V-BLAST in Lattice Reduction and Integer Forcing

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***Abstract*—Lattice-reduction-aided decision-feedback equalization (LRA DFE) and successive integer forcing are MIMO detection schemes which combine the equalization in a suited basis with the principle of successive interference cancellation (SIC). To this end, the reduction algorithm not only has to find a suited basis, but it should also provide an optimized detection order for SIC: the V-BLAST ordering, known to be optimal for conventional DFE. How these two tasks can be solved jointly has so far remained unclear in the literature. In this paper, we describe how the Lenstra-Lenstra-Lovász (LLL) reduction has to be adapted to achieve this aim. Moreover, we propose a weakened variant of the Hermite-Korkine-Zolotareff (HKZ) reduction that optimally solves both tasks jointly. Results obtained from numerical simulations complement the theoretical derivations.**

I. Introduction

For quite some time, multiple-input/multiple-output (MIMO) transmission is a major topic in communication theory. Initially employed for the point-to-point scenario, it has been extended to the multi-user case. The scenario that several users simultaneously transmit their data to one central receiver (uplink) is known as *MIMO multiple-access channel*.

To handle the multi-user interference—instead of a linear channel equalization—*decision-feedback equalization* (DFE) can be applied, incorporating the principle of *successive interference cancellation* (SIC). The *V-BLAST* algorithm [11] optimizes the detection order for SIC—of great relevance in the multi-user case—according to the worst-link performance. Some low-complexity alternatives have been proposed, e.g., [12], but with suboptimal ordering. Nevertheless, the error curves of all these strategies flatten out to diversity order one.

To overcome this diversity limitation, *lattice-reduction-aided* (LRA) equalization has been introduced [14], [10], where the channel is equalized in a *suited basis*. To this end, the channel matrix is split into a unimodular integer-interference matrix and a residual non-integer part, e.g., via the (complex-valued) Lenstra-Lenstra-Lovász (LLL) reduction [4]. Stronger criteria like Hermite-Korkine-Zolotareff (HKZ) or Minkowski (MK) reduction have been employed subsequently [16], [5]—accompanied by a higher complexity.

In LRA linear equalization, the non-integer part is linearly equalized. In LRA DFE [10], where SIC is applied instead, the situation is more complicated: the channel factorization not only has to result in a suited basis, but it should also contain a suited ordering. In the initial papers, e.g., [10], these tasks have been solved in sequence. In the sequel, schemes were proposed to combine these problems [13], [2], [7], all of them resulting in a suboptimal ordering, though.

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Recently, *integer-forcing* (IF) linear equalization [15] has become popular, which performs channel decoding followed by integer interference cancellation over a finite field [3]. IF has relaxed the unimodularity to a full-rank constraint, leading to a factorization via the *successive minima problem* (SMP) [3], [9]. This relaxation turned out be possible for the LRA receiver, too [3]. However, for *successive integer forcing* [8]—the SIC variant of IF—pure finite-field processing is not feasible: to perform SIC, the complex-valued distortions are "recovered" [8] from the finite-field elements after decoding (remapping). This actually results in the philosophy of LRA DFE, i.e., *the same factorization task* is present, cf. [3]. For successive IF, the optimality of the HKZ reduction has been stated [8], though without a direct connection to V-BLAST.

In this paper, based on an alternative interpretation of V-BLAST [6], we transfer its philosophy to LRA DFE (or successive IF). We adapt the LLL reduction to incorporate the V-BLAST ordering and clarify the relation between V-BLAST and the HKZ reduction. Moreover, we show why a weakened variant thereof—called effective HKZ reduction—results in the optimal integer matrix, which is always unimodular. Numerical results are provided that support our considerations.

The paper is structured as follows: Sec. II provides the system model for all variants of DFE. Factorization strategies for conventional DFE—including V-BLAST—are reviewed in Sec. III. In Sec. IV, we discuss how the V-BLAST strategy can be extended to LRA DFE, and Sec. V presents numerical results. The paper is summarized and concluded in Sec. VI.

II. System Model

Throughout the paper, a discrete-time complex-baseband transmission is considered, where variants of DFE are applied.[1] The related system model is depicted in Fig. 1.

K uncoordinated single-antenna transmitters send their data to a joint receiver equipped with $N \geq K$ antennas. In each time step, the transmission is mathematically expressed by

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{n} \; . \tag{1}$$

The users' transmit symbols are denoted by x_k, $k = 1, \dots, K$, in vector notation $\boldsymbol{x} = [x_1, \dots, x_K]^{\mathsf{T}} \in \mathcal{A}^K$. They are drawn from a zero-mean signal constellation $\mathcal{A}$ with variance σ_a^2, i.e., $\sigma_x^2 = \mathrm{E}\{|x_k|^2\} = \sigma_a^2$. We restrict to QAM constellations $\mathcal{A} - o \subset \mathbb{G} = \mathbb{Z} + \mathrm{j}\,\mathbb{Z}$, where $\mathbb{G}$ denotes the *Gaussian integers*, and o a constant offset, cf. [10]. The coefficients

[1]Notation: $\mathrm{E}\{\cdot\}$ denotes expectation. $\boldsymbol{A}^{\mathsf{T}}$ is the transpose and $\boldsymbol{A}^{\mathsf{H}}$ the Hermitian of a matrix $\boldsymbol{A}$. Its left pseudoinverse reads $\boldsymbol{A}^{+} = (\boldsymbol{A}^{\mathsf{H}}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathsf{H}}$, where $\boldsymbol{A}^{+\mathsf{H}} = (\boldsymbol{A}^{+})^{\mathsf{H}}$. If $\boldsymbol{A}$ is square, $\boldsymbol{A}^{-\mathsf{H}} = (\boldsymbol{A}^{-1})^{\mathsf{H}} = (\boldsymbol{A}^{\mathsf{H}})^{-1}$ is valid. $\boldsymbol{I}$ denotes the identity matrix and $\boldsymbol{0}$ the (all-)zero vector or matrix.

