Tight Bounds for HTN Planning

Ron Alford
ASEE/NRL Postdoctoral Fellow
Washington, DC, USA
ronald.alford.ctr@nrl.navy.mil

Pascal Bercher
Ulm University
Ulm, Germany
pascal.bercher@uni-ulm.de

David W. Aha
U.S. Naval Research Laboratory
Washington, DC, USA
david.aha@nrl.navy.mil

Abstract

Although HTN planning is in general undecidable, there are many syntactically identifiable sub-classes of HTN problems that can be decided. For these sub-classes, the decision procedures provide upper complexity bounds. Lower bounds were often not investigated in more detail, however. We generalize a propositional HTN formalization to one that is based upon a function-free first-order logic and provide tight upper and lower complexity results along three axes: whether variables are allowed in operator and method schemas, whether the initial task and methods must be totally ordered, and where recursion is allowed (arbitrary recursion, tail-recursion, and acyclic problems). Our findings have practical implications, both for the reuse of classical planning techniques for HTN planning, and for the design of efficient HTN algorithms.

1 Introduction

Hierarchical Task Network (HTN) planning (Ghallab, Nau, and Traverso 2004) is an automated planning formalism concerned with the completion of tasks (activities or processes). Tasks in HTN planning are either primitive, corresponding to an action that can be taken, or non-primitive. HTN problems have a set of methods that act as recipes on non-primitive tasks, decomposing them into a further set of subtasks for which to plan. A non-primitive task may even decompose into itself, either directly via a method, or indirectly via a sequence of decompositions.

This recursive structure, when combined with partially-ordered tasks, is powerful enough to encode semi-decidable problems (Erol, Hendler, and Nau 1994). However, any one of numerous restrictions on HTN planning are enough to make HTN planning decidable (Erol, Hendler, and Nau 1994; Geier and Bercher 2011; Alford et al. 2012; 2014; Höller et al. 2014). In this paper, we explore complexity and expressiveness that results from the interplay between syntactic restrictions on decomposition methods. Specifically, we examine all combinations of three characteristics: whether...

- there is no recursion, tail-recursion, or arbitrary recursion,
- the methods and initial task network are totally ordered,
- methods and operators are ground, and whether constants may be mixed with variables in method definitions.

We find that even without variables, the restricted classes of HTN planning range in expressivity from PSPACE-complete to EXPSPACE-complete. Just as in classical planning, extending these problems to include variables in the method and operator schemas gives an exponential bump in complexity. However, we identify a new (yet commonly met) restriction on HTN structure, that of constant-free methods (CFM), which often mitigates the computational impact of planning with variables. Table 1 provides a summary of our results. Erol, Hendler, and Nau (1994) provide the semi-decidability result. The rest constitute new lower bounds, new upper bounds, or both.

2 Lifted HTN Planning

In this section, we present a lifted version of the HTN planning formalism of Geier and Bercher (2011), which we extend to introduce variables.

Table 1: Complexity classes for HTN planning (only completeness results). The undecidability result (“semi-decidable”) is from Erol, Hendler, and Nau (1994).

<table>
<thead>
<tr>
<th>Hierarchy</th>
<th>Ordering</th>
<th>Variables</th>
<th>Complexity</th>
<th>Theorem</th>
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<tbody>
<tr>
<td>no recursion (acyclic)</td>
<td>total</td>
<td>no</td>
<td>PSPACE</td>
<td>4.1</td>
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<td>CFM</td>
<td>NEXPTIME</td>
<td>4.2</td>
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<tr>
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<td>EXPSPACE</td>
<td>4.1</td>
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<tr>
<td>partial</td>
<td>no</td>
<td>no</td>
<td>NEXPTIME</td>
<td>6.1</td>
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<tr>
<td>partial</td>
<td>CFM</td>
<td>NEXPTIME</td>
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<tr>
<td>partial</td>
<td>yes</td>
<td>2-NEXPTIME</td>
<td>6.1</td>
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<tr>
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<td>total</td>
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<td>PSPACE</td>
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<td>total</td>
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<tr>
<td>arbitrary recursion</td>
<td>partial</td>
<td>—</td>
<td>semi-decidable</td>
<td>—</td>
</tr>
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</table>
In HTN planning, task names represent activities to accomplish and are syntactically first-order atoms. Given a set of task names $X$, a task network is a tuple $tn = (T, \prec, \alpha)$ such that:

- $T$ is a finite nonempty set of task symbols.
- $\prec$ is a strict partial order over $T$.
- $\alpha : T \rightarrow X$ maps from task symbols to task names.

The task symbols function as place holders for task names, allowing multiple instances of a task name to exist in a task network. We say a task network is ground if all task names occurring in it are variable-free.

An HTN problem is a tuple $(L, O, M, s_I, tn_I)$, where:

- $L$ is a function-free first order language with a finite set of relations and constants.
- $O$ is a set of operators, where each operator $o$ is a tuple $(n, \chi, c)$, where $n$ is a task name (referred to as name$(o)$) not occurring in $L$, $\chi$ is a first-order logic formula called the precondition of $o$, and $c$ is a conjunction of positive and negative literals in $L$ called the effects of $o$. We refer to the set of task names in $O$ as primitive task names.
- $M$ is a set of decomposition methods, where each method $m$ is a pair $(c, tn_I)$, where $c$ is a (non-primitive or compound) task name, called the method’s head, not occurring in $O$ or $L$, and $tn_I$ is a task network, called the method’s subtasks, defined over the names in $O$ and the method heads in $M$.
- $s_I$ is the (ground) initial state and $tn_I$ is the initial task network that is defined over the names in $O$.

