- Provably complete, derives all entailed subsumptions between concept names

$\mathcal{EL}$ Classification, Example

Ontology $\mathcal{O}$:

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2. $B \sqsubseteq C$
3. $\exists R.C \sqsubseteq D$

$$\boxed{\mathbf{R}_{\sqsubseteq}\ \frac{B \sqsubseteq B}{B \sqsubseteq C} : B \sqsubseteq C \in \mathcal{O}}$$

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$\mathbf{R}_{\exists}\ \frac{E \sqsubseteq \exists R.C \quad C \sqsubseteq D}{E \sqsubseteq \exists R.D}$: $\exists R.D$ occurs in $\mathcal{O}$

$\mathbf{R}_{\sqcap}^{-}\ \frac{C \sqsubseteq D_1 \sqcap D_2}{C \sqsubseteq D_1 \quad C \sqsubseteq D_2}$

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$\mathcal{EL}$ Classification, Example

- $\mathsf{A} \sqsubseteq \mathsf{A}$
- $\mathsf{A} \sqsubseteq \exists \mathsf{R}.\mathsf{B}$
- $\mathsf{B} \sqsubseteq \mathsf{B}$
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These are all entailed atomic subsumptions

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Modern tableau-based reasoners have optimizations that reduce the number of subsumption tests and reuse results between the tests

Basic Implementation

A well-known procedure which can be used with any rules

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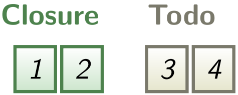

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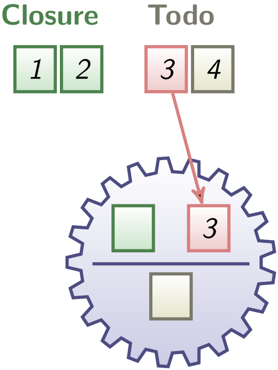

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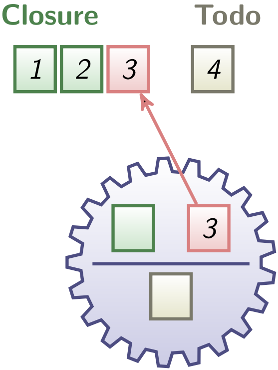

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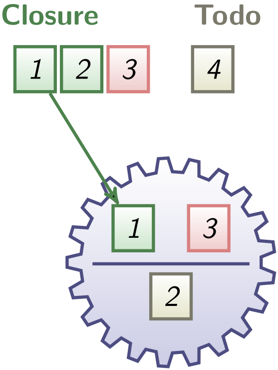

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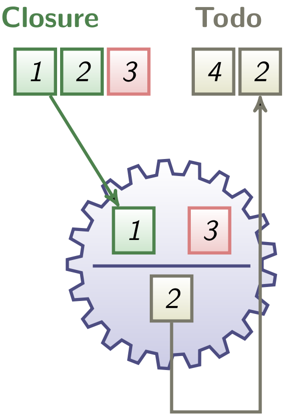

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Granularity of $\mathcal{EL}$ Reasoning

Subsumers for C and D can be computed (semi) independently

Limited interaction:

- Derived $C \sqsubseteq D$
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- distributed reasoning:
  C and D may come from different ontologies!

Problems with Tableau-based Reasoning

Consequence-based Reasoning for $\mathcal{EL}$

Practical Consequence-based $\mathcal{EL}$ Reasoning

What is Practical?

You want your reasoner to be:

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Sad truth: early implementations are almost always not practical

Evaluation Goals

Measure performance and scalability

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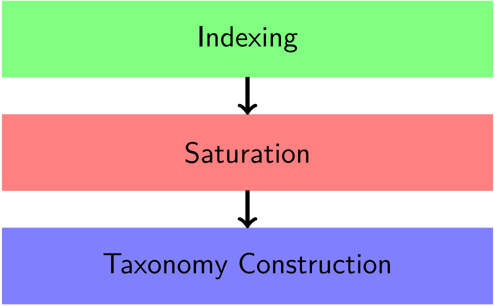
Without evaluation optimization is like shooting in the dark

Stages of $\mathcal{EL}$ Reasoning

Consider one-time classification



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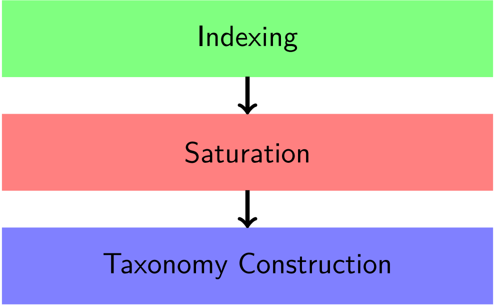

We focus on saturation and indexing

$\mathcal{EL}$ Saturation Statistics

Useful saturation metrics:

- Number of rule applications
- Time spent applying rules
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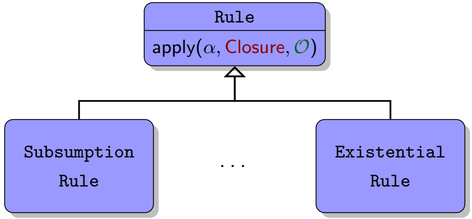
Collection mechanism must ensure:

- Can be turned on/off any time
- No need to change the rules
- Extensibility (w.r.t. new rules or new stats)

Statistics Collection in ELK

ELK rules represented as a class hierarchy



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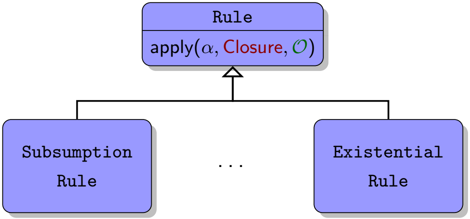

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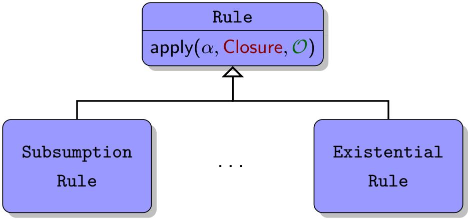

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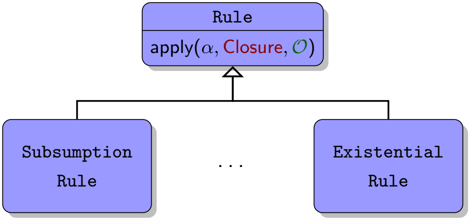

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2. `basic visit`

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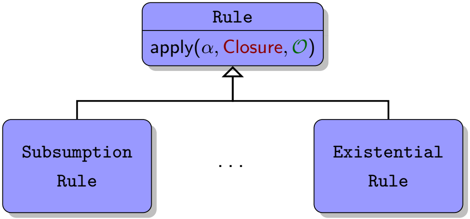

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Rule timing visitor:

1. `t = (current time)`
2. `basic visit`
3. `t = (current time) - t`

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We consider the first two issues

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Sometimes the same inference(s) can be derived more than once.

