On a conditioned Brownian motion and a maximum principle on the disk

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Abstract

Bounds for the $3G$-expression $\int_\Omega G(x,z)\,G(z,y)\,dz/G(x,y)$ play a fundamental role in potential theory. Here $G(x,y)$ is the Green function for the Laplace problem with zero Dirichlet boundary conditions on Ω. The $3G$-formula equals $\mathbb{E}_x^y(\tau_\Omega)$, the expected lifetime for a Brownian motion starting in $x \in \bar{\Omega}$, that is killed on exiting Ω and conditioned to converge to and to be stopped at $y \in \bar{\Omega}$. Although it was shown by probabilistic methods for bounded (simply connected) 2d-domains that if $x \in \partial\Omega$ then the supremum of $y \mapsto \mathbb{E}_x^y(\tau_\Omega)$ is assumed for some y at the boundary, the analogous question remained open for x in the interior. Here we are able to give an answer in the case that $B \subset \mathbb{R}^2$ is the unit disk. The dependence of this quantity on the positions of x and y is investigated and it is shown that indeed $\mathbb{E}_x^y(\tau_B)$ is maximized on $\bar{B}^2$ by opposite boundary points. The result gives also an answer to a number of questions related to the best constant for the positivity preserving property of some elliptic systems. In particular it confirms a relation with a 'sum of inverse eigenvalues' that was conjectured in [11].

1 Introduction

Let $B = \left\{x \in \mathbb{R}^2 : |x| < 1\right\}$ denote the unit disk and set

$$G_B(x,y) = \frac{1}{4\pi}\log\left(\frac{|x|^2\,|y|^2 - 2x\cdot y + 1}{|x|^2 - 2x\cdot y + |y|^2}\right) \text{ for } x,y \in B.$$

This function G_B is the Green function for

$$\begin{cases} -\Delta u = f & \text{in } B,\\ u = 0 & \text{on } \partial B, \end{cases} \tag{1}$$

that is, the solution of (1) is given by $u(x) = \int_B G_B(x,y)\,f(y)\,dy$. We will show that for every $y \in B$ the function $x \mapsto H(x,y)$ ($\equiv \mathbb{E}_x^y(\tau_B)$ for Brownian motion normalized for $-\Delta$) given by

$$H(x,y) = \int_B \frac{G_B(x,z)\,G_B(z,y)}{G_B(x,y)}dz \tag{2}$$

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is increasing away from y along the hyperbolic geodesics and along the curves of a complementary family. See Theorem 1 and Figure 1 below. As a consequence we will find that $x \mapsto H(x,y)$ has no interior maximum and we will even pinpoint the location of the maximum at the boundary.

Our aim in studying this problem was to supply an answer to some questions left open in [2], [9] and in [10], [11]. After explaining the background we will come back to this in section 1.3.

1.1 The link between analysis and probability

The model problem for the positivity preserving property of systems of second order elliptic boundary value problems that are coupled in a noncooperative way is

$$\begin{cases} -\Delta u = f - \lambda v & \text{in } \Omega, \\ -\Delta v = f & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where Ω is a bounded set in $\mathbb{R}^n$ and $\lambda \in \mathbb{R}^+$. One knows, at least for Ω that satisfy some boundary regularity, that there exists $\lambda_c(\Omega) \in (0,\infty)$ such that for all $f \geq 0$ the solution u satisfies $u \geq 0$ if and only if $\lambda \leq \lambda_c(\Omega)$. See [11], [12] and [15]. Since the solution u of (3) equals

$$u(x) = \int_{y\in\Omega} G_\Omega(x,y) \left(1 - \lambda \int_{z\in\Omega} \frac{G_\Omega(x,z)\, G_\Omega(z,y)}{G_\Omega(x,y)} dz\right) f(y)\, dy,$$

one can show that

$$\lambda_c(\Omega)^{-1} = \sup_{x,y\in\Omega} \int_{z\in\Omega} \frac{G_\Omega(x,z)\, G_\Omega(z,y)}{G_\Omega(x,y)} dz, \tag{4}$$

where G_Ω is the Green function for the Laplace problem with zero Dirichlet boundary condition on Ω. For rather general elliptic problems Cranston, Fabes and Zhao in [4] showed that the right hand side of (4) is finite. For the Laplacian such a bound has been obtained by Cranston in [6] for $n \geq 3$ and with McConnell in [5] for $n = 2$.

The link between (4) and probability theory is:

$$\mathbb{E}_x^y(\tau_\Omega) = \int_{z\in\Omega} \frac{G_\Omega(x,z)\, G_\Omega(z,y)}{G_\Omega(x,y)} dz, \tag{5}$$

where $\mathbb{E}_x^y(\tau_\Omega)$ is the expectation of the lifetime of a Brownian motion starting in x, conditioned to converge to and to be stopped at y and to be killed on exiting Ω.

The famous result from [5] states that there is a $c > 0$ such that

$$\mathbb{E}_x^y(\tau_\Omega) \leq c|\Omega| \text{ for all } \Omega \subset \mathbb{R}^2,$$

where $|\Omega|$ is the Lebesgue measure of Ω.

Some details for identity (5). A Brownian motion that starts in $x \in \Omega$ and is killed on $\partial\Omega$ has transition density given by $p_\Omega(t,x,y)$ and has expected lifetime given by

$$E_x(\tau_\Omega) = \int_\Omega G_\Omega(x,z)dz.$$

To consider Brownian motion that is conditioned to exit Ω through $\Gamma \subset \partial\Omega$ and stopped at leaving Ω, one uses the transition density $p_\Omega^h(t,x,z) = p_\Omega(t,x,z)\frac{h(z)}{h(x)}$ where h is the solution of

$$\begin{cases} -\Delta h = 0 & \text{in } \Omega, \\ h = 0 & \text{on } \partial\Omega \setminus \Gamma, \\ h = 1 & \text{on } \Gamma. \end{cases}$$