We define the semantics of lifted HTN planning through grounding. Given that $L$ is function-free with a finite set of relations and constants, we can create a ground (or propositional) HTN planning problem $P = (L, O, M, s_I, tn_I)$, where $O$ and $M$ are variable-free: Let $V$ be the set of all full assignments from variables in $L$ to constants in $L$. Then the ground methods $M$ are given by $\bigcup_{v \in V} \bigcup_{(c, tn_I) \in M} \{ (c[v], tn_I[v]) \}$, where the syntax $c[v]$ denotes syntactic variable substitution of the variables occurring in $c$ with the matching term from $v$. Note that we introduced additional methods not in $M$ that are required to ground the initial task network. The symbol $c_{top}$ not occurring in $O, L$, or in any set of methods of $M$ is a new compound task name that decomposes into the possible groundings of the original initial task network $tn_I$. The initial task network $tn_I$ of the ground HTN problem $P$ then consists of only this new task $c_{top}$. Let $O'$ be the set of quantifier-free operators obtained from $O$ by eliminating the variables with constants from $L$ (Gazend and Knoblock 1997). Then the ground operators $O$ are given by $\bigcup_{v \in V, (n, \chi, c) \in O'} \{ (n[v], \chi[v], c[v]) \}$.

The ground operators form an implicit state-transition function $\gamma : 2^L \times O \rightarrow 2^L$ for the problem, where:

- A state is any subset of the ground atoms in $L$. The finite set of states in a problem is denoted by $2^L$.
- $o$ is applicable in a state $s$ iff $s \models prec(o)$; and
- $\gamma(s, o)$ is defined iff $o$ is applicable in $s$; and
- $\gamma(s, o) = (s \setminus del(o)) \cup add(o)$.

A sequence of ground operators $\langle o_1, \ldots, o_n \rangle$ is executable in a state $s_0$ iff there exists a sequence of states $s_1, \ldots, s_n$ such that $\forall 1 \leq i \leq n \gamma(s_{i-1}, o_i) = s_i$. A ground task network $tn = (T, \prec, \alpha)$ is primitive iff it contains only task names from $O$. $tn$ is executable in a state $s_0$ iff $tn$ is primitive and there exists a total ordering $t_1, \ldots, t_n$ of $T$ consistent with $\prec$ such that $\langle \alpha(t_1), \ldots, \alpha(t_n) \rangle$ is executable starting in $s_0$.

For a ground HTN problem $P = (L, O, M, s_I, tn_I)$, we can decompose the task network $tn_I = (T, \prec, \alpha)$ if there is a task $t \in T$ such that $\alpha(t)$ is a non-primitive task name and there is a corresponding method $m = (\alpha(t), (T_m, \prec_m, \alpha_m)) \in M$. Intuitively, decomposition is done by selecting a task with a non-primitive task name, and replacing the task in the network with the task network of a corresponding method. More formally, assume without loss of generality that $T \cap T_m = \emptyset$. Then the decomposition of $t$ in $tn_I$ by $m$ into a task network $tn'$ is given by:

$T' := (T \setminus \{t\}) \cup T_m$
$\prec' := \{(t, t') \in \prec \mid t, t' \in T'\} \cup \prec_m$
$\alpha' := \{(t, n) \in \alpha \mid t \in T'\} \cup \alpha_m$

$tn' := (T', \prec', \alpha')$

A ground HTN problem $P$ is solvable iff either $tn_I$ is executable in $s_I$, or there is a sequence of decompositions of $tn_I$ to a task network $tn'$ such that $tn'$ is executable in $s_I$. Checking whether an HTN problem has a solution is the plan-existence decision problem.

Current HTN planners either solve problems using decomposition directly, or by using progression (Nau et al. 2003; Alford et al. 2012). Progression consists of a choice of two operations on just the unconstrained tasks (those without a predecessor in the task network): decomposition, or operator application. Operator application takes an unconstrained primitive task $t$ in the current task network $tn$ such that $\alpha(t)$ is applicable in $s$, removes $t$ from $tn$ to form a new task network $tn'$, and returns a new HTN problem with $\gamma(s, \alpha(t))$ as the initial state and $tn'$ as the initial task network. Hence, progression interleaves decomposition and finding a total executable order over the primitive tasks. A problem’s progression bound is the size of the largest task network reachable via any sequence of progressions.

### 3 Stratifications for propositional and constant-free method domains

Decomposition and progression respectively decide, roughly speaking, the class of acyclic HTN problems (HTN problems without recursion) and tail-recursive HTN problems (those problems where tasks can only recurse through the last task of any method). Alford et al. (2012) give syntactic tests for identifying ground method structures that are decided by decomposition and progression (identified by $\leq_1$-stratification and $\leq_r$-stratification, respectively).
In this section, we generalize the $\leq_1$- and $\leq_r$-stratification tests to identify all acyclic and tail-recursive lifted HTN problems whose methods are constant-free. Formally a constant-free method (CFM) HTN problem is one where only variables may occur as terms in the task names of the domain’s methods (both in the head and the subtasks). Fully-ground domains can be trivially transformed into CFM domains by rewriting the task names with 0-arity predicates, and so we include both fully-ground and propositional problems in the class of constant-free method problems.

Extending the stratification tests to CFM problems will yield upper complexity bounds for acyclic and tail-recursive CFM problems (NEXPTIME and EXPSpace, respectively). Since propositional HTN planning is equivalent to HTN planning with all unary predicates (and thus constant-free), these are also upper bounds for acyclic and tail-recursive propositional problems.

**Upper bounds for mostly-acyclic HTN problems**

When the method structure of a problem is acyclic, every sequence of decompositions is finite, and so almost any decomposition-based algorithm is a decision-procedure for the problem (Erol, Hendler, and Nau 1994). Alford et al. (2012) extend the class of acyclic problems to include those whose methods only allow recursion when it does not increase the size of the task network, and define a syntactic test called $\leq_1$-stratifiability to recognize ground instances of these problems. Here we extend $\leq_1$-stratification to CFM HTN problems.

A CFM HTN problem $P$ is $\leq_1$-stratifiable if there exists a total preorder (a relation that is both reflexive and transitive) $\leq_1$ on the task names in $P$ such that:

- For any task names $c_1$ and $c_2$ in $P$, if there are variable substitutions $v_1$, $v_2$ such that $c_1[v_1] = c_2[v_2]$, then $c_1 \leq_1 c_2$.
- For every method $(e_i, (T, <, \alpha))$ in $P$:
  - If $|T| > 1$, then $\forall t_i \in T \alpha(t_i) <_1 e_i$.
  - If $T = \{t\}$, then $\alpha(t) \leq_1 e_i$.