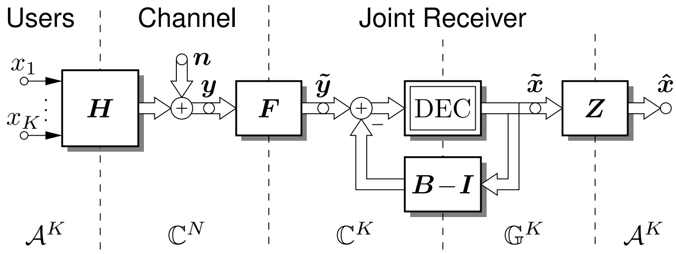

Fig. 1. System model of decision-feedback equalization for the MIMO multiple-access channel, where $\boldsymbol{Z}$ describes a channel transformation.

of the MIMO channel matrix $\boldsymbol{H} \in \mathbb{C}^{N\times K}$ are assumed to be i.i.d. zero-mean unit-variance complex Gaussian. A block-fading channel is considered, i.e., $\boldsymbol{H}$ is constant over a burst of symbols. Additive white Gaussian noise is present at each receive antenna. The noise components are combined into $\boldsymbol{n} = [n_1, \ldots, n_N]^\mathsf{T} \in \mathbb{C}^N$. They are assumed to be i.i.d. zero-mean complex Gaussian with variance σ_n^2. The vector of receive symbols finally reads $\boldsymbol{y} = [y_1, \ldots, y_N]^\mathsf{T} \in \mathbb{C}^N$.

To obtain the receiver matrices, the channel matrix has to be factorized: the *augmented channel matrix* [12] defined by $\boldsymbol{\mathcal{H}} = \left[\begin{smallmatrix}\boldsymbol{H}\\ \sqrt{\zeta}\boldsymbol{I}\end{smallmatrix}\right] \in \mathbb{C}^{(N+K)\times K}$, $\zeta = \sigma_n^2/\sigma_x^2$, is transformed into

$$\boldsymbol{\mathcal{H}}\boldsymbol{Z} = \boldsymbol{\mathcal{F}}^{+}\boldsymbol{B}\ . \tag{2}$$

The matrix $\boldsymbol{Z}$ describes a channel transformation which will be specified subsequently. The transformed channel is split into the either upper or lower triangular matrix $\boldsymbol{B} \in \mathbb{C}^{K\times K}$ with unit main diagonal[2] and the pseudoinverse of the augmented *feedforward matrix* $\boldsymbol{\mathcal{F}} = [\boldsymbol{f}_1^\mathsf{H}, \ldots, \boldsymbol{f}_K^\mathsf{H}]^\mathsf{H} \in \mathbb{C}^{K\times(N+K)}$. As its rows $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_K$ are orthogonal, $\boldsymbol{\mathcal{F}}^+$ has orthogonal columns.

Given (2), the SIC is performed in the following way: Since the cascade $\boldsymbol{\mathcal{F}}\boldsymbol{\mathcal{H}}\boldsymbol{Z} = \boldsymbol{B}$ shapes the interference to have a causal structure (described by $\boldsymbol{B}$), the receive symbols are linearly equalized to $\tilde{\boldsymbol{y}} = \boldsymbol{F}\boldsymbol{y} = [\tilde{y}_1, \ldots, \tilde{y}_K]^\mathsf{T} \in \mathbb{C}^K$. Thereby, the $K \times N$ left part of $\boldsymbol{\mathcal{F}}$ yields the feedforward matrix $\boldsymbol{F}$ according to the minimum mean-square error (MMSE) criterion.[3] Following this, the causal interference is canceled via the feedback part depicted in Fig. 1. To this end, the vector of decoded symbols is initially set to $\tilde{\boldsymbol{x}} = [\tilde{x}_1, \ldots, \tilde{x}_K]^\mathsf{T} = \boldsymbol{0}$. Then, $\tilde{x}_k = \mathrm{DEC}\{\tilde{y}_k - \boldsymbol{b}_k\tilde{\boldsymbol{x}}\}$ is calculated successively, where $\boldsymbol{b}_k$ denotes the k^{th} row of $\boldsymbol{B} = [\boldsymbol{b}_1^\mathsf{H}, \ldots, \boldsymbol{b}_K^\mathsf{H}]^\mathsf{H}$ and $\mathrm{DEC}\{\cdot\}$ the decoding operation.[4] Finally, the channel transformation is reversed, resulting in the vector of estimated transmit symbols $\hat{\boldsymbol{x}} = \boldsymbol{Z}\tilde{\boldsymbol{x}} = [\hat{x}_1, \ldots, \hat{x}_K]^\mathsf{T} \in \mathcal{A}^K$.

The feedforward part is the crucial point regarding transmission performance. It determines the mean-square error $\sigma_{e,k}^2 = \sigma_n^2\,\|\boldsymbol{f}_k^\mathsf{H}\|_2^2$ of each symbol $\tilde{y}_k$, $k = 1, \ldots, K$, before decoding (neglecting error propagation from SIC). Hence, the row norms of $\boldsymbol{\mathcal{F}}$ should be minimized,[5] i.e., $\|\boldsymbol{f}_k^\mathsf{H}\|_2^2 \to \min$.

[2]If $\boldsymbol{B}$ has a lower triangular structure, the SIC is performed in the order $k = 1, \ldots, K$. Otherwise, the order has to be reversed to $k = K, \ldots, 1$.