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Answer: No, duplicate inferences are not inserted into Closure

Duplicate Inferences

Sometimes the same inference(s) can be derived more than once.

Ontology $\mathcal{O}$:

1. $A \sqsubseteq B$
2. $A \sqsubseteq C$
3. $A \sqsubseteq D$
4. $D \sqsubseteq B \sqcap C$
5. $B \sqcap C \sqsubseteq E$

Derived superclasses of A:

- $A \sqsubseteq B$ by $\mathbf{R}_{\sqsubseteq}$
- $A \sqsubseteq C$ by $\mathbf{R}_{\sqsubseteq}$
- $A \sqsubseteq B \sqcap C$ by $\mathbf{R}_{\sqcap}^{+}$
- $\boxed{A \sqsubseteq B \text{ by } \mathbf{R}_{\sqcap}^{-}}$
- $\boxed{A \sqsubseteq C \text{ by } \mathbf{R}_{\sqcap}^{-}}$

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Lesson: $\mathbf{R}_{\sqcap}^{-}$ should not apply to conclusions of $\mathbf{R}_{\sqcap}^{+}$

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What reducing duplicate inferences mean in practice?

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Let $\mathsf{O}_\sqcap$, $\mathsf{O}_\exists$ be optimizations of $\mathsf{R}_\sqcap$, $\mathsf{R}_\exists$

	time	derived $C \sqsubseteq D$
SNOMED CT		
no optimization	26.31	47,435,318
with $\mathsf{O}_\sqcap$	25.48	41,770,050
with $\mathsf{O}_\exists$	19.75	28,438,072
with $\mathsf{O}_\sqcap$, $\mathsf{O}_\exists$	18.71	22,772,804
GALEN8		
no optimization	50.39	69,138,922
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Last Bit on Duplicate Inferences

Question: which subsumption rule generates more duplicates?

$\mathbf{R}_{\sqsubseteq} \frac{C \sqsubseteq D}{C \sqsubseteq E} : D \sqsubseteq E \in \mathcal{O}$ or $\mathbf{R}'_{\sqsubseteq} \frac{C \sqsubseteq D \quad D \sqsubseteq E}{C \sqsubseteq E}$

premise from Closure + side condition

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Inferences using $\mathbf{R}'_{\sqsubseteq}$:

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$\mathbf{R}_{\sqsubseteq}\frac{C \sqsubseteq D}{C \sqsubseteq E} : D \sqsubseteq E \in \mathcal{O}$ premise from Closure + side condition

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Redundant Inferences

Also not all unique inferences are essential for classification

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$\mathcal{O}$: $A \sqsubseteq \exists R.B$, $B \sqsubseteq C, D \sqsubseteq \exists R.C$

$\mathcal{O}$ entails no non asserted atomic subsumptions

But we derive $A \sqsubseteq \exists R.C$ using

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$\mathsf{SomeOrgan} \sqsubseteq \exists \mathsf{hasRole}.\mathsf{SomeRole}$

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Also true for $\mathbf{R}^+_\sqcap$: $\frac{C \sqsubseteq D_1 \quad C \sqsubseteq D_2}{C \sqsubseteq D_1 \sqcap D_2}$, $D_1 \sqcap D_2$ must occur on the left

Testing for Duplicate and Redundant Inferences

Some duplicate and redundant inferences are inevitable

How do you know if you have a problem?

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- # produced inferences $\gg$ # unique inferences $\rightsquigarrow$ potential problem with duplicates
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Definitions of $\gg$ and "much more" depend on ontology

Slow Rule Selection

Finding applicable rules is non-trivial

Input: Set of named classes CN
Result: Closure, a set containing all atomic subsumptions
Closure, Todo $\leftarrow \emptyset$;
for C $\in \mathrm{CN}$ **do**
  Todo.$add(\{\mathsf{C} \sqsubseteq \mathsf{C}, \mathsf{C} \sqsubseteq \top\})$

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for C $\in \mathrm{CN}$ **do**
    Todo.add($\{$C $\sqsubseteq$ C, C $\sqsubseteq \top\}$)
while ($\alpha \leftarrow$ Todo.$poll()$) $\neq$ $null$ **do**
    if $\alpha \notin$ Closure **then**
        Closure.$add(\alpha)$
        for R $\in$ *select-rules*(α, Closure) **do**
            Todo.add(conclusions of R)
return Closure