The above conditions ensure that any decomposition in $\leq_1$-stratifiable problems either replaces a task with a single task from the same stratum, or replaces a task with one or more tasks from lower strata. We can determine $\leq_1$-stratification in polynomial time with any algorithm for finding strongly connected components in a directed graph, such as Tarjan’s algorithm.

By extending $\leq_1$-stratification to CFM HTN problems, we can show that there is a strict correspondence between the stratification of a CFM problem and its grounding:

**Lemma 3.1.** A CFM HTN problem $P$ is $\leq_1$-stratifiable if and only if there exists a $\leq_1$-stratification of the grounding of $P$ of the same height.

**Proof.** Let $L$ be the language of $P$ and $P$ be the grounding of $P$. If $P$ is $\leq_1$-stratifiable, then grounding the task names of each level of a $\leq_1$-stratification for $P$ is a stratification of $P$ of the same height.

So assume $P$ is not $\leq_1$ stratifiable. Then by the negation of $\leq_1$-stratifiability, there are methods $(c_1, tn_1), \ldots, (c_k, tn_k)$ and variable assignments $v_1, \ldots, v_k$ such that $tn_1$ has more than one subtask, $tn_k[v_k]$ contains the subtask $c_1$, and each $tn_i[v_i]$ contains the subtask $c_{i+1}$.

Let $a$ be an arbitrary constant in $L$ and $v_a$ be the assignment that maps all variables in $L$ to $a$. Then each $(c_i[v_a], tn_k[v_a])$ is a ground method in $P$. Each $tn_i[v_i]$ contains the task name $c_{i+1}[v_a]$, and $tn_k[v_a]$ contains $c_1[v_a]$ as a subtask, so $P$ is also not $\leq_1$-stratifiable.

We call HTN problems mostly-acyclic if either they are CFM and $\leq_1$-stratifiable or they are non-CFM and their grounding is $\leq_1$-stratifiable. If all sequences of decompositions of a given problem are finite, we call that problem acyclic. Section 4 contains examples of both CFM and non-CFM acyclic method structures.

We can transform any propositional $\leq_1$-stratifiable problem into an acyclic problem in polynomial time as follows: Let $P = (L, \mathcal{O}, M, s_1, tn_1)$ be any $\leq_1$-stratifiable propositional HTN problem, and let $\mathcal{S}$ be the maximal $\leq_1$-stratification in the following sense: if $c_1$ and $c_2$ are task names on a stratum of $\mathcal{S}$, then $c_1 \leq c_2 \leq c_3$ in any $\leq_1$-stratification of $P$. So each task name is on a stratum by itself, or the stratum consists of a set of task names $c_1, \ldots, c_k$ such that $c_1 \leq \ldots \leq c_k \leq c_1$. Let $M_r \subseteq M$ be the methods responsible for these later constraints (each having some $c_i$ as a task head and a task network with a singular task of some $c_j$), and let $M_r \subseteq M$ be the methods leading to strictly lower strata. Then $(M \setminus M_r) \cup \bigcup_{1 \leq i \leq k} \{(c_i, tn_a) : (c_i, tn_a) \in M_a\}$ eliminates recursion at this stratum while still admitting the same set of primitive decompositions as $M$. Repeating this process on each strata eliminates recursion from the problem at the cost of a polynomial increase in size.

This leads to an upper bound for $\leq_1$-stratifiable CFM HTN problems:

**Corollary 3.2.** Plan-existence for CFM (and propositional) mostly-acyclic HTN problems is in NEXPTIME.

**Proof.** Let $P = (L, \mathcal{O}, M, s_1, tn_1)$ be a $\leq_1$-stratifiable CFM HTN problem, and let $\mathcal{S}$ be $P$’s maximal $\leq_1$-stratification. Let $m$ be the maximum number of tasks in $tn_1$ or any method, and let $k$ be the number of task names occurring in $M$.

Let $P$ be the grounding of $P$ (taking EXPTIME) with a $\leq_1$-stratification $\mathcal{S}$ of the same height as $\mathcal{S}$. By the above process, we can create $P'$ as an acyclic version of $P$, and since that process preserves any stratification, $\mathcal{S}$ is also a $\leq_1$-stratification of $P'$.

By the construction of $\mathcal{S}$ from $\mathcal{S}$, any decomposition of a task results in a set of tasks from strictly lower strata. This gives a tree-like structure to the decomposition hierarchy, with a maximum branching factor of $m$ and a max depth of $k$. So $m^k$ is a bound on the length of any sequence of decompositions of the initial task network. Thus the following is a decision procedure for $P$:

Pick and apply a sequence of decompositions of $tn_1$ of length $m^k$ or less (NEXPTIME). Guess a total ordering of the resulting network and check if it is executable in $s$. 

Grounding an HTN problem produces a worst-case size
blowup that is exponential in the arity of task names and predicates. Thus, if \( b(x) \) is an upper space or time bound for a class of ground HTN problems \( B \), then \( O(2^{b(x)}) \) is an upper bound (space or time, respectively) for problems whose groundings are in \( B \). So Corollary 3.2 implies a 2-NEXPTIME upper bound for non-CFM mostly-acyclic HTN problems. Section 5 provides matching lower bounds.

Upper bounds for tail-recursive HTN problems

Many problems are structured so that tasks can only recur through the last task of any of its associated methods. These problems are guaranteed to have a finite progression bound, and thus are decided by simple progression methods. These problems are guaranteed to have a finite upper space bounds on the size of task networks under progression for \( \leq_r \)-stratifiable problems: an exponential bound for \( \leq_r \)-stratifiable CFM problems, and a polynomial bound for totally-ordered propositional \( \leq_r \)-stratifiable problems.