[3]The zero-forcing (ZF) variant is simply obtained by setting $\zeta = 0$ in (2). This is equivalent to a factorization according to $\boldsymbol{H}\boldsymbol{Z} = \boldsymbol{F}^+\boldsymbol{B}$.

[4]In case of uncoded transmission, $\mathrm{DEC}\{\cdot\}$ is a simple quantization to $\mathbb{G}$.

[5]In the literature, e.g., [11], [2], the *equivalent* task $1/\sigma_{e,k}^2 \to \max$ is often considered, i.e., maximizing the signal-to-interference-plus-noise ratio.

III. Conventional DFE and V-BLAST

We briefly review important aspects of conventional DFE and the V-BLAST strategy, where the only degree of freedom is the detection order for SIC. To this end, in (2), $\boldsymbol{Z}$ is restricted to be a permutation matrix, i.e., $\boldsymbol{Z} \in \mathcal{P}_{K\times K}$, where $\mathcal{P}_{K\times K}$ denotes the set of permutation matrices[6] (matrices with only a single one per row and column) of size $K \times K$.

The required equalization matrices (including optimized ordering) can be obtained via the *sorted QR factorization*

$$\boldsymbol{G}\boldsymbol{T} = \boldsymbol{Q}\boldsymbol{R}\ , \tag{3}$$

where $\boldsymbol{G} = [\boldsymbol{g}_1, \ldots, \boldsymbol{g}_K]$ is the matrix to be decomposed and $\boldsymbol{T} \in \mathcal{P}_{K\times K}$. $\boldsymbol{Q} = [\boldsymbol{q}_1, \ldots, \boldsymbol{q}_K]$ is the Gram-Schmidt orthogonal matrix of $\boldsymbol{G}\boldsymbol{T}$ with orthogonal columns and $\boldsymbol{R}$ is an *upper* triangular $K \times K$ matrix with unit main diagonal.

This factorization can be realized via the Gram-Schmidt orthogonalization (GSO) with pivoting according to Algorithm 1 ($\mathit{reduce} = \text{false}$). In particular, it is a greedy strategy where the columns of $\boldsymbol{Q}$ and the rows of $\boldsymbol{R}$ are calculated successively, initially setting $\boldsymbol{Q} = \boldsymbol{G}$. In each step $k = 1, \ldots, K$, the sorting (k^{th} column of $\boldsymbol{T}$) is chosen in such a way that the k^{th} column of $\boldsymbol{Q}$ has the minimum norm among the columns $\boldsymbol{q}_k, \ldots, \boldsymbol{q}_K$ after step $k-1$ (Lines 3–6 in Algorithm 1). Subsequently, the remaining columns $\boldsymbol{q}_{k+1}, \ldots, \boldsymbol{q}_K$ are projected onto the orthogonal complement of the chosen vector $\boldsymbol{q}_k$ (Lines 7–10). Hence, $\boldsymbol{q}_k$ is a shortest vector of the projection onto the orthogonal complement of $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{k-1}$, i.e., the column norms of $\boldsymbol{Q}$ are *greedily minimized*, cf., e.g., [2], [7].

A. Detection Order of Sorted QR Decomposition

The GSO with pivoting has initially been proposed to straightly factorize $\boldsymbol{G} = \boldsymbol{\mathcal{H}}$ according to (3), i.e., $\boldsymbol{\mathcal{H}}\boldsymbol{T} = \boldsymbol{Q}\boldsymbol{R}$. This strategy is known as *sorted QR decomposition* [12]; we abbreviate the related sorting with "SQRD". It directly corresponds to (2), with $\boldsymbol{Z} = \boldsymbol{T}$, $\boldsymbol{\mathcal{F}}^+ = \boldsymbol{Q}$, and $\boldsymbol{B} = \boldsymbol{R}$.

Unfortunately, SQRD has a significant drawback: since the column norms of $\boldsymbol{Q} = \boldsymbol{\mathcal{F}}^+$ are greedily minimized in order $k = 1, \ldots, K$, the related row norms $\|\boldsymbol{f}_k^\mathsf{H}\|_2^2$ are maximized.[7]

B. Detection Order of V-BLAST

SQRD has been proposed as a low-complexity alternative to classical V-BLAST [11]. In the V-BLAST algorithm, the rows of $\boldsymbol{\mathcal{F}}$ are successively calculated via the inversion of an "updated" version of $\boldsymbol{\mathcal{H}}$. In particular, in the k^{th} step, the $k-1$ columns of $\boldsymbol{\mathcal{H}}$ which correspond to the already incorporated data symbols are set to zero. The shortest row of its inverse is taken as the k^{th} row of $\boldsymbol{\mathcal{F}}$; $\boldsymbol{Z}$ and $\boldsymbol{B}$ are updated accordingly.

In [6], it was shown that the classical V-BLAST procedure is equivalent to a GSO with pivoting, if $\boldsymbol{G} = \boldsymbol{\mathcal{H}}^{+\mathsf{H}}$ instead of $\boldsymbol{G} = \boldsymbol{\mathcal{H}}$ is decomposed according to (3). Specifically, this corresponds to the factorization

$$\boldsymbol{\mathcal{H}}^{+\mathsf{H}}\boldsymbol{Z}^{-\mathsf{H}} = \boldsymbol{\mathcal{F}}^\mathsf{H}\boldsymbol{B}^{-\mathsf{H}}\ , \tag{4}$$

[6]If $\boldsymbol{Z}$ is a permutation matrix, $\boldsymbol{Z} = \boldsymbol{Z}^{-\mathsf{H}}$ is valid.

[7]Although this strategy is generally not optimal, an acceptable performance is reached in practice [12], [2] as the SIC is conducted in reversed order $k = K, \ldots, 1$ ($\boldsymbol{B} = \boldsymbol{R}$ is *upper* triangular).