Slow Rule Selection

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```
Input: Set of named classes CN
Result: Closure, a set containing all atomic subsumptions
Closure, Todo ← ∅;
for C ∈ CN do
    Todo.add({C ⊑ C, C ⊑ ⊤})
while (α ← Todo.poll()) ≠ null do
    if α ∉ Closure then
        Closure.add(α)
        for R ∈ select-rules(α, Closure) do
            Todo.add(conclusions of R)
return Closure
```

In the worst-case select-rules(...) requires $O(|\text{Closure}|)$

Need efficient rule lookups

Rule Lookups

Assume that the initialization rules are applied eagerly

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Requires indexing of both Closure and $\mathcal{O}$

Rules as Functions

$$\mathbf{R}_{\overline{\eta}} : \gamma \text{ occurs in } \mathcal{O} \rightsquigarrow \begin{cases} \eta, & \gamma \text{ occurs in } \mathcal{O} \\ \emptyset & \text{otherwise} \end{cases}$$

Rules as Functions

$$\mathbf{R}\frac{-}{\eta} : \gamma \text{ occurs in } \mathcal{O} \rightsquigarrow \begin{cases} \eta, & \gamma \text{ occurs in } \mathcal{O} \\ \emptyset & \text{otherwise} \end{cases}$$

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Currying unifies the last two cases

$$\mathbf{R}(\alpha, \beta) = \mathbf{R'} : \alpha \mapsto \left(\beta \mapsto \begin{cases} \eta, & \gamma \text{ occurs in } \mathcal{O} \\ \emptyset & \text{otherwise} \end{cases} \right)$$

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Rules can be indexed as: $\alpha \mapsto (\beta \mapsto (\gamma \mapsto \eta))$

$\mathcal{EL}$ Rules as Functions

$\mathbf{R_0}$: $\{C \sqsubseteq C\}$

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$$\mathbf{R}_\sqsubseteq : C \sqsubseteq D \mapsto \begin{cases} C \sqsubseteq E, & D \sqsubseteq E \in \mathcal{O} \\ \emptyset & \text{otherwise} \end{cases}$$

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$\mathbf{R}^+_\sqcap$: $C \sqsubseteq D_1, C \sqsubseteq D_2 \mapsto \begin{cases} C \sqsubseteq D_1 \sqcap D_2, & D_1 \sqcap D_2 \text{ occurs in } \mathcal{O} \\ \emptyset & \text{otherwise} \end{cases}$

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$$\mathbf{R}_\top : \{C \sqsubseteq \top\}$$

$$\mathbf{R}_\sqcap^- : C \sqsubseteq D_1 \sqcap D_2 \mapsto \{C \sqsubseteq D_1, C \sqsubseteq D_2\}$$

$$\mathbf{R}_\sqsubseteq : C \sqsubseteq D \mapsto \begin{cases} C \sqsubseteq E, & D \sqsubseteq E \in \mathcal{O} \\ \emptyset & \text{otherwise} \end{cases}$$

$$\mathbf{R}_\sqcap^+ : C \sqsubseteq D_1, C \sqsubseteq D_2 \mapsto \begin{cases} C \sqsubseteq D_1 \sqcap D_2, & D_1 \sqcap D_2 \text{ occurs in } \mathcal{O} \\ \emptyset & \text{otherwise} \end{cases}$$

$$\mathbf{R}_\exists : E \sqsubseteq \exists \mathsf{R}.C, C \sqsubseteq D \mapsto \begin{cases} E \sqsubseteq \exists \mathsf{R}.D, & \exists \mathsf{R}.D \text{ occurs in } \mathcal{O} \\ \emptyset & \text{otherwise} \end{cases}$$

$\mathcal{EL}$ Rule Indexing

$$\mathbf{R}(\alpha, \beta) = \mathbf{R'} : \alpha \mapsto \left(\beta \mapsto \begin{cases} \eta & \gamma \text{ occurs in } \mathcal{O} \\ \emptyset & \text{otherwise} \end{cases} \right)$$

Implement $\alpha \mapsto (\beta \mapsto (\gamma \mapsto \eta))$ or $\alpha \mapsto (\gamma \mapsto \eta)$ for $\mathcal{EL}$

$\mathcal{EL}$ Rule Indexing

$$\mathbf{R}(\alpha, \beta) = \mathbf{R}' : \alpha \mapsto \left(\beta \mapsto \begin{cases} \eta & \gamma \text{ occurs in } \mathcal{O} \\ \emptyset & \text{otherwise} \end{cases} \right)$$

Implement $\alpha \mapsto (\beta \mapsto (\gamma \mapsto \eta))$ or $\alpha \mapsto (\gamma \mapsto \eta)$ for $\mathcal{EL}$

Trivial for $\mathbf{R_0}, \mathbf{R}_\top$, and $\mathbf{R}^-_\sqcap$ (no side conditions)

$\mathcal{EL}$ Rule Indexing

$$\mathbf{R}(\alpha, \beta) = \mathbf{R}' : \alpha \mapsto \left(\beta \mapsto \begin{cases} \eta & \gamma \text{ occurs in } \mathcal{O} \\ \emptyset & \text{otherwise} \end{cases}\right)$$

Implement $\alpha \mapsto (\beta \mapsto (\gamma \mapsto \eta))$ or $\alpha \mapsto (\gamma \mapsto \eta)$ for $\mathcal{EL}$

$\mathbf{R}_{\sqsubseteq}\frac{C \sqsubseteq D}{C \sqsubseteq E} : D \sqsubseteq E \in \mathcal{O}$

told-subsumers $(\alpha \mapsto \gamma)$: $D \mapsto \{E \mid D \sqsubseteq E \in \mathcal{O}\}$

$\mathcal{EL}$ Rule Indexing

$$\mathbf{R}(\alpha, \beta) = \mathbf{R}' : \alpha \mapsto \left(\beta \mapsto \begin{cases} \eta & \gamma \text{ occurs in } \mathcal{O} \\ \emptyset & \text{otherwise} \end{cases} \right)$$