A CFM HTN problem \( \mathcal{P} \) is \( \leq_r \)-stratifiable if there exists a total preorder \( \leq_r \) on the task names in \( \mathcal{P} \) such that:

- For each pair of task names \( c_1 \) and \( c_2 \) in \( \mathcal{P} \), if there are variable substitutions \( v_1, v_2 \) such that \( c_1[v_1] = c_2[v_2] \), then \( c_1 \leq_r c_2 \leq_r c_1 \).
- For every method \( (c, (T, \prec, \alpha)) \) in \( \mathcal{P} \):
  - If there is a task \( t_r \in T \) such that all other tasks are predecessors \( t \prec t_r \), then \( \alpha(t_r) \leq_r c \). We call \( t_r \) the last task of \( (T, \prec, \alpha) \).
  - For all non-last tasks \( t \in T \), \( \alpha(t) <_r c \).

The above conditions ensure that methods in \( \leq_r \)-stratifiable problems can only recurse through their last task, yet it still allows hierarchies of tasks. This is a strict generalization of \( \leq_1 \)-stratifiability from the previous section, and of regular HTN problems from Erol, Hendler, and Nau (1994), which only allow at most one non-primitive task in every method (and the initial task network) that also has to occur as the last task in the respective method (the initial task network, respectively). As with \( \leq_1 \)-stratifiability, there is a strict correspondence between a CFM problem’s \( \leq_r \)-stratifiability and its grounding’s \( \leq_r \)-stratifiability:

**Lemma 3.3.** A CFM HTN problem \( \mathcal{P} \) is \( \leq_r \)-stratifiable if and only if there exists a \( \leq_r \)-stratification of the grounding of \( \mathcal{P} \) of the same height.

We omit a formal proof, since it is structurally identical to the one of Lemma 3.1. We call both \( \leq_r \)-stratifiable CFM problems and non-CFM problems whose groundings are \( \leq_r \)-stratifiable *tail-recursive*.

This allows us to calculate a bound on the size of task networks reachable under progression in a bottom-up fashion. Let \( \mathcal{P} = (\mathcal{L}, \partial, \mathcal{M}, s_1, t_n) \) be a tail-recursive CFM HTN problem where \( S_1, \ldots, S_n \) is a \( \leq_r \)-stratification of \( \mathcal{P} \). WLOG, assume that \( S_1 \) contains only primitive task names. The contribution of any primitive task in \( t_n \) (those with names in \( S_1 \)) to the progression bound is 1. The contribution of tasks of \( t_n \) with names in \( S_2 \) is then bounded by the number of tasks in the largest method corresponding to a task name in \( S_2 \). Upper strata are bounded by the size of the largest corresponding task network multiplied by the bound for the next lower stratum. This gives an exponential worst-case progression bound on \( \leq_r \)-stratifiable problems:

**Lemma 3.4.** If \( \mathcal{P} \) is a tail-recursive CFM HTN problem with initial tasks, \( r \) is the largest number of tasks in any method in \( \mathcal{P} \) and \( h \) the height of \( \mathcal{P} \)’s \( \leq_r \)-stratification, then \( k \cdot r^h \) is a progression bound for \( \mathcal{P} \).

This implies upper bounds for all tail-recursive problems:

**Corollary 3.5.** Plan-existence for propositional and CFM HTN \( \leq_r \)-stratifiable problems is in EXPSPACE. Plan-existence for non-CFM HTN problems whose groundings are \( \leq_r \)-stratifiable is in 2-EXPSPACE.

Consider the case that each task network in \( \mathcal{P} \) is totally-ordered. Then any progression of \( \mathcal{P} \) leaves the initial task network totally ordered. Moreover, decomposition of the first task in the network can only grow the list with tasks from lower strata. This gives a progression bound for totally-ordered tail-recursive problems:

**Lemma 3.6.** If \( \mathcal{P} \) is a totally-ordered tail-recursive CFM HTN problem with \( k \) initial tasks, \( r \) the largest number of tasks in any method in \( \mathcal{P} \), and \( h \) the height of \( \mathcal{P} \)’s \( \leq_r \)-stratification, then \( k + r \cdot h \) is a progression bound for \( \mathcal{P} \).

This gives a PSPACE upper bound for propositional totally-ordered tail-recursive problems. CFM \( \leq_r \)-stratifiable problems, whether ordered or not, are dominated by the size of the state and not the task network, and so have an EXPSPACE upper bound. Via grounding, non-CFM totally-ordered tail-recursive problems (those whose groundings are \( \leq_r \)-stratifiable) have an exponential progression bound, which matches their worst-case state size, and so are also in EXPSPACE.

Erol, Hendler, and Nau (1994) give encodings of both propositional and lifted classical planning into regular HTN problems, where every method has at most one non-primitive task, and that task must be the last task in the method. The lifted encoding uses no constants in the methods, so both are \( \leq_r \)-stratifiable. Since propositional planning is PSPACE-complete (Bylander 1994) and lifted planning is EXPSPACE-complete (Erol, Nau, and Subrahmanian 1995), this gives our first completeness results of the paper:

**Theorem 3.7.** Plan-existence for propositional totally-ordered tail-recursive problems is PSPACE-complete. Plan-existence for tail-recursive CFM HTN problems and totally-ordered non-CFM tail-recursive problems is EXPSPACE-complete.

4 Hierarchies and counting in HTNs

Whereas tail-recursive HTN problems allow us to express tasks that may repeat an arbitrary number of times, the number of repeats is fixed in advance for acyclic and mostly-acyclic problems. In this section, we will show how to express in polynomial space tasks that occur in exponential
and double-exponential numbers of times in any solution. This will give us immediate lower bounds for totally-ordered acyclic problems, and will also be used in Section 6 for the lower bounds of partially-ordered problems.

First, we show how to repeat a task an exponential number of times with a set of propositional \( \leq_1 \)-stratifiable methods: Let \( k \geq 0 \) and let \( o_0 \) be some task name. Let \( M_{o_k} = \{ (o_1, t_{n_1}), \ldots, (o_k, t_{n_k}) \} \) be task names such that each \( t_{n_i} = (T, \prec, \alpha) \) contains two tasks \( t_1, t_2 \), such that \( t_1 \prec t_2 \) and \( \alpha(t_1) = \alpha(t_2) = o_{i-1} \). Thus \( o_1 \) decomposes into two copies of \( o_0 \), \( o_2 \) decomposes into four, and so on until \( o_k \) decomposes into \( 2^k \) copies of \( o_0 \). With \( M_{o_k} \), any sequence of \( o_1 \) of length \( 2^{k+1} - 1 \) or less can be expressed in a task network by taking the appropriate subset of \( \{ o_1, \ldots, o_k \} \).