Algorithm 1 GSO with pivoting (*reduce* = false), optionally with CLLLdeep reduction (*reduce* = true).

$[\boldsymbol{Q},\ \boldsymbol{R},\ \boldsymbol{T}] = \mathrm{GSO\_CLLLD}(\boldsymbol{G},\ \mathit{reduce},\ \delta)$ ▷ $\boldsymbol{G} = [\boldsymbol{g}_1, \ldots, \boldsymbol{g}_K]$

$\boldsymbol{Q} = \boldsymbol{G},\ \boldsymbol{R} = \boldsymbol{I},\ \boldsymbol{T} = \boldsymbol{I},\ k = 1$ ▷ $\boldsymbol{R} \in \mathbb{C}^{K\times K}$, $\boldsymbol{T} \in \mathbb{G}^{K\times K}$
while $k \le K$ **do**
  $\mu^{(k)} = \mathrm{argmin}_{m=k,\ldots,K} \|\boldsymbol{q}_m\|_2^2$ ▷ $\boldsymbol{Q} = [\boldsymbol{q}_1, \ldots, \boldsymbol{q}_K]$
  if $\mu^{(k)} \neq k$ **then** ▷ insert column if necessary
    $[\boldsymbol{Q},\ \boldsymbol{R},\ \boldsymbol{T}] = \mathrm{INSERTION}(\mu^{(k)},\ k,\ \boldsymbol{Q},\ \boldsymbol{R},\ \boldsymbol{T})$
  end if
  for $l = k+1, \ldots, K$ **do** ▷ GSO
    $r_{k,l} = \boldsymbol{q}_k^{\mathsf{H}} \boldsymbol{q}_l / \|\boldsymbol{q}_k\|_2^2$ ▷ k^{th} row and l^{th} column of $\boldsymbol{R}$
    $\boldsymbol{q}_l = \boldsymbol{q}_l - r_{k,l}\, \boldsymbol{q}_k$ ▷ projection onto orth. compl. of $\boldsymbol{q}_k$
  end for
  if *reduce* **then** ▷ CLLLdeep reduction if desired
    $[k,\ \boldsymbol{Q},\ \boldsymbol{R},\ \boldsymbol{T}] = \mathrm{CLLLD\_REDUCE}(k,\ \delta,\ \boldsymbol{Q},\ \boldsymbol{R},\ \boldsymbol{T})$
  end if
  $k = k + 1$ ▷ next step
end while

$[\boldsymbol{Q},\ \boldsymbol{R},\ \boldsymbol{T}] = \mathrm{INSERTION}(i,\ j,\ \boldsymbol{Q},\ \boldsymbol{R},\ \boldsymbol{T})$

insert column i between column $j-1$ and j in $\boldsymbol{Q}$, the upper $j-1$ rows of $\boldsymbol{R}$, and $\boldsymbol{T}$
delete old column i in $\boldsymbol{Q}$, the upper $j-1$ rows of $\boldsymbol{R}$, and $\boldsymbol{T}$

$[k,\ \boldsymbol{Q},\ \boldsymbol{R},\ \boldsymbol{T}] = \mathrm{CLLLD\_REDUCE}(k,\ \delta,\ \boldsymbol{Q},\ \boldsymbol{R},\ \boldsymbol{T})$

for $l = k-1, k-2, \ldots, 1$ **do** ▷ size reduction of k^{th} column
  if $|\mathrm{Re}\{r_{l,k}\}| > 1/2$ **or** $|\mathrm{Im}\{r_{l,k}\}| > 1/2$ **then**
    $\boldsymbol{t}_k = \boldsymbol{t}_k - \lfloor r_{l,k} \rceil \cdot \boldsymbol{t}_l$ ▷ $\boldsymbol{T} = [\boldsymbol{t}_1, \ldots, \boldsymbol{t}_K]$
    $\boldsymbol{r}_k = \boldsymbol{r}_k - \lfloor r_{l,k} \rceil \cdot \boldsymbol{r}_l$ ▷ $\boldsymbol{R} = [\boldsymbol{r}_1, \ldots, \boldsymbol{r}_K]$
  end if
end for
for $m = 1, \ldots, k-1$ **do** ▷ check if deep insertion needed
  if $\sum_{l=m}^{k} |r_{l,k}|^2 \|\boldsymbol{q}_l\|_2^2 < \delta \cdot \|\boldsymbol{q}_m\|_2^2$ **then**
    for $l = K, K-1, \ldots, m+1$ **do**
      $\boldsymbol{q}_l = \boldsymbol{q}_l + \sum_{i=m}^{\min(k,l-1)} r_{i,l}\, \boldsymbol{q}_i$ ▷ remove projections
    end for
    $[\boldsymbol{Q},\ \boldsymbol{R},\ \boldsymbol{T}] = \mathrm{INSERTION}(k,\ m,\ \boldsymbol{Q},\ \boldsymbol{R},\ \boldsymbol{T})$
    for $l = m+1, \ldots, K$ **do** ▷ GSO
      $r_{m,l} = \boldsymbol{q}_m^{\mathsf{H}} \boldsymbol{q}_l / \|\boldsymbol{q}_m\|_2^2$
      $\boldsymbol{q}_l = \boldsymbol{q}_l - r_{m,l}\, \boldsymbol{q}_m$ ▷ new projection
    end for
    $k = m$ ▷ go back to m^{th} step
    break
  end if
end for

i.e., Algorithm 1 results in $\boldsymbol{T} = \boldsymbol{Z}^{-\mathsf{H}}$, $\boldsymbol{Q} = \boldsymbol{\mathcal{F}}^{\mathsf{H}}$, and $\boldsymbol{R} = \boldsymbol{B}^{-\mathsf{H}}$. Briefly speaking, instead of inverting the channel K times as in classical V-BLAST, it is inverted once in the beginning. Thereby, the projections of the GSO correspond to the strategy of inverting the updated versions of $\boldsymbol{\mathcal{H}}$ with erased columns. For the mathematical details, we refer to [6]. Most important, the GSO with pivoting then greedily minimizes the columns of $\boldsymbol{Q} = \boldsymbol{\mathcal{F}}^{\mathsf{H}}$, and hence the row norms $\|\boldsymbol{f}_k^{\mathsf{H}}\|_2^2$ in detection order $k = 1, \ldots, K$ according to the V-BLAST philosophy (since $\boldsymbol{B} = \boldsymbol{R}^{-\mathsf{H}}$ is now *lower* triangular).