This is a so-called Doob's conditioned Brownian motion, see [7, Part 2, Chap. X]. The expected lifetime is given by

$$E_x^h(\tau_\Omega) = \int_\Omega G_\Omega(x,z)\frac{h(z)}{h(x)}dz. \tag{6}$$

We want to consider the expectation for the time that Brownian motion spends going from x to y and staying inside Ω. This can be approximated by the expected lifetime for the following conditioned Brownian motion. One considers the domains $\Omega_\varepsilon = \Omega \setminus B_\varepsilon(y)$ and the functions $h_{y,\varepsilon}$ such that

$$\begin{cases} -\Delta h_{y,\varepsilon} = 0 & \text{in } \Omega, \\ h_{y,\varepsilon} = 1 & \text{on } \partial B_\varepsilon(y), \\ h_{y,\varepsilon} = 0 & \text{on } \partial\Omega, \end{cases}$$

with the expected lifetime given by (6) replacing h by $h_{y,\varepsilon}$ and G_Ω by G_{Ω_ε}. The expectation of the time we are interested in becomes the expected lifetime of the Brownian motion starting at x and conditioned to leave $\Omega \setminus \{y\}$ at $\{y\}$. This is now given by

$$E_x^y(\tau_{\Omega\setminus\{y\}}) = \lim_{\varepsilon\to 0} E_x^{h_{y,\varepsilon}}(\tau_{\Omega_\varepsilon}). \tag{7}$$

For x and y in the interior, using that

$$\frac{h_{y,\varepsilon}(z)}{h_{y,\varepsilon}(x)} \to \frac{G_\Omega(z,y)}{G_\Omega(x,y)},$$

and that $G_{\Omega_\varepsilon} \to G_\Omega$ holds in dimension $n > 1$, identity (5) follows from (6) and (7).

In the particular case of $y \in \partial\Omega$ a similar procedure leads to

$$\mathbb{E}_x^y(\tau_\Omega) = \int_{z\in\Omega} G_\Omega(x,z)\frac{K_\Omega(y,z)}{K_\Omega(y,x)}dz, \tag{8}$$

where $K_\Omega(y,\cdot)$ is the Poisson kernel for $y \in \partial\Omega$, namely the function such that $u(x) = \int_{y\in\partial\Omega} K_\Omega(x,y)\,g(y)\,d\sigma_y$ solves

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

For sufficiently regular domains the expression in (8) is a continuous extension of (5) to $\Omega \times \bar{\Omega}$. Note that in the above we have used the analyst's $-\Delta$ instead of $-\frac{1}{2}\Delta$.

1.2 Notation and main result

Since the remainder is concerned with the unit disk in $\mathbb{R}^2$ we will skip the subscript Ω and write $G(x,y) = G_B(x,y)$ etc. In 2 dimensions the direct relation between conformal maps and Green functions is best exploited using $\mathbb{C}$ instead of $\mathbb{R}^2$. For the sake of clear notation we will use boldface for this complex alternative:

$$\begin{array}{rll} \text{for } x \in \mathbb{R}^2 & \text{set} & \mathbf{x} = x_1 + i x_2, \\ \text{for } h : \mathbb{R}^2 \to \mathbb{R}^2 & \text{set} & \mathbf{h}(\mathbf{x}) = h_1(x) + i h_2(x). \end{array}$$

The explicit expressions of the Green function and of the Poisson kernel in the disk can now be written as

$$\begin{aligned} G(x,y) &= \frac{1}{4\pi} \log \left(\frac{|\bar{\mathbf{y}}\mathbf{x} - 1|^2}{|\mathbf{x} - \mathbf{y}|^2} \right), \text{ where } x, y \in B, \\ K(x,y) &= \frac{1}{2\pi} \frac{1 - |\mathbf{y}|^2}{|\mathbf{x} - \mathbf{y}|^2}, \text{ where } x \in \partial B, y \in B. \end{aligned}$$

By using dominated convergence and taking limits one can extend the definition of H in (2) up to the closure $\bar{B} \times \bar{B}$. The complete definition of H then reads:

$$H(x,y) = \begin{cases} \dfrac{\int_B G(x,z)\, G(z,y)\, dz}{G(x,y)} & \text{if } x, y \in B \text{ with } x \neq y, \\ 0 & \text{if } x = y \in \bar{B}, \\ \dfrac{\int_B K(x,z)\, G(z,y)\, dz}{K(x,y)} & \text{if } x \in \partial B,\ y \in B, \\ \dfrac{\int_B K(y,z)\, G(z,x)\, dz}{K(y,x)} & \text{if } x \in B,\ y \in \partial B, \\ \pi |x - y|^2 \int_B K(x,z)\, K(y,z)\, dz & \text{if } x, y \in \partial B \text{ with } x \neq y. \end{cases}$$

This function H lies in $C\left(\bar{B} \times \bar{B}\right)$ and is strictly positive on $\bar{B}^2 \backslash \left\{(x,x)\, ;\, x \in \bar{B}\right\}$. The only delicate part is the case $x = y \in \partial B$ for which we refer to formula (12) below.

A precise formulation of the result is the following:

Theorem 1 *For all $y \in \bar{B}$ the function $x \mapsto H(x,y)$ is*

i. increasing along ‘the hyperbolic geodesics through y’ *in increasing euclidean distance;*

ii. increasing along the orthogonal trajectories of ‘the hyperbolic geodesics through y’ *in increasing euclidean distance.*

Remark 1.1 For B ‘the hyperbolic geodesics through y’ are the circles through y that intersect ∂B perpendicular. The orthogonal trajectories are again circles. See Figure 1.

Remark 1.2 For $y \in \partial B$ part i of Theorem 1 has been proved by Griffin, McConnell and Verchota in [9].

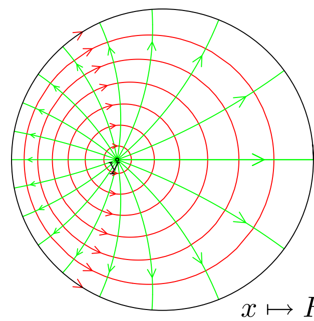

Figure 1: *The geodesics through y in green (light) and the orthogonal trajectories in red (dark).*

Corollary 2 *One directly finds that:*

i. $\sup_{x\in\bar{B}} H(x,y) = H(-y/|y|, y)$ *for any* $y \in \bar{B}\setminus\{0\}$;

ii. $\sup_{x\in\bar{B}} H(x,0) = H(e,0)$ *with* $e = (1,0)$;

iii. and $\sup_{x,y\in\bar{B}} H(x,y) = H(-e,e)$.

Remark 2.1 Since the problem has a rotational symmetry one finds that e above might be replaced by any $a \in \partial B$.

1.3 Earlier related results

Critical numbers related to (4) have been studied before in a number of papers. Caristi and Mitidieri in [2] considered the radially symmetric case (in any dimension n), that is, system (3) for radially symmetric functions and hence with $-\Delta$ replaced by $-r^{1-n}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right)$. They showed that the corresponding $H_{radial}(r,s)$ is maximal for (r,s) being extremal which means $r = 0$ and $s = 1$ or vice versa. The critical number that they find for this radial case is as follows:

$$\sup_{r,s\in[0,1]} H_{radial}(r,s) = \frac{1}{2n}.$$

In the one-dimensional case they also considered $\frac{\partial^2}{\partial x^2} + c$ without assuming symmetry.