We can also encode doubly-exponential repeats of tasks with non-CFM \( \leq_1 \)-stratifiable methods: Let \( k \) be a positive integer and let \( o \) be some task name, and let \( 0, 1 \) be arbitrary, distinct constants from \( L \). Given a new \( k \)-arity predicate \( o_e \), we will give a set of task names and methods that form a counter from \( o(n, \ldots, 1) \) down to \( o(0, \ldots, 0) \). Let \( o_{e_1}, \ldots, o_{e_k} \) be task names such that each \( o_{e_i} \) has the form \( o(v_1, v_2, \ldots, v_m, 0, \ldots, 0) \) where each \( v_m \) is a variable. So \( o_{e_1} = o(v_1, v_2, 0, \ldots, 0) \), \( o_{e_2} = o(v_1, v_2, v_3, 1, 0) \), \( o_{e_k-1} = o(v_1, 0, 1, 0, \ldots, 0) \), and \( o_{e_k} = o(1, 0, 0, \ldots, 0) \). Similarly, let \( o_{e_1}', o_{e_2}', \ldots, o_{e_k}' \) be task names of the form \( o(v_1, v_2, \ldots, v_{m+1}, 0, \ldots, 1) \), so \( o_{e_1}' = o(v_1, v_2, 0) \), \( o_{e_2}' = o(v_1, v_2, v_3, 0) \), \( o_{e_k-1}' = o(v_1, 0, 0, 1, 1) \), and \( o_{e_k}' = o(0, 1, 1, 1) \). So if \( v \) is an assignment of \( v_1, \ldots, v_k \) to \( \{ 0, 1 \} \) and we view \( o_{e_i}[v] \) as a binary number \( j \), then \( o_{e_i}'[v] = j - 1 \).

Let \( M_{o_k} = \{ (o_{e_0}, t_{n_{o_0}}), \ldots, (o_{e_k}, t_{n_{o_k}}) \} \), where: \( o_{e_0} = o(0, \ldots, 0) \), \( t_{n_{o_0}} \) has two copies of \( o \) as subtasks, and each \( t_{n_{o_k}} \) has two ordered copies of \( o_{e_k}' \). Grounding \( M_{o_k} \) to \( \{ 0, 1 \} \) produces a set of ground methods with a \( \leq_1 \)-stratification of height \( 2^{k+1} \) (including \( o \)). Groundings that include other constants are essentially truncated counters with no methods that lead to \( o_0 \) (though they are still \( \leq_1 \)-stratifiable and disjoint with the grounding to \( \{ 0, 1 \} \)).

Thus, \( o(0, \ldots, 0) \) decomposes into two copies of \( o \), \( o(0, \ldots, 0, 1) \) decomposes first into two copies of \( o(0, \ldots, 0) \) (and then four of \( o \)), and so on, until \( o(1, \ldots, 1) \) has \( 2^{k+1} - 1 \) copies of \( o \).

Let \( K \) be any polynomially-bounded sum of integers of the form \( 2^i \) and \( 2^{2^i} \) for \( i < k \). The above two counting results allow us to express \( K \) repetitions of a task \( o \) in space polynomial in \( k \). We will use the shorthand \( K \cdot o \) as the task name that decomposes into such a sequence, along with implying the existence of the supporting methods.

Erol, Hendler, and Nau (1994) use encoding of classical planning into regular HTN problems (a subset of tail-recursive problems, but not acyclic) to achieve lower bounds of PSPACE- and EXPSPACE-complete for propositional and lifted regular HTN problems, respectively. Here we sketch how to adapt this proof to acyclic problems:

**Theorem 4.1.** Plan-existence for totally-ordered mostly-acyclic propositional HTN problems is PSPACE-complete. For non-CFM totally-ordered mostly-acyclic problems, plan-existence is EXPSPACE-complete.

**Proof.** The upper bounds of PSPACE and EXPSPACE, respectively, are established by Theorem 3.7, since acyclic problems are by definition tail-recursive.
• Operators from $O_{init}$ initialize the ‘tape’ (a set of cell relations indexed by a binary counter)
• Each operator $o \in O_{step}$ mimics a single transition of $M$.
• Each operator $o \in O_{done}$ adds the literal $\text{done}(t)$ to the state whenever the machine is in an accepting state.

Let accepted be an operator which has a precondition of $`\text{done}(t)`$, and let $O = O_{init} \cup O_{step} \cup O_{done} \cup \{\text{accepted}\}$.

We define $M$ to be a set of methods such that $sim()$ is a non-primitive task with methods that decompose it into any operator in $O_{step} \cup O_{done}$, and $M$ contains methods for implementing $2^k \cdot \text{init}$ and $2^k \cdot \text{sim}$. Let $tn_1$ be the initial task network which contains three tasks, $t_1 < t_2 < t_3$ such that $\alpha(t_1) = 2^k \cdot \text{init}$, $\alpha(t_2) = 2^k \cdot \text{sim}$, and $\alpha(t_3) = \text{accepted}$. Then the acyclic CFM problem $P = (L, O, M, s, tn_1)$ is solvable if and only if there is a run of $M$ that finds an accepting state within $2^k$ steps. Thus, acyclic CFM planning is $\text{NEXPTIME}$-hard.

5 Alternating Turing machines for totally-ordered problems

Erol, Hendler, and Nau (1994) show that while arbitrary recursion when combined with partially-ordered tasks is undecidable, arbitrary recursion with totally ordered tasks in $\text{EXPTIME}$ for propositional problems and $2\text{-EXPTIME}$ otherwise. Here we show that those bounds are tight by encoding space-bounded alternating Turing machines.