In the initial V-BLAST paper [11], it has been proven that this sorting strategy minimizes the maximum norm of $\boldsymbol{\mathcal{F}}$, i.e.,

$$\boldsymbol{Z}_{\text{V-BLAST}} = \underset{\boldsymbol{Z}^{-\mathsf{H}} \in \mathcal{P}_{K\times K}}{\mathrm{argmin}} \ \max_{k=1,\ldots,K} \|\boldsymbol{f}_k^{\mathsf{H}}\|_2^2 \,. \tag{5}$$

Hence, V-BLAST sorting enables an optimal worst-link performance, usually dominating the overall system performance.

Algorithm 2 Lattice-reduction-aided variant of GSO or alternatively effective CHKZ lattice basis reduction.

$[\boldsymbol{Q},\ \boldsymbol{R},\ \boldsymbol{T}] = \mathrm{GSO\_LRA}(\boldsymbol{G})$ ▷ $\boldsymbol{G} = [\boldsymbol{g}_1, \ldots, \boldsymbol{g}_K]$

$\boldsymbol{Q} = \boldsymbol{G},\ \boldsymbol{R} = \boldsymbol{I},\ \boldsymbol{T} = \boldsymbol{I},\ k = 1$ ▷ $\boldsymbol{R} \in \mathbb{C}^{K\times K}$, $\boldsymbol{T} \in \mathbb{G}^{K\times K}$
while $k \le K$ **do**
  $\boldsymbol{\mu}^{(k)} = \mathrm{SHORTEST\_VECTOR}(k,\ \boldsymbol{Q})$
  if $\|\boldsymbol{\mu}^{(k)}\|_2^2 \neq \|\boldsymbol{q}_k\|_2^2$ **then** ▷ basis update if necessary
    $[\boldsymbol{Q},\ \boldsymbol{R},\ \boldsymbol{T}] = \mathrm{BASIS\_UPDATE}(\boldsymbol{\mu}^{(k)},\ k,\ \boldsymbol{Q},\ \boldsymbol{R},\ \boldsymbol{T})$
  end if
  for $l = k+1, \ldots, K$ **do** ▷ GSO
    $r_{k,l} = \boldsymbol{q}_k^{\mathsf{H}} \boldsymbol{q}_l / \|\boldsymbol{q}_k\|_2^2$ ▷ k^{th} row and l^{th} column of $\boldsymbol{R}$
    $\boldsymbol{q}_l = \boldsymbol{q}_l - r_{k,l}\, \boldsymbol{q}_k$ ▷ projection onto orth. compl. of $\boldsymbol{q}_k$
  end for
  $k = k + 1$
end while

$[\boldsymbol{\mu}^{(k)}] = \mathrm{SHORTEST\_VECTOR}(k,\ \boldsymbol{Q})$

$\boldsymbol{Q}^{(k)} = [\boldsymbol{q}_k, \ldots, \boldsymbol{q}_K]$
find a shortest vector $\boldsymbol{\mu}^{(k)}$ in the lattice $\boldsymbol{\Lambda}(\boldsymbol{Q}^{(k)})$

$[\boldsymbol{Q},\ \boldsymbol{R},\ \boldsymbol{T}] = \mathrm{BASIS\_UPDATE}(\boldsymbol{\mu}^{(k)},\ k,\ \boldsymbol{Q},\ \boldsymbol{R},\ \boldsymbol{T})$

update $\boldsymbol{q}_k, \ldots, \boldsymbol{q}_K$, the upper $k-1$ columns of $\boldsymbol{r}_k, \ldots, \boldsymbol{r}_K$, and $\boldsymbol{t}_k, \ldots, \boldsymbol{t}_K$ in such a way that $\boldsymbol{q}_k = \boldsymbol{\mu}^{(k)}$ and the lattice $\boldsymbol{\Lambda}(\boldsymbol{QR}) = \boldsymbol{\Lambda}(\boldsymbol{G})$ is preserved

IV. Lattice-Reduction-Aided DFE and V-BLAST

We extend our considerations to *lattice-reduction-aided* DFE [10], [2]. In LRA equalization, the (augmented) channel is interpreted as the generator matrix $\boldsymbol{G} = \boldsymbol{\mathcal{H}}$ of a complex lattice $\boldsymbol{\Lambda}(\boldsymbol{G})$ [9, Eq. (1)]. Via lattice basis reduction, the equalization is performed in a *reduced basis* $\boldsymbol{\mathcal{H}}_{\mathrm{r}} = \boldsymbol{\mathcal{H}}\boldsymbol{Z}$, where $\boldsymbol{Z} \in \mathbb{G}^{K\times K}$ is a unimodular integer matrix ($|\det(\boldsymbol{Z}) = 1|$) describing the change of basis. The unimodularity ensures the existence of a (Hermitian) integer inverse[8] $\boldsymbol{Z}^{-\mathsf{H}} \in \mathbb{G}^{K\times K}$.