Implement $\alpha \mapsto (\beta \mapsto (\gamma \mapsto \eta))$ or $\alpha \mapsto (\gamma \mapsto \eta)$ for $\mathcal{EL}$

$\mathbf{R}_{\sqsubseteq} \frac{C \sqsubseteq D}{C \sqsubseteq E} : D \sqsubseteq E \in \mathcal{O}$

told-subsumers $(\alpha \mapsto \gamma)$: $D \mapsto \{E \mid D \sqsubseteq E \in \mathcal{O}\}$

When processing $C \sqsubseteq D$ all $\{C \sqsubseteq E\}$ are derived with one look-up

This is rule grouping

Indexing $\mathbf{R}^{+}_{\sqcap}$

$\mathbf{R}^{+}_{\sqcap}\ \frac{C \sqsubseteq D_1 \quad C \sqsubseteq D_2}{C \sqsubseteq D_1 \sqcap D_2}$: $D_1 \sqcap D_2$ occurs in $\mathcal{O}$

$$\alpha = \mathsf{C} \sqsubseteq \mathsf{D}_1 \qquad \beta = \mathsf{C} \sqsubseteq \mathsf{D}_2$$

$$\gamma = \mathsf{D}_1 \sqcap \mathsf{D}_2 \text{ occurs in } \mathcal{O} \qquad \eta = \mathsf{C} \sqsubseteq \mathsf{D}_1 \sqcap \mathsf{D}_2$$

Indexing $\mathbf{R}_{\sqcap}^{+}$

$\mathbf{R}_{\sqcap}^{+}\ \frac{C \sqsubseteq D_1 \quad C \sqsubseteq D_2}{C \sqsubseteq D_1 \sqcap D_2}$: $D_1 \sqcap D_2$ occurs in $\mathcal{O}$

$$\alpha = \mathsf{C} \sqsubseteq \mathsf{D}_1 \qquad \beta = \mathsf{C} \sqsubseteq \mathsf{D}_2$$
$$\gamma = \mathsf{D}_1 \sqcap \mathsf{D}_2 \text{ occurs in } \mathcal{O} \qquad \eta = \mathsf{C} \sqsubseteq \mathsf{D}_1 \sqcap \mathsf{D}_2$$

subsumers $\quad \alpha \mapsto \beta \quad \mathsf{C} \mapsto \quad \{\mathsf{D}_2 \mid \mathsf{C} \sqsubseteq \mathsf{D}_2 \in \text{Closure}\}$

Indexing $\mathbf{R}_{\sqcap}^{+}$

$\mathbf{R}_{\sqcap}^{+}\ \frac{C \sqsubseteq D_1 \quad C \sqsubseteq D_2}{C \sqsubseteq D_1 \sqcap D_2}$: $D_1 \sqcap D_2$ occurs in $\mathcal{O}$

$\alpha = \mathsf{C} \sqsubseteq \mathsf{D}_1$ $\qquad \beta = \mathsf{C} \sqsubseteq \mathsf{D}_2$

$\gamma = \mathsf{D}_1 \sqcap \mathsf{D}_2$ occurs in $\mathcal{O}$ $\qquad \eta = \mathsf{C} \sqsubseteq \mathsf{D}_1 \sqcap \mathsf{D}_2$

subsumers	$\alpha \mapsto \beta$	$\mathsf{C} \mapsto$	$\{\mathsf{D}_2 \mid \mathsf{C} \sqsubseteq \mathsf{D}_2 \in \mathsf{Closure}\}$
conjunctions	$\beta \mapsto \gamma$	$\mathsf{D}_2 \mapsto$	$\{\mathsf{D}_1 \mapsto \mathsf{D}_1 \sqcap \mathsf{D}_2 \mid \mathsf{D}_1 \sqcap \mathsf{D}_2$ occurs in $\mathcal{O}\}$

Indexing $\mathbf{R}_\sqcap^+$

$\mathbf{R}_\sqcap^+ \frac{C \sqsubseteq D_1 \quad C \sqsubseteq D_2}{C \sqsubseteq D_1 \sqcap D_2} : D_1 \sqcap D_2$ occurs in $\mathcal{O}$

$$\alpha = \mathsf{C} \sqsubseteq \mathsf{D}_1 \qquad \beta = \mathsf{C} \sqsubseteq \mathsf{D}_2$$
$$\gamma = \mathsf{D}_1 \sqcap \mathsf{D}_2 \text{ occurs in } \mathcal{O} \qquad \eta = \mathsf{C} \sqsubseteq \mathsf{D}_1 \sqcap \mathsf{D}_2$$

subsumers	$\alpha \mapsto \beta$	$\mathsf{C} \mapsto$	$\{\mathsf{D}_2 \mid \mathsf{C} \sqsubseteq \mathsf{D}_2 \in \text{Closure}\}$
conjunctions	$\beta \mapsto \gamma$	$\mathsf{D}_2 \mapsto$	$\{\mathsf{D}_1 \mapsto \mathsf{D}_1 \sqcap \mathsf{D}_2 \mid \mathsf{D}_1 \sqcap \mathsf{D}_2$ occurs in $\mathcal{O}\}$
Result:		$\eta \in$	$\{\mathsf{C} \sqsubseteq \mathsf{D}_1 \sqcap \mathsf{D}_2 \mid \mathsf{D}_2 \in \text{subsumers}(\mathsf{C}), \mathsf{D}_1 \in \text{conjunctions}(\mathsf{D}_2)\}$

Selecting and Applying $\mathbf{R}^+_\sqcap$, Example

$\mathbf{R}^+_\sqcap \frac{C \sqsubseteq D_1 \quad C \sqsubseteq D_2}{C \sqsubseteq D_1 \sqcap D_2}$: $D_1 \sqcap D_2$ occurs in $\mathcal{O}$

Ontology $\mathcal{O}$

1. $A \sqsubseteq B$
2. $B \sqsubseteq C$
3. $A \sqsubseteq D$
4. $D \sqcap C \sqsubseteq E$

Inferences

Selecting and Applying $\mathbf{R}_{\sqcap}^{+}$, Example

$\mathbf{R}_{\sqcap}^{+} \frac{C \sqsubseteq D_1 \quad C \sqsubseteq D_2}{C \sqsubseteq D_1 \sqcap D_2}$: $D_1 \sqcap D_2$ occurs in $\mathcal{O}$

Ontology $\mathcal{O}$

1. $A \sqsubseteq B$
2. $B \sqsubseteq C$
3. $A \sqsubseteq D$
4. $D \sqcap C \sqsubseteq E$

Inferences

- $A \sqsubseteq$ B by $\mathbf{R}_{\sqsubseteq}$

Selecting and Applying $\mathbf{R}_{\sqcap}^{+}$, Example

$\mathbf{R}_{\sqcap}^{+}\ \frac{C \sqsubseteq D_1 \quad C \sqsubseteq D_2}{C \sqsubseteq D_1 \sqcap D_2}$: $D_1 \sqcap D_2$ occurs in $\mathcal{O}$

Ontology $\mathcal{O}$

1. $A \sqsubseteq B$
2. $B \sqsubseteq C$
3. $A \sqsubseteq D$
4. $D \sqcap C \sqsubseteq E$

Inferences

- $A \sqsubseteq$ B by $\mathbf{R}_{\sqsubseteq}$
- $A \sqsubseteq$ D by $\mathbf{R}_{\sqsubseteq}$
  $\{C \mapsto C \sqcap D\} \in$ conjunctions(D)
  but $C \notin$ subsumers(A), $\mathbf{R}_{\sqcap}^{+}$ doesn't apply

Selecting and Applying $\mathbf{R}^{+}_{\sqcap}$, Example