Maximal lifetime on the disk. Griffin, McConnell and Verchota in [9] considered H for general simply connected 2-dimensional domains Ω but fixed $y \in \partial\Omega$. Two of their main results for such Ω are

$$\sup_{x\in\bar{\Omega},y\in\partial\Omega} H(x,y) = \sup_{x,y\in\partial\Omega} H(x,y)$$

and that (with our 'analytic' normalization)

$$\sup_{x,y\in\bar{\Omega}} H(x,y) \le \frac{1}{2\pi}|\Omega|.$$

For $\Omega = B$ and $y \in \partial B$ they sharpen this estimate:

$$\sup_{x\in\bar{B},y\in\partial B} H(x,y) \leq 2\log 2 - 1 = \frac{2\log 2 - 1}{\pi}|B|.$$

The numerical values are $\frac{1}{2\pi} = .159155...$ and $\frac{2\log 2-1}{\pi} = .12296....$ Our result improves the last estimate by

$$\sup_{x,y\in\bar{B}} H(x,y) = \sup_{x\in\bar{B},y\in\partial B} H(x,y) \leq 2\log 2 - 1,$$

thereby giving an estimate for the lifetime inequality on a disk with a small hole which is sharper than $1/(2\pi)$ (which corresponds to $1/\pi$ in [9, Remark 5.7]).

Domain optimization. In [10] Kawohl and coauthor showed that the disk does not give the smallest bound for H among all convex planar sets of equal area. Indeed, they considered a sector-like domain S, with $|S| = |B|$, and proved that:

$$\sup_{x\in\bar{S},y\in\partial S} H(x,y) < \sup_{x\in\bar{B},y\in\partial B} H(x,y).$$

The question remains open if

$$\sup_{x,y\in\bar{S}} H(x,y) < \sup_{x,y\in\bar{B}} H(x,y)\ ? \tag{9}$$

In the present paper we show that $\sup_{x\in\bar{B},y\in\partial B} H(x,y) = \sup_{x,y\in\bar{B}} H(x,y)$ holds. We expect the last identity to hold for all planar domains Ω. Let us put it as a conjecture.

Conjecture 3 *If Ω is a (simply connected) planar domain, then*

$$\sup_{x,y\in\Omega} H(x,y) = \sup_{x,y\in\partial\Omega} H(x,y)\ .$$

The obvious consequence of this conjecture is (9). We want to remark that such a result is not likely to hold on a manifold. Consider for example the surface of a ball with a small hole near the pole, see Fig.2. Taking y near the north pole one expects the maximum of H to be attained at an interior point near the south pole.

Relation with eigenvalues In one dimension critical numbers for sign-changing in (3) were studied by Schröder [14]. The precise result was revisited in [11]. Due to the fact that in one dimension the boundary consists of isolated points one recovers an eigenvalue problem for the critical number.

A relation between that critical number and the Dirichlet eigenvalues in an interval $I \subset \mathbb{R}$ is

$$\sup_{x,y\in I} H(x,y) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} = 2\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\lambda_k}. \tag{10}$$

Note that for the unit interval $I = (0,1)$ these eigenvalues are $\lambda_k = \pi^2 k^2$,

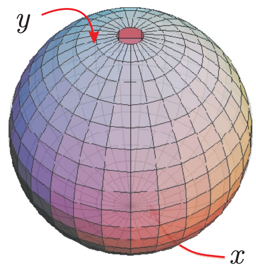

Figure 2: *Sphere with a small hole near the north pole.*

For the disk one finds

$$\sup_{x,y\in B} H(x,y) = 4\sum_{\nu=1}^{\infty}(-1)^{\nu-1}\sum_{k=1}^{\infty}\frac{m_{\nu,k}}{\lambda_{\nu,k}}, \tag{11}$$

where $\lambda_{\nu,k}$ is the eigenvalue for the eigenfunction with $k-1$ circular nodal lines and ν radial nodal lines, and where $m_{\nu,k}$ is the multiplicity, that is, $m_{\nu,k}=1$ for $\nu=0$ and $m_{\nu,k}=2$ for $\nu\geq 1$. The numbers for the two right hand sides above can be found in [11].

1.4 Scheme for the proofs

In section 2 we will consider the case where one of the points lies on the boundary. As mentioned before the case with one point at the boundary has been previously studied by Griffin, McConnell and Verchota in [9]. We will need a more precise characterization of H and in doing so we will recover some of their results. Instead of using power series in $\mathbb{C}$ our basic tools will be conformal mappings, a monotonicity result for a convolution (see Proposition 4) and the maximum principle.

Since the function under consideration is symmetric, $H(x,y)=H(y,x)$, the behaviour of $x\in B\mapsto H(x,y)$ with $y\in\partial B$ can be used for the behaviour of $x\in\partial B\mapsto H(x,y)$ with $y\in B$. Using such a result on the boundary and by several applications of the maximum principle one is able to transfer a inequality valid on the boundary to the interior. This is done in section 3 and will lead to our main result.

Most of the steps consist of deriving estimates for some tailor-made functions. Since all these technicalities might blur the line of arguments we hope to clarify our approach by complementing each intermediate result for a increasing direction of $x\mapsto H(x,y)$ (or a related function) by a sketch.

2 The proof for one point lying on the boundary

Assuming $y\in\partial B$ we may suppose without loss of generality that $y=e=(1,0)$. The numerator $\int_B K(e,z)\,G(z,x)\,dz$ equals:

$$E(x) := -\frac{1-\mathbf{x}\bar{\mathbf{x}}}{8\pi}\left(\frac{\log(1-\mathbf{x})}{\mathbf{x}}+\frac{\log(1-\bar{\mathbf{x}})}{\bar{\mathbf{x}}}+1\right) \text{ for } x\in\bar{B}\backslash\{e\} \text{ and } E(e)=0.$$

Indeed, since $z \mapsto K(e,z) \in L^p(B)$ for $p \in (1,2)$ the Dirichlet problem for the Poisson equation $-\Delta u = K(e,\cdot)$ in B with $u=0$ on ∂B has a unique solution in $W^{2,p}(B) \cap W_0^{1,p}(B)$ by [8, Theorem 9.15]. Since G is the kernel for the solution operator from $L^p(B)$ to $W^{2,p}(B) \cap W_0^{1,p}(B)$ this Dirichlet problem is solved by $u(x) = \int_B K(e,z)\, G(z,x)\, dz$. Next one checks straightforwardly that E lies in $W^{2,p}(B) \cap C_0\left(\bar{B}\right)$ for $p \in (1,2)$ and by [1, Theorem IX.17] it follows that $E \in W^{2,p}(B) \cap W_0^{1,p}(B)$. Since $-\Delta E = -4\frac{\partial}{\partial \mathbf{x}}\frac{\partial}{\partial \bar{\mathbf{x}}}E = K(e,\cdot)$ in B one finds $E=u$, the unique solution. The expression for E can also be deduced from an explicit formula for $\int_B G(x,z)G(z,y)dz$ with $x,y \in B$, which is given in [13].