We can divide the most important reduction criteria (cf., e.g., [16], [9]) into two categories: the ones which directly minimize the norms of the reduced basis $\boldsymbol{G}_{\mathrm{r}}$, particularly the MK reduction and the (possibly non-unimodular) solution to the SMP—and the criteria which consider its QR factorization $\boldsymbol{G}_{\mathrm{r}} = \boldsymbol{GT} = \boldsymbol{QR}$, especially the CLLL or CHKZ reduction.[9]

In analogy to the sorted QR factorization (3), it is quite evident that the QR-based reduction schemes directly yield the matrices required for DFE—with the difference that $\boldsymbol{T} \in \mathbb{G}^{K\times K}$ is a unimodular integer matrix instead of a permutation matrix. In particular, following the strategy of the sorted QR decomposition by reducing $\boldsymbol{G} = \boldsymbol{\mathcal{H}}$, we obtain the channel factorization (2). Following the V-BLAST philosophy via $\boldsymbol{G} = \boldsymbol{\mathcal{H}}^{+\mathsf{H}}$ instead, a factorization according to (4) is present. In accordance with Sec. III, the lattice $\boldsymbol{\Lambda}(\boldsymbol{\mathcal{H}}^{+\mathsf{H}})$—called the *dual lattice* [6] of $\boldsymbol{\Lambda}(\boldsymbol{\mathcal{H}})$—is considered and the lattice-reduction algorithm directly operates on $\boldsymbol{Q} = \boldsymbol{\mathcal{F}}^{\mathsf{H}}$.

In LRA DFE, *two* tasks have to be solved: i) $\boldsymbol{Z}$ should describe the transformation to a "suited basis"; ii) $\boldsymbol{Z}$ should

[8] Recently, this unimodularity constraint has been relaxed [15], [3]. Indeed, lattice decoding is still possible if $\boldsymbol{Z}$ is *any* matrix with full-rank (Hermitian) integer inverse, i.e., $\boldsymbol{Z}^{-\mathsf{H}} \in \mathbb{G}^{K\times K}$ and $\mathrm{rank}(\boldsymbol{Z}^{-\mathsf{H}}) = K$. To this end, the factorization approach (4) is compulsory, cf. [3].

[9] Since complex lattices are present, we consider the complex variants thereof [9], which are indicated by a "C" in the abbreviations.

TABLE I
OVERVIEW ON THE SORTING AND OPTIMALITY OF DIFFERENT FACTORIZATION APPROACHES FOR DECISION-FEEDBACK EQUALIZATION.

	Based on $\boldsymbol{\mathcal{H}}$		Based on $\boldsymbol{\mathcal{H}}^{+\mathsf{H}}$	
	Sort.	Opt.	Sorting	Optimality
GSO with pivoting	SQRD	✗	V-BLAST	$\underset{\boldsymbol{Z}^{-\mathsf{H}}\in\mathcal{P}_{K\times K}}{\operatorname{argmin}} \max_k \lVert\boldsymbol{f}_k^{\mathsf{H}}\rVert_2^2$
(eff.) CHKZ	SQRD	✗	V-BLAST	$\underset{\substack{\boldsymbol{Z}^{-\mathsf{H}}\in\mathbb{G}^{K\times K}\\ \operatorname{rank}(\boldsymbol{Z}^{-\mathsf{H}})=K}}{\operatorname{argmin}} \max_k \lVert\boldsymbol{f}_k^{\mathsf{H}}\rVert_2^2$
CLLLdeep, $\delta=1$	SQRD	✗	V-BLAST	✗
CLLLdeep, $\delta<1$	✗	✗	✗	✗
(eff.) CLLL, CMK	✗	✗	✗	✗
CSMP	not possible		✗	✗

include an optimized sorting for SIC. Since these tasks seem to be contrary, in the initial papers, e.g., [10], they have been solved in sequence: after lattice basis reduction, an additional sorted QR factorization based on $\boldsymbol{G}_{\mathrm{r}}$ (e.g., V-BLAST) has been performed, cascading both transformation matrices. This may not only cause redundant computations, but is obviously a handicap for optimally solving both tasks. In the literature, combined factorization approaches have been proposed, e.g., in [13], [2], [7], though, not resulting in the (optimal) V-BLAST sorting. Below, we show how this is achieved.[10]

A. LLL Reduction and V-BLAST Sorting

First, we consider the most famous type of reduction: the (C)LLL reduction [4]. A basis $\boldsymbol{G}=\boldsymbol{Q}\boldsymbol{R}$ is CLLL-reduced, if $\boldsymbol{R}=[r_{l,k}]$, with $r_{k,k}=1$, is *size-reduced* according to

$$|\mathrm{Re}\{r_{l,k}\}|\leq 0.5\cap|\mathrm{Im}\{r_{l,k}\}|\leq 0.5\,,\quad 1\leq l<k\leq K\,,\tag{6}$$

and if the *Lovász condition* with quality parameter $0.5<\delta\leq 1$

$$\sum\nolimits_{l=k-1}^{k}|r_{l,k}|^2\|\boldsymbol{q}_l\|_2^2\geq\delta\cdot\|\boldsymbol{q}_{k-1}\|_2^2\,,\quad 1<k\leq K\,,\tag{7}$$

is fulfilled.[11] In order to achieve a CLLL-reduced basis with optimized sorting in accordance with Sec. III, each column $\boldsymbol{q}_k$, $k=1,\ldots,K$, additionally has to constitute a shortest vector of the projection onto the orthogonal complement of $\boldsymbol{q}_1,\ldots,\boldsymbol{q}_{k-1}$. Unfortunately, the Lovász condition only compares column $\boldsymbol{q}_k$ with $\boldsymbol{q}_{k-1}$; even if $\delta=1$ (optimal CLLL reduction [9]) it is too weak to keep the shortest-vector property after the size reduction [4] of $r_{k-1,k}$. This problem has been solved in [7]: the Lovász condition is "extended" to all previous positions $m=1,\ldots,k-1$, i.e.,

$$\sum\nolimits_{l=m}^{k}|r_{l,k}|^2\|\boldsymbol{q}_l\|_2^2\geq\delta\cdot\|\boldsymbol{q}_m\|_2^2\,,\quad 1\leq m<k\leq K\,,\tag{8}$$

is demanded, requiring a full size reduction. If $\delta=1$ is chosen, (8) turns out to be the sorting condition of the GSO with pivoting, as shown in [7] along with a proof of convergence. Hence, the shortest-vector property is kept.