$\mathbf{R}^{+}_{\sqcap}\ \frac{C \sqsubseteq D_1 \quad C \sqsubseteq D_2}{C \sqsubseteq D_1 \sqcap D_2}$: $D_1 \sqcap D_2$ occurs in $\mathcal{O}$

Ontology $\mathcal{O}$

1. $A \sqsubseteq B$
2. $B \sqsubseteq C$
3. $A \sqsubseteq D$
4. $D \sqcap C \sqsubseteq E$

Inferences

- $A \sqsubseteq$ B by $\mathbf{R}_{\sqsubseteq}$
- $A \sqsubseteq$ D by $\mathbf{R}_{\sqsubseteq}$
  $\{C \mapsto C \sqcap D\} \in$ conjunctions(D)
  but $C \notin$ subsumers(A), $\mathbf{R}^{+}_{\sqcap}$ doesn't apply
- $A \sqsubseteq$ C by $\mathbf{R}_{\sqsubseteq}$
  $\{D \mapsto C \sqcap D\} \in$ conjunctions(C)
  and $D \in$ subsumers(A), sooo

Selecting and Applying $\mathbf{R}^{+}_{\sqcap}$, Example

$\mathbf{R}^{+}_{\sqcap}\ \frac{C \sqsubseteq D_1 \quad C \sqsubseteq D_2}{C \sqsubseteq D_1 \sqcap D_2}$: $D_1 \sqcap D_2$ occurs in $\mathcal{O}$

Ontology $\mathcal{O}$

1. $A \sqsubseteq B$
2. $B \sqsubseteq C$
3. $A \sqsubseteq D$
4. $D \sqcap C \sqsubseteq E$

Inferences

- $A \sqsubseteq$ B by $\mathbf{R}_{\sqsubseteq}$
- $A \sqsubseteq$ D by $\mathbf{R}_{\sqsubseteq}$
  $\{C \mapsto C \sqcap D\} \in$ conjunctions(D)
  but $C \notin$ subsumers(A), $\mathbf{R}^{+}_{\sqcap}$ doesn't apply
- $A \sqsubseteq$ C by $\mathbf{R}_{\sqsubseteq}$
  $\{D \mapsto C \sqcap D\} \in$ conjunctions(C)
  and $D \in$ subsumers(A), sooo
- $A \sqsubseteq C \sqcap D$ by $\mathbf{R}^{+}_{\sqcap}$

Selecting and Applying $\mathbf{R}_{\sqcap}^{+}$, Example

$\mathbf{R}_{\sqcap}^{+}\ \frac{C \sqsubseteq D_1 \quad C \sqsubseteq D_2}{C \sqsubseteq D_1 \sqcap D_2}$: $D_1 \sqcap D_2$ occurs in $\mathcal{O}$

Ontology $\mathcal{O}$

1. $A \sqsubseteq B$
2. $B \sqsubseteq C$
3. $A \sqsubseteq D$
4. $D \sqcap C \sqsubseteq E$

Inferences

- $A \sqsubseteq$ B by $\mathbf{R}_{\sqsubseteq}$
- $A \sqsubseteq$ D by $\mathbf{R}_{\sqsubseteq}$
  $\{C \mapsto C \sqcap D\} \in$ conjunctions(D)
  but $C \notin$ subsumers(A), $\mathbf{R}_{\sqcap}^{+}$ doesn't apply
- $A \sqsubseteq$ C by $\mathbf{R}_{\sqsubseteq}$
  $\{D \mapsto C \sqcap D\} \in$ conjunctions(C)
  and $D \in$ subsumers(A), sooo
- $A \sqsubseteq C \sqcap D$ by $\mathbf{R}_{\sqcap}^{+}$
- $A \sqsubseteq E$ by $\mathbf{R}_{\sqsubseteq}$

Existential-based Indexing $\mathbf{R}_\exists$

$\mathbf{R}_\exists \frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq C}{A \sqsubseteq \exists R.C}$: $\exists R.C$ occurs in $\mathcal{O}$

Processing $E \sqsubseteq \exists R.C$

$$\alpha = \mathsf{A} \sqsubseteq \exists \mathsf{R}.\mathsf{B} \qquad \beta = \mathsf{B} \sqsubseteq \mathsf{C}$$
$$\gamma = \exists \mathsf{R}.\mathsf{C} \text{ occurs in } \mathcal{O} \qquad \eta = \mathsf{A} \sqsubseteq \exists \mathsf{R}.\mathsf{C}$$

Existential-based Indexing $\mathbf{R}_\exists$

$\mathbf{R}_\exists \frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq C}{A \sqsubseteq \exists R.C}$: $\exists R.C$ occurs in $\mathcal{O}$

Processing $E \sqsubseteq \exists R.C$

$$\alpha = \mathsf{A} \sqsubseteq \exists \mathsf{R}.\mathsf{B} \qquad \beta = \mathsf{B} \sqsubseteq \mathsf{C}$$

$$\gamma = \exists \mathsf{R}.\mathsf{C} \text{ occurs in } \mathcal{O} \qquad \eta = \mathsf{A} \sqsubseteq \exists \mathsf{R}.\mathsf{C}$$

subsumptions $\alpha \mapsto \beta \quad \mathsf{B} \mapsto \{\mathsf{C} \mid \mathsf{B} \sqsubseteq \mathsf{C} \in \text{Closure}\}$

Existential-based Indexing $\mathbf{R}_\exists$

$\mathbf{R}_\exists \frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq C}{A \sqsubseteq \exists R.C}$: $\exists R.C$ occurs in $\mathcal{O}$

Processing $E \sqsubseteq \exists R.C$

$$\alpha = \mathsf{A} \sqsubseteq \exists \mathsf{R}.\mathsf{B} \qquad \beta = \mathsf{B} \sqsubseteq \mathsf{C}$$
$$\gamma = \exists \mathsf{R}.\mathsf{C} \text{ occurs in } \mathcal{O} \qquad \eta = \mathsf{A} \sqsubseteq \exists \mathsf{R}.\mathsf{C}$$

subsumptions	$\alpha \mapsto \beta$	$\mathsf{B} \mapsto$	$\{\mathsf{C} \mid \mathsf{B} \sqsubseteq \mathsf{C} \in \text{Closure}\}$
existentials	$\beta \mapsto \gamma$	$\mathsf{C} \mapsto$	$\{\mathsf{R} \mid \exists \mathsf{R}.\mathsf{C} \text{ occurs in } \mathcal{O}\}$

Existential-based Indexing $\mathbf{R}_\exists$

$\mathbf{R}_\exists \frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq C}{A \sqsubseteq \exists R.C}$: $\exists R.C$ occurs in $\mathcal{O}$

Processing $E \sqsubseteq \exists R.C$

$$\alpha = \mathsf{A} \sqsubseteq \exists \mathsf{R}.\mathsf{B} \qquad \beta = \mathsf{B} \sqsubseteq \mathsf{C}$$
$$\gamma = \exists \mathsf{R}.\mathsf{C} \text{ occurs in } \mathcal{O} \qquad \eta = \mathsf{A} \sqsubseteq \exists \mathsf{R}.\mathsf{C}$$