Dividing $E(x)$ by $K(e,x)$ yields:

$$H(x,e) = -\frac{(1-\mathbf{x})(1-\bar{\mathbf{x}})}{4}\left(\frac{\log(1-\mathbf{x})}{\mathbf{x}} + \frac{\log(1-\bar{\mathbf{x}})}{\bar{\mathbf{x}}} + 1\right), \tag{12}$$

for $x \in \bar{B}\backslash\{e\}$ and by continuity $H(e,e)=0$. We remark that log denotes the analytic extension of the standard logarithm to $\mathbb{C}\backslash(-\infty,0]$ and that the function $\mathbf{x} \mapsto \frac{\log(1-\mathbf{x})}{\mathbf{x}}$ is extended by -1 for $\mathbf{x}=0$.

2.1 In the halfplane

We consider the conformal map from the ball B onto the halfplane $\mathbb{R}^+ \times \mathbb{R}$ that maps $(-1,0)$ to $(0,0)$ and $(0,0)$ to $(1,0)$. This map is given by $\mathbf{h}(\mathbf{x}) = \frac{1+\mathbf{x}}{1-\mathbf{x}}$. Note that $h(e)=\infty$. We let X denote an element of $\mathbb{R}^+ \times \mathbb{R}$, or in complex notation $\mathbf{X} = \mathbf{X}_1 + i\mathbf{X}_2 \in \mathbb{R}^+ + i\mathbb{R}$. The inverse of $\mathbf{h}$ is also a conformal map and is defined by $\mathbf{h}^{-1}(\mathbf{X}) = \frac{\mathbf{X}-1}{\mathbf{X}+1}$.

It follows from a property of conformal maps that

$$H(x,e) = \int_{\mathbb{R}^+\times\mathbb{R}} \frac{K\left(e,h^{-1}(Z)\right)}{K(e,x)} G_{\mathbb{R}^+\times\mathbb{R}}(Z,h(x)) \left|\left(\mathbf{h}^{-1}\right)'(Z_1+iZ_2)\right|^2 dZ_1 dZ_2,$$

where $G_{\mathbb{R}^+\times\mathbb{R}}(X,Y) = \frac{1}{4\pi}\log\left(1+\dfrac{4X_1Y_1}{|X-Y|^2}\right)$. Next, by defining the function

$$\tilde{H}(X) := H(x,e) \ \text{ for } X = h(x),$$

one finds

$$\tilde{H}(X) = \frac{1}{4\pi}\int_{\mathbb{R}^+\times\mathbb{R}} \frac{Z_1}{X_1}\log\left(1+\frac{4X_1Z_1}{|X-Z|^2}\right)\frac{4}{\left((1+Z_1)^2+Z_2^2\right)^2} dZ_1 dZ_2.$$

We will show that $X_2 \longmapsto \tilde{H}(X_1,X_2)$ is decreasing for $X_2>0$. In doing that we need:

Proposition 4 *Let $f,g \in L^2(\mathbb{R})$, $f,g \geq 0$, $f(t) = f(|t|)$, $g(t) = g(|t|)$ and f,g decreasing for $t>0$. Then*

$$t \mapsto \int_{\mathbb{R}} f(x)\, g(x+t)\, dx, \tag{13}$$

is decreasing on $\mathbb{R}^+$.

Proof. We suppose first that additionally $g \in \mathcal{C}_0^\infty(\mathbb{R})$. One has

$$\frac{\partial}{\partial t}\int_{\mathbb{R}} f(x)\, g(x+t)\, dx = \int_{-\infty}^{+\infty} f(x)\, g'(x+t)\, dx$$

$$= \int_{-\infty}^{-t} f(x)\, g'(x+t)\, dx + \int_{-t}^{+\infty} f(x)\, g'(x+t)\, dx.$$

Using that $g'(x+t) = -g'(-x-t)$, one gets

$$\frac{\partial}{\partial t}\int_{\mathbb{R}} f(x)\, g(x+t)\, dx = -\int_{-\infty}^{-t} f(x)\, g'(-x-t)\, dx + \int_{-t}^{+\infty} f(x)\, g'(x+t)\, dx.$$

Changing the coordinates one obtains

$$\frac{\partial}{\partial t}\int_{\mathbb{R}} f(x)\, g(x+t)\, dx = \int_{+\infty}^{0} f(-y-t)\, g'(y)\, dy + \int_{0}^{+\infty} f(y-t)\, g'(y)\, dy$$

$$= \int_{0}^{+\infty} g'(y)\left(f(y-t) - f(-y-t)\right) dy.$$

Now for $t > 0$, one has $|y-t| < |-y-t|$. Hence the function (13) is decreasing.

The preceding arguments yields the result also for g as in the hypothesis. We observe that such g may be approximated in $L^2(\mathbb{R})$ by $(g_k)_{k\in\mathbb{N}} \subset \mathcal{C}_0^\infty(\mathbb{R})$ having the additional properties above. This is achieved by using an even and in positive x-direction decreasing mollifier in $\mathcal{C}_0^\infty(\mathbb{R})$. ■

Corollary 5 *The relations*

$$\max_{X_2\in\mathbb{R}} \tilde{H}(X_1, X_2) = \tilde{H}(X_1, 0) \;\; \text{and} \;\; X_2 \frac{\partial}{\partial X_2}\tilde{H}(X_1, X_2) \le 0,$$

hold for every $X_1 \in [0, +\infty)$.