The condition (8) is actually known from the *(C)LLL with deep insertions* (CLLLdeep), which has been proposed for joint factorization earlier [2], [7]. The GSO with pivoting can easily be extended to perform a CLLLdeep reduction, cf. Algorithm 1 ($reduce$ = true): following the k^{th} projection of the GSO, the related size reduction (Lines 1–6 in CLLD_REDUCE) is performed. Then, condition (8) is checked for all previous columns $\boldsymbol{q}_1,\ldots,\boldsymbol{q}_{k-1}$. If it is violated at index m, i.e., a shorter vector is found, all previously calculated projections up to this position are removed. After the insertion of this shorter vector at index m, its projection is calculated. The GSO finally continues with step $k=m+1$.

In [7], the authors concluded that the optimal CLLLdeep reduction ($\delta=1$) not only results in a CLLL-reduced basis of $\boldsymbol{\Lambda}(\boldsymbol{\mathcal{H}})$, but that $\boldsymbol{Z}$ also includes the SQRD ordering. Taking advantage of the alternative interpretation of V-BLAST, we can transfer this to the dual lattice $\boldsymbol{\Lambda}(\boldsymbol{\mathcal{H}}^{+\mathsf{H}})$: the optimal CLLLdeep reduction of $\boldsymbol{G}=\boldsymbol{\mathcal{H}}^{+\mathsf{H}}$ always results in an integer matrix inherently containing the V-BLAST ordering.

B. LRA V-BLAST and its Relation to HKZ Reduction

Nevertheless, the optimal CLLLdeep reduction only ensures that the CLLL-reduced basis fulfills the sorting condition. It does not guarantee that $\boldsymbol{Z}\in\mathbb{G}^{K\times K}$ optimizes the worst-link performance in the V-BLAST sense in analogy to (5).

To achieve this aim, we transfer the greedy philosophy of the GSO with pivoting to lattices as shown in Algorithm 2: instead of choosing a shortest vector of $\boldsymbol{Q}^{(k)}=[\boldsymbol{q}_k,\ldots,\boldsymbol{q}_K]$ after step $k-1$, a *shortest vector in the lattice* $\boldsymbol{\Lambda}(\boldsymbol{Q}^{(k)})$ is taken (Line 3). This can be implemented via the sphere decoder [1] using the real-valued representation of a lattice [10], or via [5, Algorithm 2]. Since this vector is chosen to be the k^{th} column of $\boldsymbol{Q}$, the basis of the lattice has to be updated accordingly (Line 5), e.g., via the transformation strategy in [16]. Then, in accordance with the GSO with pivoting, the columns $\boldsymbol{q}_{k+1},\ldots,\boldsymbol{q}_K$ are projected onto the orthogonal complement of $\boldsymbol{q}_k$, and the algorithm continues with step $k+1$.

This strategy, an "LRA GSO", is closely related to the CHKZ reduction: a basis $\boldsymbol{G}=\boldsymbol{Q}\boldsymbol{R}$ is CHKZ-reduced [5], if i) $\boldsymbol{R}$ is size-reduced according to (6), and if ii) $\boldsymbol{q}_k$ is a shortest vector in $\boldsymbol{\Lambda}(\boldsymbol{G}^{(k)})$, $\boldsymbol{G}^{(k)}=[\boldsymbol{0},\ldots,\boldsymbol{0},\boldsymbol{q}_k,\ldots,\boldsymbol{q}_K]\boldsymbol{R}$, for $k=1,\ldots,K$. The bases obtained from Algorithm 2 turn out to fulfill the second condition: as $\boldsymbol{R}$ is gradually obtained in Algorithm 2, $\boldsymbol{Q}^{(k)}\widehat{=}\boldsymbol{G}^{(k)}$ after step $k-1$. In contrast, the size-reduction property is not present, which is actually not of relevance since the size reduction does not operate on $\boldsymbol{Q}$ (cf. effective CLLL). Moreover—different from (7)—condition ii) does not depend on it. We call this strategy of skipping the ineffective size reduction *effective (eff.) CHKZ reduction*.

If we reduce $\boldsymbol{G}=\boldsymbol{\mathcal{H}}$ with Algorithm 2, it is quite evident that $\boldsymbol{T}=\boldsymbol{Z}$ contains the SQRD ordering. If $\boldsymbol{G}=\boldsymbol{\mathcal{H}}^{+\mathsf{H}}$ is taken instead, we not only obtain the V-BLAST sorting, but the columns of $\boldsymbol{T}=\boldsymbol{Z}^{-\mathsf{H}}$ are greedily chosen to minimize the columns of $\boldsymbol{\mathcal{F}}^{\mathsf{H}}$. Thus, the reduced basis has an optimal worst-link performance in analogy to (5). As the shortest-vector

[10] Evidently, the successive minimization of the norms of $\boldsymbol{G}_{\mathrm{r}}$ (CMK/CSMP) is in contradiction to this task. We hence restrict to the QR-based criteria.