subsumptions	$\alpha \mapsto \beta$	$\mathsf{B} \mapsto$	$\{\mathsf{C} \mid \mathsf{B} \sqsubseteq \mathsf{C} \in$ Closure$\}$
existentials	$\beta \mapsto \gamma$	$\mathsf{C} \mapsto$	$\{\mathsf{R} \mid \exists \mathsf{R}.\mathsf{C}$ occurs in $\mathcal{O}\}$
	Result	$\eta \in$	$\{\mathsf{A} \sqsubseteq \exists \mathsf{R}.\mathsf{C} \mid \mathsf{C} \in$ subsumptions(B), $\mathsf{R} \in$ existentials$(\mathsf{C})\}$

Subsumption-based Indexing $\mathbf{R}_\exists$

$\mathbf{R}_\exists \frac{B \sqsubseteq C \quad A \sqsubseteq \exists R.B}{A \sqsubseteq \exists R.C}$: $\exists R.C$ occurs in $\mathcal{O}$

Processing $B \sqsubseteq C$

$$\alpha = B \sqsubseteq C \qquad \beta = A \sqsubseteq \exists R.B$$
$$\gamma = \exists R.C \text{ occurs in } \mathcal{O} \qquad \eta = A \sqsubseteq \exists R.C$$

Subsumption-based Indexing $\mathbf{R}_\exists$

$\mathbf{R}_\exists \frac{B \sqsubseteq C \quad A \sqsubseteq \exists R.B}{A \sqsubseteq \exists R.C}$: $\exists R.C$ occurs in $\mathcal{O}$

Processing $B \sqsubseteq C$

$$\begin{aligned} \alpha &= B \sqsubseteq C & \beta &= A \sqsubseteq \exists R.B \\ \gamma &= \exists R.C \text{ occurs in } \mathcal{O} & \eta &= A \sqsubseteq \exists R.C \end{aligned}$$

backward-links $\quad \alpha \mapsto \beta \quad B \mapsto \quad \{R \mapsto A \mid A \sqsubseteq \exists R.B \in \text{Closure}\}$

Subsumption-based Indexing $\mathbf{R}_\exists$

$\mathbf{R}_\exists \frac{B \sqsubseteq C \quad A \sqsubseteq \exists R.B}{A \sqsubseteq \exists R.C}$: $\exists R.C$ occurs in $\mathcal{O}$

Processing $B \sqsubseteq C$

$$\begin{array}{ll} \alpha = B \sqsubseteq C & \beta = A \sqsubseteq \exists R.B \\ \gamma = \exists R.C \text{ occurs in } \mathcal{O} & \eta = A \sqsubseteq \exists R.C \end{array}$$

backward-links	$\alpha \mapsto \beta$	$B \mapsto$	$\{R \mapsto A \mid A \sqsubseteq \exists R.B \in \text{Closure}\}$
existentials	$\beta \mapsto \gamma$	$C \mapsto$	$\{R \mid \exists R.C \text{ occurs in } \mathcal{O}\}$

Subsumption-based Indexing $\mathbf{R}_\exists$

$\mathbf{R}_\exists \frac{B \sqsubseteq C \quad A \sqsubseteq \exists R.B}{A \sqsubseteq \exists R.C}$: $\exists R.C$ occurs in $\mathcal{O}$

Processing $B \sqsubseteq C$

$$\alpha = B \sqsubseteq C \qquad \beta = A \sqsubseteq \exists R.B$$
$$\gamma = \exists R.C \text{ occurs in } \mathcal{O} \qquad \eta = A \sqsubseteq \exists R.C$$

backward-links	$\alpha \mapsto \beta$	$B \mapsto$	$\{R \mapsto A \mid A \sqsubseteq \exists R.B \in \text{Closure}\}$
existentials	$\beta \mapsto \gamma$	$C \mapsto$	$\{R \mid \exists R.C \text{ occurs in } \mathcal{O}\}$
	Result	$\eta \in$	$\{A \sqsubseteq \exists R.C \mid A \in \text{backward-links}(B, R)\}, R \in \text{existentials}(C)$

Selecting and Applying $\mathbf{R}_\exists$, Example

$\mathbf{R}_\exists \frac{E \sqsubseteq \exists \mathsf{R}.C \quad C \sqsubseteq D}{E \sqsubseteq \exists \mathsf{R}.D}$: $E \sqsubseteq \exists \mathsf{R}.D$ occurs in $\mathcal{O}$

Ontology $\mathcal{O}$: $\mathsf{A} \sqsubseteq \mathsf{B}$ $\quad \mathsf{B} \sqsubseteq \exists \mathsf{R}.(\mathsf{C} \sqcap \mathsf{D})$ $\quad \mathsf{C} \sqsubseteq \mathsf{E}$ $\quad \exists \mathsf{R}.\mathsf{E} \sqsubseteq \mathsf{X}$

Closure:

Selecting and Applying $\mathbf{R}_\exists$, Example

$\mathbf{R}_\exists \frac{E \sqsubseteq \exists \mathsf{R}.C \quad C \sqsubseteq D}{E \sqsubseteq \exists \mathsf{R}.D}$: $E \sqsubseteq \exists \mathsf{R}.D$ occurs in $\mathcal{O}$

Ontology $\mathcal{O}$: $\mathsf{A} \sqsubseteq \mathsf{B} \quad \mathsf{B} \sqsubseteq \exists \mathsf{R}.(\mathsf{C} \sqcap \mathsf{D}) \quad \mathsf{C} \sqsubseteq \mathsf{E} \quad \exists \mathsf{R}.\mathsf{E} \sqsubseteq \mathsf{X}$

Closure:

$\mathsf{A} \sqsubseteq \mathsf{B}$ by $\mathbf{R}_\sqsubseteq$

Selecting and Applying $\mathbf{R}_{\exists}$, Example

$\mathbf{R}_{\exists}\ \frac{E \sqsubseteq \exists \mathsf{R}.C \quad C \sqsubseteq D}{E \sqsubseteq \exists \mathsf{R}.D}$: $E \sqsubseteq \exists \mathsf{R}.D$ occurs in $\mathcal{O}$

Ontology $\mathcal{O}$: $\mathsf{A} \sqsubseteq \mathsf{B}$ $\quad \mathsf{B} \sqsubseteq \exists \mathsf{R}.(\mathsf{C} \sqcap \mathsf{D})$ $\quad \mathsf{C} \sqsubseteq \mathsf{E}$ $\quad \exists \mathsf{R}.\mathsf{E} \sqsubseteq \mathsf{X}$

Closure:

$\mathsf{A} \sqsubseteq \mathsf{B}$ by $\mathbf{R}_{\sqsubseteq}$