Proof. For every $X_1 \in \mathbb{R}^+$, one has

$$\tilde{H}(X) = \frac{1}{\pi}\int_{\mathbb{R}^+} \frac{Z_1}{X_1}\int_{\mathbb{R}} \log\left(1 + \frac{4X_1Z_1}{|X-Z|^2}\right)\frac{1}{\left((1+Z_1)^2 + Z_2^2\right)^2} dZ_2 dZ_1.$$

Hence defining

$$\begin{aligned} f(Z_2) &= \log\left(1 + \tfrac{4X_1Z_1}{(X_1-Z_1)^2+Z_2^2}\right), \\ g(Z_2) &= \tfrac{1}{\left((1+Z_1)^2+Z_2^2\right)^2}, \end{aligned}$$

we can write

$$\int_{\mathbb{R}} \log\left(1 + \tfrac{4X_1Z_1}{|X-Z|^2}\right)\tfrac{1}{\left((1+Z_1)^2+Z_2^2\right)^2} dZ_2 = \int_{\mathbb{R}} f(Z_2 - X_2)\, g(Z_2)\, dZ_2.$$

Applying Proposition 4 one gets that the function $\tilde{H}$ is decreasing for X_2 positive and increasing for X_2 negative for every $X_1 \in \mathbb{R}^+$. The claim follows using the regularity of the function. The case $X_1 = 0$ goes similarly by proceeding to the limit. ■

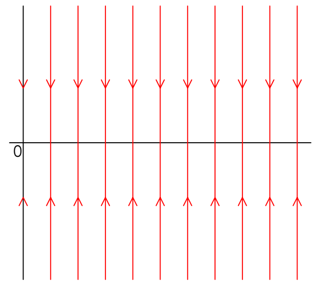

Figure 3: *Illustration of Corollary 5; arrows denote increasing directions of* $X \mapsto \tilde{H}(X)$.

2.2 Back in the disk

Using the properties of conformal mapping, see [3, Sect. III.3], from the increasing direction of $\tilde{H}$ we get an increasing direction of $H(x, e)$. The lines $\mathbf{h}^{-1}(\{\mathbf{X}_1 = k_1\})$, varying k_1 in $\mathbb{R}^+$, are circles inside the disk which are tangent to ∂B in $(1, 0)$. Hence, we have for every (x_1, x_2) that the function H is increasing in the direction

$$v_{(x_1,x_2)} = \left(-x_2, \frac{2x_1 - x_1^2 - 1 + x_2^2}{2(1 - x_1)}\right), \text{ if } x_2 > 0, \tag{14}$$

and in the $-v_{(x_1,x_2)}$–direction, if $x_2 < 0$. In particular we obtain that

$$x_2 \frac{\partial}{\partial \theta} H(x, e) := x_2 \left(-x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}\right) H(x, e) \geq 0 \text{ when } |x| = 1. \tag{15}$$

Here we write $x_1 = |x| \cos \theta$ and $x_2 = |x| \sin \theta$.

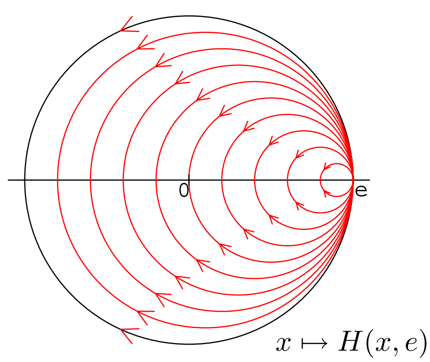

Figure 4: *The result of Corollary 5 transformed back to the disk; arrows denote increasing directions of* $x \mapsto H(x, e)$.

Since we will proceed through properties of the differential equation for H let us fix the following formula.

Lemma 6 *For* $a, b \in C^2$ *with* $b \neq 0$ *the following identity holds*

$$-\Delta\left(\frac{a}{b}\right) - 2\frac{\nabla b}{b} \cdot \nabla\left(\frac{a}{b}\right) + \frac{-\Delta b}{b}\left(\frac{a}{b}\right) = \frac{-\Delta a}{b}. \tag{16}$$

Having $e \in \partial B$ one finds $-\Delta K(x,e) = 0$ and $-\Delta\left(\int_B G(x,z)K(z,e)dz\right) = K(x,e)$ in B and by (16) the function H satisfies:

$$-\Delta H\,(x,e) - 2\frac{\nabla K\,(x,e)}{K\,(x,e)} \cdot \nabla H\,(x,e) = 1 \text{ when } x \in B.$$

Let us consider the derivative with respect to the angle $\frac{\partial}{\partial\theta}H$. Since $\frac{\partial}{\partial\theta} = x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}$, we get

$$\nabla\frac{\partial}{\partial\theta}H \;=\; \nabla\left(\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right)H\right) = \left(\mathcal{R} + \frac{\partial}{\partial\theta}\right)\nabla H,$$

with $\mathcal{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since $\frac{\partial}{\partial\theta}$ and Δ commute and since $\mathcal{R}$ is skew-symmetric, one obtains that $\frac{\partial}{\partial\theta}H\,(x,e)$ satisfies

$$\begin{aligned}
-\Delta\frac{\partial}{\partial\theta}H - 2\nabla\log(K)\cdot\nabla\frac{\partial}{\partial\theta}H \;&=\; -\frac{\partial}{\partial\theta}\Delta H - 2\frac{\nabla K}{K}\cdot\left(\mathcal{R} + \frac{\partial}{\partial\theta}\right)\nabla H \;= \\
&=\; \frac{\partial}{\partial\theta}\left(-\Delta H - 2\frac{\nabla K}{K}\cdot\nabla H\right) + 2\left(\frac{\partial}{\partial\theta}\frac{\nabla K}{K}\right)\cdot\nabla H - 2\frac{\nabla K}{K}\cdot\mathcal{R}\nabla H \\
&=\; 0 + 2\left(\left(\frac{\partial}{\partial\theta} + \mathcal{R}\right)\nabla\log K\right)\cdot\nabla H \\
&=\; 2\left(\nabla\frac{\partial}{\partial\theta}\log K\right)\cdot\nabla H.
\end{aligned}$$

By the symmetry one observes that $\frac{\partial}{\partial\theta}H(x,e) = 0$ in $\{x \in B : x_2 = 0\}$. Furthermore it follows from (15) that $\frac{\partial}{\partial\theta}H(x,e) \geq 0$ in $\{x \in \partial B : x_2 > 0\}$. A priori one knows that $x \mapsto \frac{\partial}{\partial\theta}H\,(x,e)$ is in $\mathcal{C}^2(B^+) \cap \mathcal{C}(\bar{B}^+ \setminus \{e\})$ and only the behavior near e remains to be studied. Using the explicit formula of $H(x,e)$ given by (12), we will prove the following:

Lemma 7 *The following identity holds*

$$\lim_{\substack{x\to e,\\ x\in B^+}} \frac{\partial}{\partial\theta}H(x,e) = 0.$$