[11] The size reduction has no direct impact on LRA DFE, as it doesn't operate on $\boldsymbol{Q}$ (cf. Lines 1–6 of CLLLD_REDUCE in Algorithm 1). Though, it is needed for the Lovász condition (7), where the size reduction of the elements of $\boldsymbol{R}$ above its main diagonal suffices, called *effective (C)LLL reduction* [7].

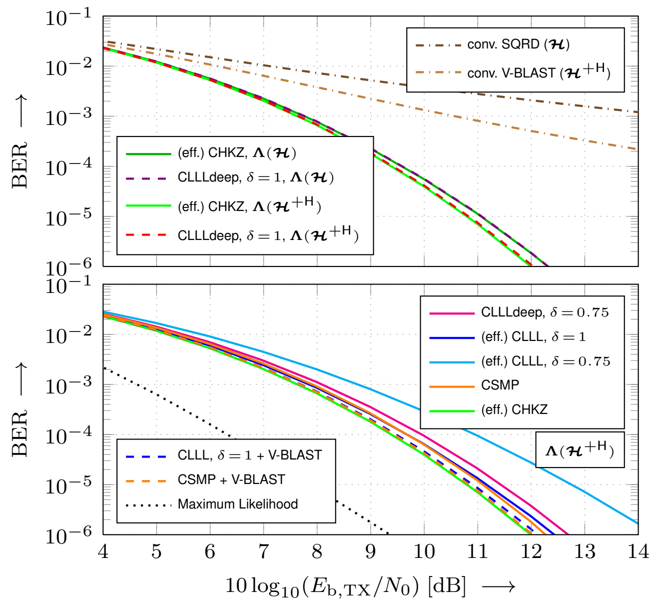

Fig. 2. BER of (LRA) MMSE DFE for $N = K = 8$ and uncoded 16QAM (Gray labeling) in dependency of $E_{\mathrm{b,TX}}/N_0$ in dB. Variation of the factorization strategy. Top: comparison of SQRD and V-BLAST sorting ($\mathcal{H}$ vs. $\mathcal{H}^{+\mathsf{H}}$). Bottom: comparison of CHKZ reduction with other strategies, $\Lambda(\mathcal{H}^{+\mathsf{H}})$.

transformations in Algorithm 2 are changes of basis, $\boldsymbol{Z}^{-\mathsf{H}}$ is unimodular—the relaxation $\mathrm{rank}(\boldsymbol{Z}^{-\mathsf{H}}) = K$ is not required.

The optimality of the (full) CHKZ reduction of $\boldsymbol{G} = \mathcal{H}^{+\mathsf{H}}$ has recently been stated for successive IF in [8]. Instead of deriving the optimality with a discussion on the relations to V-BLAST as above, the authors mathematically proved that

$$\boldsymbol{Z}_{\mathsf{CHKZ}} = \underset{\substack{\boldsymbol{Z}^{-\mathsf{H}} \in \mathbb{G}^{K \times K} \\ \mathrm{rank}(\boldsymbol{Z}^{-\mathsf{H}}) = K}}{\mathrm{argmin}} \; \max_{k=1,\ldots,K} \|\boldsymbol{f}_k^{\mathsf{H}}\|_2^2 \, . \tag{9}$$

The proof is lead in analogy to one of (5) in [11]. In Table I, the properties of all mentioned strategies are summarized.

V. Numerical Results

We finally provide numerical results for the high-diversity case $N = K = 8$. In Fig. 2, the average bit-error rate (BER) of uncoded 16QAM transmission is shown over the signal-to-noise ratio (SNR) as TX-side energy per bit over the noise power density $E_{\mathrm{b,TX}}/N_0 = \sigma_x^2/(\sigma_n^2 \log_2(16))$. It has been averaged over a large number of channel/noise realizations.

First, we consider Fig. 2 Top, where we compare the SQRD with the V-BLAST sorting strategy ($\mathcal{H}$ vs. $\mathcal{H}^{+\mathsf{H}}$). Considering conventional DFE, both curves flatten out to diversity one, but V-BLAST achieves a gain of several dB in the high-SNR regime. In contrast, all curves of LRA DFE exhibit diversity order eight. Thereby, the right choice of the lattice is more important than the reduction criterion: whereas the dual lattice enables a gain of 0.2–0.3 dB, the CLLLdeep reduction of $\mathcal{H}^{+\mathsf{H}}$ with $\delta = 1$ is already almost optimal ((eff.) CHKZ, $\mathcal{H}^{+\mathsf{H}}$).

In Fig. 2 Bottom, we restrict to $\Lambda(\mathcal{H}^{+\mathsf{H}})$ and compare the (eff.) CHKZ reduction—which is 2.5–3 dB away from maximum-likelihood detection via the sphere decoder [1]—to other approaches with non-optimal sorting. In particular, the CLLLdeep reduction with standard parameter $\delta = 0.75$ performs about 0.5 dB worse than the one with $\delta = 1$, but is still significantly better than the (eff.) CLLL with $\delta = 0.75$. The unsorted QR factorization of the non-unimodular solution to the CSMP—optimal for LRA linear equalization—performs about the same like the CLLL with $\delta = 1$. When cascading both approaches with a V-BLAST (re)sorting, a gain in SNR is achieved, but the optimal integer matrix is not always obtained.

VI. Summary and Conclusion

We have discussed how the philosophy of V-BLAST can be transferred to LRA DFE (or successive IF). Via the dual-lattice approach, the optimal CLLLdeep-reduced basis inherently contains the optimal ordering for SIC. Besides, a weakened CHKZ reduction—called effective CHKZ reduction—achieves the optimal worst-link performance among all full-rank integer transformations. Simulations have revealed that the optimal CLLLdeep and (eff.) CHKZ reduction are almost equivalent.

Future research could deal with complexity comparisons or the dualization to the MIMO broadcast channel, cf. [10], [2].

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