$\mathsf{A} \sqsubseteq \exists \mathsf{R}.(\mathsf{C} \sqcap \mathsf{D})$ by $\mathbf{R}_{\sqsubseteq}$ using $\mathsf{B} \sqsubseteq \exists \mathsf{R}.(\mathsf{C} \sqcap \mathsf{D})$

Selecting and Applying $\mathbf{R}_\exists$, Example

$\mathbf{R}_\exists \frac{E \sqsubseteq \exists R.C \quad C \sqsubseteq D}{E \sqsubseteq \exists R.D}$: $E \sqsubseteq \exists R.D$ occurs in $\mathcal{O}$

Ontology $\mathcal{O}$: $A \sqsubseteq B$ $\quad B \sqsubseteq \exists R.(C \sqcap D)$ $\quad C \sqsubseteq E$ $\quad \exists R.E \sqsubseteq X$

Closure:

$A \sqsubseteq B$	by	$\mathbf{R}_\sqsubseteq$
$A \sqsubseteq \exists R.(C \sqcap D)$	by	$\mathbf{R}_\sqsubseteq$ using $B \sqsubseteq \exists R.(C \sqcap D)$
		A added to backward-links$(C \sqcap D, R)$

Selecting and Applying $\mathbf{R}_\exists$, Example

$\mathbf{R}_\exists \frac{E \sqsubseteq \exists \mathsf{R}.C \quad C \sqsubseteq D}{E \sqsubseteq \exists \mathsf{R}.D}$: $E \sqsubseteq \exists \mathsf{R}.D$ occurs in $\mathcal{O}$

Ontology $\mathcal{O}$: $\mathsf{A} \sqsubseteq \mathsf{B}$ $\quad \mathsf{B} \sqsubseteq \exists \mathsf{R}.(\mathsf{C} \sqcap \mathsf{D})$ $\quad \mathsf{C} \sqsubseteq \mathsf{E}$ $\quad \exists \mathsf{R}.\mathsf{E} \sqsubseteq \mathsf{X}$

Closure:

$\mathsf{A} \sqsubseteq \mathsf{B}$	by	$\mathbf{R}_\sqsubseteq$
$\mathsf{A} \sqsubseteq \exists \mathsf{R}.(\mathsf{C} \sqcap \mathsf{D})$	by	$\mathbf{R}_\sqsubseteq$ using $\mathsf{B} \sqsubseteq \exists \mathsf{R}.(\mathsf{C} \sqcap \mathsf{D})$
		A added to backward-links$(\mathsf{C} \sqcap \mathsf{D}, \mathsf{R})$
$\mathsf{C} \sqcap \mathsf{D} \sqsubseteq \mathsf{C} \sqcap \mathsf{D}$	by	$\mathbf{R_0}$

Selecting and Applying $\mathbf{R}_\exists$, Example

$\mathbf{R}_\exists \frac{E \sqsubseteq \exists R.C \quad C \sqsubseteq D}{E \sqsubseteq \exists R.D}$: $E \sqsubseteq \exists R.D$ occurs in $\mathcal{O}$

Ontology $\mathcal{O}$: $A \sqsubseteq B \quad B \sqsubseteq \exists R.(C \sqcap D) \quad C \sqsubseteq E \quad \exists R.E \sqsubseteq X$

Closure:

$A \sqsubseteq B$	by	$\mathbf{R}_\sqsubseteq$
$A \sqsubseteq \exists R.(C \sqcap D)$	by	$\mathbf{R}_\sqsubseteq$ using $B \sqsubseteq \exists R.(C \sqcap D)$
		A added to backward-links$(C \sqcap D, R)$
$C \sqcap D \sqsubseteq C \sqcap D$	by	$\mathbf{R_0}$
$C \sqcap D \sqsubseteq C$	by	$\mathbf{R}_\sqcap^-$

Selecting and Applying $\mathbf{R}_\exists$, Example

$\mathbf{R}_\exists \frac{E \sqsubseteq \exists R.C \quad C \sqsubseteq D}{E \sqsubseteq \exists R.D}$: $E \sqsubseteq \exists R.D$ occurs in $\mathcal{O}$

Ontology $\mathcal{O}$: $A \sqsubseteq B$ $\quad B \sqsubseteq \exists R.(C \sqcap D)$ $\quad C \sqsubseteq E$ $\quad \exists R.E \sqsubseteq X$

Closure:

$A \sqsubseteq B$ by $\mathbf{R}_\sqsubseteq$

$A \sqsubseteq \exists R.(C \sqcap D)$ by $\mathbf{R}_\sqsubseteq$ using $B \sqsubseteq \exists R.(C \sqcap D)$
A added to backward-links$(C \sqcap D, R)$

$C \sqcap D \sqsubseteq C \sqcap D$ by $\mathbf{R_0}$

$C \sqcap D \sqsubseteq C$ by $\mathbf{R}^-_\sqcap$

$A \sqsubseteq \exists R.C$???

Selecting and Applying $\mathbf{R}_\exists$, Example

$\mathbf{R}_\exists \frac{E \sqsubseteq \exists R.C \quad C \sqsubseteq D}{E \sqsubseteq \exists R.D}$: $E \sqsubseteq \exists R.D$ occurs in $\mathcal{O}$

Ontology $\mathcal{O}$: $A \sqsubseteq B$ $\quad B \sqsubseteq \exists R.(C \sqcap D)$ $\quad C \sqsubseteq E$ $\quad \exists R.E \sqsubseteq X$

Closure:

$A \sqsubseteq B$	by	$\mathbf{R}_\sqsubseteq$
$A \sqsubseteq \exists R.(C \sqcap D)$	by	$\mathbf{R}_\sqsubseteq$ using $B \sqsubseteq \exists R.(C \sqcap D)$
		A added to backward-links$(C \sqcap D, R)$
$C \sqcap D \sqsubseteq C \sqcap D$	by	$\mathbf{R}_0$
$C \sqcap D \sqsubseteq C$	by	$\mathbf{R}_\sqcap^-$
$A \sqsubseteq \exists R.C$???	No	$R \notin$ existentials(C)

Selecting and Applying $\mathbf{R}_\exists$, Example

$\mathbf{R}_\exists \frac{E \sqsubseteq \exists \mathsf{R}.C \quad C \sqsubseteq D}{E \sqsubseteq \exists \mathsf{R}.D}$: $E \sqsubseteq \exists \mathsf{R}.D$ occurs in $\mathcal{O}$

Ontology $\mathcal{O}$: $\mathsf{A} \sqsubseteq \mathsf{B}$ $\quad \mathsf{B} \sqsubseteq \exists \mathsf{R}.(\mathsf{C} \sqcap \mathsf{D})$ $\quad \mathsf{C} \sqsubseteq \mathsf{E}$ $\quad \exists \mathsf{R}.\mathsf{E} \sqsubseteq \mathsf{X}$

Closure:

$\mathsf{A} \sqsubseteq \mathsf{B}$	by	$\mathbf{R}_\sqsubseteq$
$\mathsf{A} \sqsubseteq \exists \mathsf{R}.(\mathsf{C} \sqcap \mathsf{D})$	by	$\mathbf{R}_\sqsubseteq$ using $\mathsf{B} \sqsubseteq \exists \mathsf{R}.(\mathsf{C} \sqcap \mathsf{D})$