Proof. Since $\frac{\partial}{\partial\theta} = i\left(\mathbf{x}\frac{\partial}{\partial\mathbf{x}} - \bar{\mathbf{x}}\frac{\partial}{\partial\bar{\mathbf{x}}}\right)$, one gets

$$\begin{aligned}
\frac{\partial}{\partial\theta}H\,(x,e) \;&=\; i\frac{(1-\bar{\mathbf{x}})}{4}\left(\log(1-\mathbf{x}) + \frac{\mathbf{x}}{\bar{\mathbf{x}}}\log(1-\bar{\mathbf{x}}) + \mathbf{x}\right) \\
&\quad -i\frac{(1-\mathbf{x})(1-\bar{\mathbf{x}})}{4}\left(-\frac{\log(1-\mathbf{x})}{\mathbf{x}} - \frac{1}{1-\mathbf{x}}\right) \\
&\quad -i\frac{(1-\mathbf{x})}{4}\left(\frac{\bar{\mathbf{x}}}{\mathbf{x}}\log(1-\mathbf{x}) + \log(1-\bar{\mathbf{x}}) + \bar{\mathbf{x}}\right) \\
&\quad +i\frac{(1-\mathbf{x})(1-\bar{\mathbf{x}})}{4}\left(-\frac{\log(1-\bar{\mathbf{x}})}{\bar{\mathbf{x}}} - \frac{1}{1-\bar{\mathbf{x}}}\right)
\end{aligned}$$

$$= i\frac{\log(1-\mathbf{x})}{4\mathbf{x}}(1 - 2\bar{\mathbf{x}} + \mathbf{x}\bar{\mathbf{x}}) - i\frac{\log(1-\bar{\mathbf{x}})}{4\bar{\mathbf{x}}}(1 - 2\mathbf{x} + \mathbf{x}\bar{\mathbf{x}}) - i\frac{\bar{\mathbf{x}} - \mathbf{x}}{2}.$$

One observes that

$$\lim_{\substack{x \to e,\\ x \in B^+}} \frac{\partial}{\partial \theta} H(x,e) = 0.$$

■

Hence $\frac{\partial}{\partial\theta} H(\cdot, e) \in \mathcal{C}^2(B) \cap \mathcal{C}(\bar{B})$ and that $\frac{\partial}{\partial\theta} H(\cdot, e)$ satisfies the boundary value problem

$$\begin{cases} -\Delta \frac{\partial}{\partial\theta} H - 2\frac{\nabla K}{K} \cdot \nabla \frac{\partial}{\partial\theta} H = 2\nabla \frac{\partial}{\partial\theta} \log K \cdot \nabla H & \text{in } B^+, \\ \frac{\partial}{\partial\theta} H \geq 0 & \text{on } \partial B^+. \end{cases} \tag{17}$$

Proposition 8 *The inequality* $x_2 \dfrac{\partial}{\partial\theta} H(x,e) \geq 0$ *holds for all* $x \in B$.

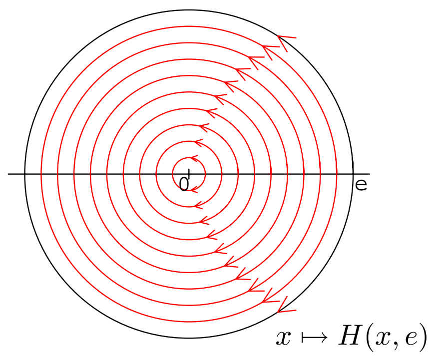

Figure 5: *For $y = e \in \partial B$ the function $x \mapsto H(x,y)$ is increasing along semicircles to the left.*

Proof. Since $K(x,e) = \dfrac{1 - |x|^2}{|x-e|^2}$ we have

$$\begin{aligned} \frac{\partial}{\partial\theta} \log K &= -\frac{\frac{\partial}{\partial\theta} |x-e|^2}{|x-e|^2} = -\frac{\left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right)\left((x_1-1)^2 + x_2^2\right)}{|x-e|^2} \\ &= 2\frac{x_2(x_1-1) - x_1 x_2}{|x-e|^2} = \frac{-2x_2}{|x-e|^2}, \end{aligned}$$

and

$$\begin{aligned} \nabla \frac{\partial}{\partial\theta} \log K &= \nabla \left(\frac{-2x_2}{|x-e|^2} \right) = \frac{-2}{|x-e|^2}(0,1) + \frac{2x_2}{|x-e|^4}(2(x_1-1), 2x_2) \\ &= \frac{2}{|x-e|^4}\left(2x_2(x_1-1), 2x_2^2 - |x-e|^2\right) \\ &= 2\frac{\left(2x_2(x_1-1), x_2^2 - (x_1-1)^2\right)}{|x-e|^4} \\ &= \frac{4(1-x_1)}{|x-e|^4}\left(-x_2, \frac{2x_1 - x_1^2 - 1 + x_2^2}{2(1-x_1)}\right). \end{aligned}$$

We see that $\nabla\frac{\partial}{\partial\theta}\log K\left(x_1,x_2\right)$ has the direction of $v_{(x_1,x_2)}$ as defined in (14) . Hence the term

$$\nabla\frac{\partial}{\partial\theta}\log K\cdot\nabla H,$$

is non-negative. Applying Theorem A to (17) the claim follows. ∎

3 The proof for both points in the interior

3.1 Tangential directions

We consider now y in the interior. Without loss of generality we may suppose that $y=(-s,0)$ with $s\in(0,1)$. The case $s=0$ gives the radial symmetric case which has been considered previously by Caristi and Mitidieri in [2].

Let us fix x at the boundary and consider $H\left(x,y\right)$. Let $C_s=\{y:|y|=s\}$. From the previous section it follows that the maximum of $H\left(x,\cdot\right)$ in C_s is attained in $y=-sx$. This is equivalent to ask for $y=(-s,0)$ that $x=(1,0)$. So using the symmetry of the problem we can say that

$$x_2\frac{\partial}{\partial\theta}H\left(x,(-s,0)\right)\le 0\text{ when }x\in\partial B. \tag{18}$$

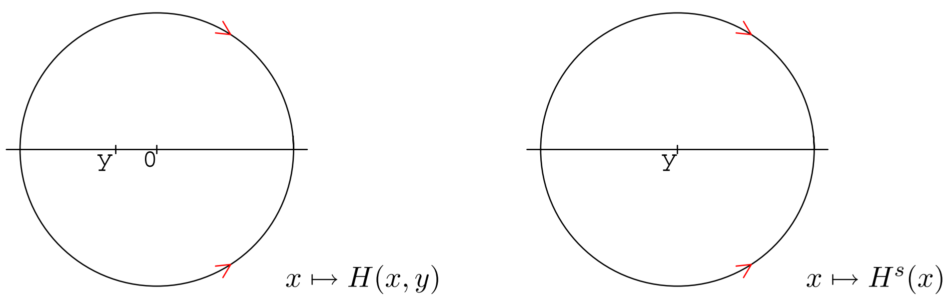

Figure 6: *Using the symmetry between x and y we may conclude that for any $y\in B$, the function $x\mapsto H(x,y)$ (left) is increasing along ∂B from the nearest boundary point of y to the most distant boundary point. Putting $y=(-s,0)$ with $s>0$ it means increasing to the right along ∂B. Also the function $x\mapsto H^s(x)$ (right) is increasing to the right along ∂B.*