An alternating Turing machine (ATM) is syntactically identical to a nondeterministic Turing machine (NTM). However, where an NTM accepts if any run of the machine accepts, an ATM accepts only if all runs of the machine accept. The classes of problems that run in polynomial or exponential space on an ATM are $\text{APSPACE}$ and $\text{AEXPSPACE}$, respectively. Since $\text{APSPACE}=\text{EXPTIME}$ and $\text{AEXPSPACE}=2\text{-EXPTIME}$ (Chandra, Kozen, and Stockmeyer 1981), an encoding of a space-bounded ATM gives lower time bounds for totally-ordered HTN planning:

**Theorem 5.1.** Propositional totally-ordered HTN planning is $\text{EXPTIME}$-complete. CFM and non-CFM HTN planning is $2\text{-EXPTIME}$-complete.

**Proof.** Erol, Hendler, and Nau (1994) established the $\text{EXPTIME}$ and $2\text{-EXPTIME}$ upper bounds. Alford et al. (2012) confirm these upper bounds with a set-theoretic formulation of HTN planning.

Let $A$ be an ATM, denoted by $A = (S, \Sigma, \Gamma, \delta, q_0, F)$, where $K$ is a finite state of state symbols, $F \subseteq S$ is the set of accepting states, $\Gamma$ is the set of tape symbols with $\Sigma \subseteq \Gamma$ being the allowable input symbols, $q_0 \in S$ is the initial state, and $\delta$ is the transition function, mapping from $S \times \Gamma$ to $\mathcal{P}(S \times \Gamma \times \{\text{Left}, \text{Right}\})$.

We will use the same initial state $s$ and operators $O_{init}$ and $O_{step}$ that were used in the proof of Theorem 4.2. We use the operators $O_{done}$ with the modification that they have no effect, just the precondition of test whether the machine is in an accepting state. To this, we add a set of inverting operators $O^{-1}_{step}$ such that for each $o \in O_{step}$, there is a $o^{-1} \in O^{-1}_{step}$ such that $\gamma^o(s, o) \cdot o^{-1} = s$ for every state $s$ in which $o$ is applicable.

Let $O = O_{init} \cup O_{done} \cup O_{step} \cup O^{-1}_{step}$. The set of methods $M$ is defined as follows:

- For each operator $o \in O_{init}$, we have a method $(\text{init}, tn)$, where $tn$ contains just the task $o$.
- For each operator $o \in O_{done}$, we have a method $(\text{sim}, tn)$, where $tn$ contains just the task $o$.
- For each state $s$ and tape symbol $c$, we have a method $(\text{sim}, tn) \in M$. Let $T = \delta(s, c)$, $T$ denotes a set of transitions, and $A$ must halt on each of these. So let $o_1, \ldots, o_n$ be the operators from $O_{step}$ associated with the transitions in $T$. Then $tn$ is the tasks network with the totally-ordered tasks $o_1, sim, o_1^{-1}$, $o_2, sim, o_2^{-1}$, $\ldots$, $o_n, sim, o_n^{-1}$.

Let $tn_1$ be the initial task network which contains the totally-ordered tasks $2^k \cdot \text{init}$ and $\text{sim}$. Let $P$ be the totally-ordered CFM HTN problem given by $P = (L, O, M, s, tn_1)$. Notice that, although the methods for the $\text{sim}$ task change the state, they always revert it before they’re done: If the machine is already in an accepting state, the first set of methods leave the state unchanged. Otherwise, if there is a transition in $\delta$ for the current state and tape symbol, then there is a corresponding method in $M$. The method applies an operator, runs the $\text{sim}$ task to conclusion, reverts the operators, and so on until each operator has been applied and the $\text{sim}$ tasks run. So all possible runs of $A$ (which we bound to run in $\text{AEXPSPACE}$) are verified. Since $A$ was arbitrary, totally-ordered CFM planning is $2\text{-EXPTIME}$-hard.

The encoding from Erol, Nau, and Subrahmanian (1991) uses predicates of logarithmic arity in the size of the bound. The only use of variables in our encoding was for these operators, so if we had chosen a polynomial space bound, the grounding of our encoding would have been polynomial in size. Thus, the same proof works with the polynomial bound to encode an $\text{APSPACE}$ machine, so totally-ordered propositional planning is $\text{EXPTIME}$-hard.

6 Interactions with partial-orders and hierarchies

In Section 4, we described acyclic counting techniques that generated large numbers of tasks. When the methods and initial tasks were ordered, progression-based algorithms had to deal with only a small portion of those tasks at a time. However, when the tasks are partially ordered, tasks can interact with each other in intricate ways. Here we adapt the $\text{APSPACE}$-completeness proof of reachability for communicating hierarchical state machines from Alur, Kannan, and Yannakakis (1999) to obtain lower bounds for partially-ordered HTN problems that match the upper bounds we provide in Section 3. Since the proofs will be nearly identical to each other, we collapse them into one theorem and prove only the completeness bound for the partially-ordered lifted acyclic HTN problems:

**Theorem 6.1.** Plan-existence for partially-ordered propositional acyclic HTN planning is $\text{NEXPTIME}$-complete; for partially-ordered propositional tail-recursive HTN planning is $\text{EXPSPACE}$-complete; for partially-ordered lifted acyclic
HTN planning is \(2\text{-NEXPTIME}\)-complete; and for partially-ordered lifted tail-recursive HTN planning is \(2\text{-EXPSPACE}\)-complete.

**Proof.** Upper bounds were established by Lemma 3.1.

For the lower bound, let \(N = (S, \Sigma, \Gamma, \delta, q_0, F)\) be a non-deterministic Turing machine (NTM). Given a positive integer \(k\), we will encode \(N\) into a lifted acyclic HTN planning problem \(P\) such that \(P\) is solvable if and only if there is a run of \(N\) in an accepting state after \(2^k\) steps.