		A added to backward-links$(\mathsf{C} \sqcap \mathsf{D}, \mathsf{R})$
$\mathsf{C} \sqcap \mathsf{D} \sqsubseteq \mathsf{C} \sqcap \mathsf{D}$	by	$\mathbf{R_0}$
$\mathsf{C} \sqcap \mathsf{D} \sqsubseteq \mathsf{C}$	by	$\mathbf{R}_\sqcap^-$
$\mathsf{A} \sqsubseteq \exists \mathsf{R}.\mathsf{C}$???	No	$\mathsf{R} \notin$ existentials(C)
$\mathsf{C} \sqcap \mathsf{D} \sqsubseteq \mathsf{E}$	by	$\mathbf{R}_\sqsubseteq$

Selecting and Applying $\mathbf{R}_\exists$, Example

$\mathbf{R}_\exists \frac{E \sqsubseteq \exists R.C \quad C \sqsubseteq D}{E \sqsubseteq \exists R.D}$: $E \sqsubseteq \exists R.D$ occurs in $\mathcal{O}$

Ontology $\mathcal{O}$: $A \sqsubseteq B \quad B \sqsubseteq \exists R.(C \sqcap D) \quad C \sqsubseteq E \quad \exists R.E \sqsubseteq X$

Closure:

$A \sqsubseteq B$	by	$\mathbf{R}_\sqsubseteq$
$A \sqsubseteq \exists R.(C \sqcap D)$	by	$\mathbf{R}_\sqsubseteq$ using $B \sqsubseteq \exists R.(C \sqcap D)$
		A added to backward-links$(C \sqcap D, R)$
$C \sqcap D \sqsubseteq C \sqcap D$	by	$\mathbf{R_0}$
$C \sqcap D \sqsubseteq C$	by	$\mathbf{R}_\sqcap^-$
$A \sqsubseteq \exists R.C$???	No	$R \notin$ existentials(C)
$C \sqcap D \sqsubseteq E$	by	$\mathbf{R}_\sqsubseteq$
$A \sqsubseteq \exists R.E$???		

Selecting and Applying $\mathbf{R}_\exists$, Example

$\mathbf{R}_\exists \frac{E \sqsubseteq \exists R.C \quad C \sqsubseteq D}{E \sqsubseteq \exists R.D}$: $E \sqsubseteq \exists R.D$ occurs in $\mathcal{O}$

Ontology $\mathcal{O}$: $A \sqsubseteq B$ $\quad B \sqsubseteq \exists R.(C \sqcap D)$ $\quad C \sqsubseteq E$ $\quad \exists R.E \sqsubseteq X$

Closure:

$A \sqsubseteq B$	by	$\mathbf{R}_\sqsubseteq$
$A \sqsubseteq \exists R.(C \sqcap D)$	by	$\mathbf{R}_\sqsubseteq$ using $B \sqsubseteq \exists R.(C \sqcap D)$
		A added to backward-links$(C \sqcap D, R)$
$C \sqcap D \sqsubseteq C \sqcap D$	by	$\mathbf{R}_0$
$C \sqcap D \sqsubseteq C$	by	$\mathbf{R}_\sqcap^-$
$A \sqsubseteq \exists R.C$???	No	$R \notin$ existentials(C)
$C \sqcap D \sqsubseteq E$	by	$\mathbf{R}_\sqsubseteq$
$A \sqsubseteq \exists R.E$???	Yes	$R \in$ existentials(E),
		$A \in$ backward-links$(C \sqcap D, R)$

Selecting and Applying $\mathbf{R}_\exists$, Example

$\mathbf{R}_\exists \frac{E \sqsubseteq \exists \mathsf{R}.C \quad C \sqsubseteq D}{E \sqsubseteq \exists \mathsf{R}.D}$: $E \sqsubseteq \exists \mathsf{R}.D$ occurs in $\mathcal{O}$

Ontology $\mathcal{O}$: $\mathsf{A} \sqsubseteq \mathsf{B}$ $\quad \mathsf{B} \sqsubseteq \exists \mathsf{R}.(\mathsf{C} \sqcap \mathsf{D})$ $\quad \mathsf{C} \sqsubseteq \mathsf{E}$ $\quad \exists \mathsf{R}.\mathsf{E} \sqsubseteq \mathsf{X}$

Closure:

$\mathsf{A} \sqsubseteq \mathsf{B}$	by	$\mathbf{R}_\sqsubseteq$
$\mathsf{A} \sqsubseteq \exists \mathsf{R}.(\mathsf{C} \sqcap \mathsf{D})$	by	$\mathbf{R}_\sqsubseteq$ using $\mathsf{B} \sqsubseteq \exists \mathsf{R}.(\mathsf{C} \sqcap \mathsf{D})$
		A added to backward-links$(\mathsf{C} \sqcap \mathsf{D}, \mathsf{R})$
$\mathsf{C} \sqcap \mathsf{D} \sqsubseteq \mathsf{C} \sqcap \mathsf{D}$	by	$\mathbf{R_0}$
$\mathsf{C} \sqcap \mathsf{D} \sqsubseteq \mathsf{C}$	by	$\mathbf{R}_\sqcap^-$
$\mathsf{A} \sqsubseteq \exists \mathsf{R}.\mathsf{C}$???	No	$\mathsf{R} \notin$ existentials(C)
$\mathsf{C} \sqcap \mathsf{D} \sqsubseteq \mathsf{E}$	by	$\mathbf{R}_\sqsubseteq$
$\mathsf{A} \sqsubseteq \exists \mathsf{R}.\mathsf{E}$???	Yes	$\mathsf{R} \in$ existentials(E),
		$\mathsf{A} \in$ backward-links$(\mathsf{C} \sqcap \mathsf{D}, \mathsf{R})$
$\mathsf{A} \sqsubseteq \mathsf{X}$	by	$\mathbf{R}_\sqsubseteq$ using $\exists \mathsf{R}.\mathsf{E} \sqsubseteq \mathsf{X}$

So Is It Practical?

Short answer: yes

- $<$10s to classify SNOMED CT ($>$200s for tableau)
- 10s for GALEN (∞ for tableau)

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There is still room for improvement

- around 23,000,000 inferences made to classify SNOMED CT
- . . . but only 300,000 concepts, few subsumers per each
- even more economical classification might be possible

Take Home Message

Consequence-based reasoning is different from tableau reasoning

- Uses natural deduction (rules) instead of building a model
- Never tries to test a subsumption that doesn't hold

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- Horn-$\mathcal{SHIQ}$ (the language of Full GALEN)
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Tractable does not necessarily mean pactical!

- Even $O(n^2)$ is fatal if it is typical case
- Converse: intractable does not always mean impractical