We consider a conformal map $\mathbf{k}_s$ from the disk onto the disk that maps y into 0:

$$\mathbf{k}_s\left(\mathbf{x}\right)=\frac{\mathbf{x}+s}{1+s\mathbf{x}}.$$

Proceeding as before we will now study the function

$$H^s\left(x\right):=H\left(\mathbf{k}_s^{-1}(\mathbf{x}),\mathbf{k}_s^{-1}(\mathbf{0})\right),$$

which due to the behaviour of conformal mappings transforms into

$$H^s\left(x\right)=\int_B\frac{G\left(x,z\right)G\left(z,0\right)}{G\left(x,0\right)}\left|\left(\mathbf{k}_s^{-1}\right)'\left(z_1+iz_2\right)\right|^2dz_1dz_2.$$

We have $\mathbf{k}_s^{-1}(\mathbf{z}) = \frac{\mathbf{z}-s}{1-s\mathbf{z}}$ and $\left|\left(\mathbf{k}_s^{-1}\right)'(z_1+iz_2)\right|^2 = \frac{\left(1-s^2\right)^2}{|e-sz|^4}$, hence

$$H^s(x) = \int_B \frac{G(x,z)\,G(z,0)}{G(x,0)} \frac{\left(1-s^2\right)^2}{|e-sz|^4} dz_1 dz_2.$$

One gets for $x \neq 0$ that the function H^s satisfies

$$-\Delta H^s(x) + \frac{2}{r\,|\log r|}\frac{\partial}{\partial r} H^s(x) = \frac{\left(1-s^2\right)^2}{|e-sx|^4}.$$

Proposition 9 *The inequality $x_2 \frac{\partial}{\partial\theta} H^s(x) \le 0$ holds for all $x \in B$.*

Proof. By symmetry one may assume $x \in B^+$. We consider the function $\Theta(x) := \frac{\partial}{\partial\theta} H^s(x)$ or to be more specific

$$\Theta(x) = x_1 \frac{\partial}{\partial x_2} H^s(x) - x_2 \frac{\partial}{\partial x_1} H^s(x).$$

Since Δ and $\frac{\partial}{\partial\theta}$ commute, one finds

$$\begin{aligned} -\Delta\Theta(x) &= -\frac{\partial}{\partial\theta}\Delta H^s(x) = \frac{\partial}{\partial\theta}\left[-\frac{2}{r|\log r|}\frac{\partial}{\partial r}H^s(x) + \frac{\left(1-s^2\right)^2}{|e-sx|^4}\right] \\ &= -\frac{2}{r|\log r|}\frac{\partial}{\partial r}\Theta(x) - 4\frac{\left(1-s^2\right)^2}{|e-sx|^6} s x_2. \end{aligned}$$

A priori $\Theta \in \mathcal{C}^2(\bar{B} \setminus \{0\})$ holds and only the behavior of Θ in 0 remains to be studied. We have

$$\frac{\partial}{\partial\theta} H^s(x) = \frac{1}{G(x,0)}\frac{\partial}{\partial\theta} R(x),$$

where $R(x)$ satisfies

$$\begin{cases} -\Delta R(x) = -\frac{\left(1-s^2\right)^2}{4\pi|e-sx|^4}\log|x|^2 & \text{in } B, \\ R(x) = 0 & \text{on } \partial B. \end{cases} \tag{19}$$

Since the right hand side of (19) is in $L^p(B)$ for every $p \in (1,+\infty)$, one gets $R \in W^{2,p}(B)$ and hence, using the Sobolev imbedding theorem it follows that

$$R \in \mathcal{C}^{1,\alpha}(\bar{B}) \text{ for every } \alpha \in (0,1). \tag{20}$$

Setting $\Omega = B_{\frac{1}{2}}(0)$, we have $\frac{\partial}{\partial\theta}R$ and $G^{-1}(\cdot,0) \in \mathcal{C}(\bar{\Omega})$ (where we extend $G^{-1}(\cdot,0)$ in 0 by 0). Hence $\Theta \in \mathcal{C}^2(B^+) \cap \mathcal{C}^0(\bar{B}^+)$.

Using (18) and the fact that H^s is symmetric in $x_2 = 0$, we find that $\Theta(x) \le 0$ on ∂B^+. We may summarize:

$$\begin{cases} -\Delta\Theta(x) + \frac{2}{r|\log r|}\frac{\partial}{\partial r}\Theta(x) = -4\frac{\left(1-s^2\right)^2}{|e-sx|^6} s x_2 & \text{in } B^+, \\ \Theta(x) \le 0 & \text{on } \partial B^+. \end{cases}$$

The claim follows applying the maximum principle, see Theorem A. ∎

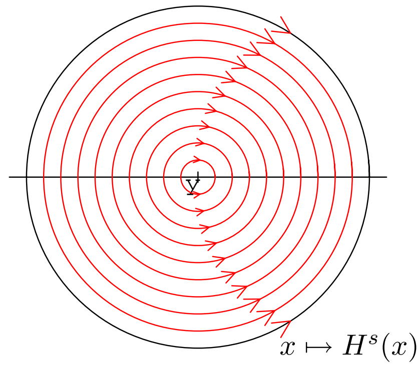

Figure 7: *A conformal mapping changed $H(x,y)$ to $H^s(x)$ and put y in the center. By Proposition 9 the mapping $x \mapsto H^s(x)$ is increasing to the right along all semicircles around* 0.