Let \(K = 2^k\). Since \(N\) has a \(K\) size space bound, we can view a configuration of \(N\) as the position of the head, the state, and a string \(w\) over \(\Gamma\) of length \(K\) representing the tape. Then, if \(w_0, \ldots, w_K\) are the tape configurations of an accepting run of \(N\), then the string \(W = \#w_0\#w_1\# \ldots \#w_K\), where \# is a separator, represents a checkable proof that \(N\) halts on this input in \(K\) steps. To check the proof, we need to make sure that each \(w_i\) follows from the \(w_{i-1}\) before it, which we will check character by character. Specifically, if we are checking the \(j^{th}\) character of \(w_i\), then the \(j^{th}\) character of \(w_{i+1}\) (or exactly \(K + 1\) characters later in \(W\)) is either the same as it was in \(w_i\), or the head was over the \(j^{th}\) position and the \(j^{th}\) character of \(w_{i+1}\) follows from some legal transition from \(\delta\). Without loss of generality, we will assume that \(\Gamma\) contains the character \# used for separation, and that \(\delta\) defines no transitions for it. We also assume that \(N\) always has a transition from any accepting state back to an accepting state. Further, we assume that the tape is initially blank (WLOG, since there is a polynomial transformation from an NTM with input to one with a blank input). Let 0 be the default tape character.

Our encoding of this check will have an entirely propositional state language \(L\):

- There is a set of propositions for each state in \(S\). Only one of these propositions will be true at any point in time, encoding the state of \(N\) for configuration \(w_i\) up until we check the character underneath the head, when we switch to the state for the next tape configuration, \(w_{i+1}\).
- There is a set of propositions for each tape symbol in \(\Gamma\). Only one will be true at a time.
- There are three pairs of propositions used to synchronize tasks: \(\text{head\_step},\ \text{head\_stepped},\ \text{check\_step},\ \text{check\_stepped},\ \text{sync\_step},\ \text{and\ sync\_stepped}\). Of each pair, at most one will be true at a time. We will describe how these are used shortly.

Let \(O\) be the set of propositional operators defined below:

- For each character \(c \in \Gamma\) there are two operators: \(\text{assert}_c\) and \(\text{check}_c\), where \(\text{assert}_c\) adds \(c\) to the state while retracting every other character in \(\Gamma\), and \(\text{check}_c\), which has \(c\) as precondition and no effects.
- For all transitions \((s', c', \text{Left}) \in \delta(s, c)\), there is an operator \(\text{step}_{s,s',c',\text{Left}}\), which has a precondition of \(\text{head} \wedge s \wedge c\) and has an effect of retracting \(k\) and asserting \(c'\). \(\text{step}_{s,s',c',\text{Left}}\) is defined similarly for transitions that move the head right.
- There is an operator \(\text{done}\), which has a precondition of \(\bigvee_{k \in F} k\) and no effects.
- There is an operator \(\text{no\_head}\), which has \(\neg\text{head}\) as a precondition and no effects, as well as operators \(\text{assert\_head}\) and \(\text{retract\_head}\) for asserting and retracting the \text{head}\ proposition.
- For the \(\text{head\_step},\ \text{head\_stepped}\) propositions, we define four operators: \(\text{call\_head}\) which has no precondition, asserts \(\text{head\_step}\); \(\text{start\_head}\) has the precondition of \(\text{head\_step}\) which it retracts; \(\text{respond\_head}\) has no precondition and asserts \(\text{head\_stepped}\); and \(\text{wait\_head}\) has \(\text{head\_stepped}\) as its precondition which it retracts. Operators for the other \text{step/stepped}\ pairs are defined similarly.

Let \(M\) be the following set of methods:

- For the \(\text{call\_head},\ \text{wait\_head}\) operators, we introduce a method \((\text{step\_head}, tn)\), where \(tn\) contains the totally-ordered tasks \(\text{call\_head}\) and \(\text{head\_wait}\). \(\text{step\_tasks}\) are defined similarly for the rest of the \(\text{call/wait}\) operators.
- For each \(c \in \Gamma\), a method \((\text{produce}, tn)\), where \(tn\) contains the totally-ordered tasks \(\text{produce}\_c\) and \(\text{produce}\_c\_0\). We will define the consumers for the step tasks below. We also have two specialized versions of the method, \((\text{produce}\_#, tn)\) and \((\text{produce}_0, tn)\) specifically for asserting \# and 0, but are otherwise the same.
- We add a method \((\text{produce}, tn)\) where \(tn\) contains \(\text{check\_done}\), which will only be applicable if a previous \text{produce} gave a valid character that moved the machine into an accepting state.
- We add a method \((\text{config\_producers}, tn)\) which contains the task \((K+1)\_\text{produce},\) which decomposes into enough \text{produce} tasks to ensure that the configuration \(w_{i+1}\) is a valid successor of the current configuration \(w_{i}\). We also add \((\text{config\_producers}_\text{init}, tn_{\text{init}})\) to encode the initial tape configuration. \(tn_{\text{init}}\) has two ordered subtasks: \(\text{produce}\_#\) and \(K \cdot \text{produce}_0\).
- A method \((\text{head}, tn)\) for asserting the head, where \(tn\) contains the tasks \(\text{start\_head},\ \text{assert\_head},\ \text{and\ respond\_head}\). A method for the task \(\text{no\_head}\) is defined similarly with \(\text{retract\_head}\).
- A method \((\text{heads}, tn)\), where \(tn\) contains the totally-ordered tasks \(\text{no\_head},\ \text{head},\ (K + 2) \cdot \text{no\_head}\). This will place the head on the second character and, by default, move it one to the right in every subsequent configuration of \(N\). We will show how to adjust for this when checking the transitions.
- A method \((\text{wait}, tn)\) where \(tn\) contains four totally-ordered tasks which sequentially call the operators \(\text{start\_check},\ \text{respond\_check},\ \text{start\_sync},\ \text{and\ respond\_sync}\). A \text{wait} task will then eat up one \text{call\_check} followed by one \text{call\_sync}.
- For each \(c \in \Gamma\), we have a method \((\text{check}, tn)\) that, when the head is not currently present, will ensure that character \(K + 1\) chars later is identical. \(tn\) contains eight totally-ordered tasks: \(\text{start\_check},\ \text{no\_head},\ \text{check}_c,\ \text{respond\_check},\ K \cdot \text{wait},\ \text{start\_check},\ \text{check}_c,\ \text{and\ respond\_check}\).
• For each transition \((s', c', \text{Left}) \in \delta(s, c)\), we have the task \((\text{check}, tn)\), where \(tn\) sets the new state and ensures that the character \(K + 1\) chars later follows a legal transition. \(tn\) contains the following totally-ordered tasks: \(\text{start\_check}, \text{step\_s, s', c'}, 2 \cdot \text{step\_head}, \text{respond\_check}, K \cdot \text{wait}, \text{start\_check}, \text{check}, c' \) and \(\text{respond\_check}\). Notice how we call \(\text{step\_head}\) twice to make the head move left (instead of moving to the right by default). We define \(\text{check}\) methods for \(\text{Right}\) moving transitions similarly, omitting the \(2 \cdot \text{step\_head}\) subtask.

• We define a task \(\text{config\_checks}(v_1, \ldots, v_k)\) with methods similar to the doubly-exponential counters of Section 4. \(\text{config\_checks}(1, 0, \ldots, 0)\) will be responsible for launching \(K\) \(\text{check}\) tasks, starting in each cell of the configuration. Instead of the two subtasks of the \(OE\) counting methods, each method \((\text{config\_checks}, tn)\) has three tasks \(t_1, t_2 < t_3\) (so \(t_1\) is not ordered with respect to the rest), where \(\alpha(t_1) = \alpha(t_3) = \text{config\_checks}'\), and \(\alpha(t_2) = \text{wait}'\), where \(\text{config\_checks}'\) and \(\text{wait}'\) are the appropriate decrement of \(\text{config\_checks}\). We will add a method \((\text{config\_checks}+1, tn)\) that launches exactly \(K + 1\) checks for each of the cells of the configuration.

Let \(P = (\mathcal{L}, \mathcal{O}, \mathcal{M}, s_0, t_1)\) be an HTN problem with the above \(\mathcal{L}, \mathcal{O}, \mathcal{M}\). Let \(s\) contain the proposition for \(N\)'s initial state \(q_0\), and let \(tn_1\) contain the tasks \(t_1 < t_2 < t_{\text{done}}\), \(t_3, t_4\), and \(t_5 < t_6\), where:

\[
\begin{align*}
\alpha(t_1) &= \text{config\_producers}_{\text{init}} \\
\alpha(t_2) &= K \cdot \text{config\_producers} \\
\alpha(t_3) &= (K + 1) \cdot \text{heads} \\
\alpha(t_4) &= \alpha(t_6) = K \cdot \text{config\_checks} + 1 \\
\alpha(t_5) &= (K + 1) \cdot \text{wait} \\
\alpha(t_{\text{done}}) &= \text{done}
\end{align*}
\]

\(t_1\) and \(t_2\) are the driving tasks of the problem, asserting one character in \(W\) at a time, and driving the rest of the tasks with \(\text{call\_head}\) and \(\text{call\_check}\). \(t_1\) lays out the initial tape cells, with the separator first followed by \(K\) \(0\) cells. \(t_2\) does \(K\) sequential copies of the unconstrained producer. The producer for each odd numbered \(v_i\) is validated by the \(\text{check}\) steps started by \(t_4\), while the even numbered ones \((i > 0)\) are validated by the \(\text{check}\) tasks from \(t_6\), which were forced to wait one full configuration before starting by \(t_5\). Once a producer validates a cell under the head of a configuration that leads to an accepting state, the \(\text{check\_done}\) operator can be applied, and the rest of the tasks can short circuit via the \(\text{done}\) operator. Thus, \(P\) simultaneously generates and checks a witness \(W\) that \(N\) halts in \(2^k\) steps.

Since \(P\) is solvable iff there is a run of \(N\) that terminates \(2^k\), partially-ordered lifted acyclic HTN planning is \(2\)-\(NEXPTIME\)-complete. Replacing the top level tasks with tail-recursive tasks would encode a strictly space-bound NTM. The only variables used in the encoding were for counting \(K\) repetitions of tasks. Replacing \(K\) with a merely-exponential bound would let us encode \(N\) in a fully propositional problem. Using combinations of either of these modifications (tail-recursive top level tasks or using \(K = 2^k\)) gives the remaining lower bounds of the theorem.

\[\Box\]

7 Conclusions

We proposed a straight-forward extension for propositional HTN planning to a lifted representation that is based upon a function-free first-order logic. We studied how the variables/constants, the (partial) order of tasks, and various variants of recursion interact w.r.t. the complexity of the plan existence problem. Our results have straight-forward implications for other hierarchical planning formalisms, such as hierarchical goal networks (HGNs), that have a direct correspondence with HTN planning (Shivashankar et al. 2012), and for hybrid planning, a framework that fuses HTN planning with partial-order causal-link (POCL) planning (Biundo and Schattenberg 2001).

Apart from giving deeper theoretical insights of the complexity and expressiveness of HTN planning, our work also has implications on the design of future HTN algorithms. For example, the T0PHTN algorithm from Alford et al. (2012) uses a mixture of progression and problem-decomposition, and is able to decide every totally-ordered propositional problem in \(EXPTIME\). However, it also takes exponential space. The progression-based algorithm from the same paper (PHTN) can be made space efficient on the tail-recursive subset of these problems, taking only polynomial space. This leaves an obvious gap in the literature that could be filled with an algorithm that can decide both the classes of problems efficiently.

Some HTN problems can also be solved via compiling them into non-hierarchical planning problems. Alford, Kuter, and Nau (2009) describe such a translation for totally-ordered tail-recursive problems. It should be straightforward to extend this technique to tail-recursive problems with arbitrary ordering. However, our results show that any such translation for partially-ordered problems must yield an exponential blow-up in the general case.

In future work, we want to extend our results to provide tight complexity bounds for hierarchical planning with task insertion (TIHTN planning) – a variant of hierarchical planning that allows to insert tasks into task networks without the need to decompose abstract tasks (Geier and Bercher 2011; Shivashankar et al. 2013). Other interesting problems include extending the \(NP\)-completeness results for both propositional acyclic regular HTN problems (Alford et al. 2014) and for HTN plan verification (Behnke, Höller, and Biundo 2015) into our lifted HTN formalism.

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