3.2 Radial directions

In order to prove Theorem 1, it remains to show that the function $H^s(x_1,0)$ is increasing on the interval $(0,1)$. We will show that the function H^s is increasing in radial direction by using the maximum principle. First we will show that the function H^s satisfies a zero Neumann boundary condition:

Lemma 10 *The identity $\frac{\partial}{\partial r}H^s(x) = 0$ holds for all $x \in \partial B$.*

Proof. We write

$$H^s(x) = \frac{R(x)}{G(x,0)},$$

with $R(x) = \int_B G(x,z)\,G(z,0)\,\frac{(1-s^2)^2}{|e-sz|^4}dz_1dz_2$ and observe that $R(x) = G(x,0) = 0$ for $x \in \partial B$. Moreover

$$-\Delta R(x) = G(x,0)\,\frac{(1-s^2)^2}{|e-sx|^4} \text{ and } -\Delta G(x,0) = 0 \text{ for } x \neq 0,\ x \in B.$$

Since $-\Delta = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial^2\phi}$, we find that at the boundary

$$-\frac{\partial^2}{\partial r^2}R(x) = \frac{\partial}{\partial r}R(x), \tag{21}$$

$$-\frac{\partial^2}{\partial r^2}G(x,0) = \frac{\partial}{\partial r}G(x,0). \tag{22}$$

Using the series expansion near the boundary for $R(x)$ and $G(x,0)$, we get for $x \in \partial B$:

$$\begin{aligned}
\lim_{B\ni\xi\to x}\frac{\partial}{\partial r}H^s(\xi) &= \lim_{B\ni\xi\to x}\frac{\frac{\partial}{\partial r}G(\xi,0)}{G(\xi,0)}\left(\frac{\frac{\partial}{\partial r}R(\xi)}{\frac{\partial}{\partial r}G(\xi,0)} - \frac{R(\xi)}{G(\xi,0)}\right)\\
&= \lim_{B\ni\xi\to x}\frac{1}{|\xi|-1}\left(\frac{\frac{\partial}{\partial r}R(\xi)+(|\xi|-1)\frac{\partial^2}{\partial r^2}R(\xi)+..}{\frac{\partial}{\partial r}G(\xi,0)+(|\xi|-1)\frac{\partial^2}{\partial r^2}G(\xi,0)+..} - \frac{\frac{\partial}{\partial r}R(\xi)+\frac{|\xi|-1}{2}\frac{\partial^2}{\partial r^2}R(\xi)+..}{\frac{\partial}{\partial r}G(\xi,0)+\frac{|\xi|-1}{2}\frac{\partial^2}{\partial r^2}G(\xi,0)+..}\right)\\
&= \frac{1}{2}\frac{\frac{\partial^2}{\partial r^2}R(x)\,\frac{\partial}{\partial r}G(x,0) - \frac{\partial^2}{\partial r^2}G(x,0)\,\frac{\partial}{\partial r}R(x)}{\left(\frac{\partial}{\partial r}G(x,0)\right)^2},
\end{aligned}$$

which is zero by using (21) and (22). ■

Proposition 11 *The inequality* $r\frac{\partial}{\partial r}H^s(x) \geq 0$ *holds for all* $x \in B$.

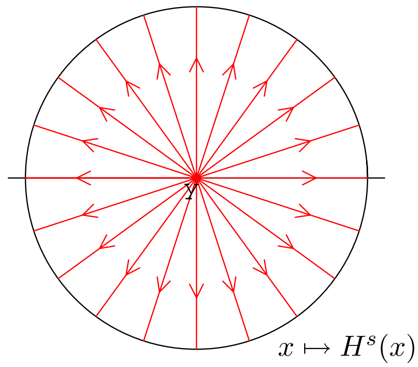

Figure 8: *A conformal mapping changed* $H(x,y)$ *to* $H^s(x)$ *and, roughly spoken, put* y *in the center. Here is the result from Proposition 11: the function* $x \mapsto H^s(x)$ *is radially increasing. The combination with Figure 7 and the inverse conformal mapping lead to Figure 1.*

Proof. The function H^s satisfies

$$-\Delta H^s(x) = \frac{\left(1-s^2\right)^2}{|e-sx|^4} + \frac{4}{|x|^2\left(\log|x|^2\right)} x \cdot \nabla H^s(x). \tag{23}$$

Let us define $\Xi(x) := r\frac{\partial}{\partial r}H^s(x) = x \cdot \nabla H^s(x)$. One has

$$-\Delta \Xi(x) = -2\Delta H^s(x) - x_1 \frac{\partial}{\partial x_1}\Delta H^s(x) - x_2 \frac{\partial}{\partial x_2}\Delta H^s(x) = \ldots$$

and by (23)

$$\begin{aligned}
\ldots &= 2\tfrac{\left(1-s^2\right)^2}{|e-sx|^4} + \tfrac{8}{|x|^2\left(\log|x|^2\right)} x \cdot \nabla H^s(x) + x \cdot \nabla\left(\tfrac{\left(1-s^2\right)^2}{|e-sx|^4} + \tfrac{4}{|x|^2\left(\log|x|^2\right)} x \cdot \nabla H^s(x)\right) \\
&= 2\tfrac{\left(1-s^2\right)^2}{|e-sx|^4} + \tfrac{8}{|x|^2\left(\log|x|^2\right)}\Xi(x) + 4sx_1\tfrac{\left(1-s^2\right)^2}{|e-sx|^6}(1-sx_1) \\
&\quad + \tfrac{4x_1}{|x|^2\left(\log|x|^2\right)}\tfrac{\partial}{\partial x_1}\Xi(x) - \tfrac{8x_1^2}{|x|^4\left(\log|x|^2\right)}\Xi(x) - \tfrac{8x_1^2}{|x|^4\left(\log^2|x|^2\right)}\Xi(x) \\
&\quad - 4s^2x_2^2\tfrac{\left(1-s^2\right)^2}{|e-sx|^6} + \tfrac{4x_2}{|x|^2\left(\log|x|^2\right)}\tfrac{\partial}{\partial x_2}\Xi(x) - \tfrac{8x_2^2}{|x|^4\log|x|^2}\Xi(x) - \tfrac{8x_2^2}{|x|^4\left(\log^2|x|^2\right)}\Xi(x),
\end{aligned}$$

that gives

$$-\Delta\Xi(x) - \tfrac{4x\cdot\nabla\Xi(x)}{|x|^2\log|x|^2} + \tfrac{8\Xi(x)}{|x|^2\left(\log^2|x|^2\right)} = 2\tfrac{\left(1-s^2\right)^2}{|e-sx|^4}\left(\frac{1-s|x|^2}{|e-sx|^2}\right). \tag{24}$$

One sees that the right hand side of (24) is non-negative. Furthermore, since

$$\Xi(x) = \frac{r}{G(x,0)}\frac{\partial}{\partial r}R(x) - \frac{R(x)}{(G(x,0))^2}r\frac{\partial}{\partial r}G(x,0),$$

with $R \in \mathcal{C}^{1,\alpha}(\bar{B})$ (from (20)), one has that $\Xi(0) = 0$ and that Ξ is continuous in B. With help of the preceding Lemma 10 we get that $\Xi \in \mathcal{C}^0(\bar{B})$. Hence, summarizing we have

$$\begin{cases} -\Delta\Xi\,(x) - \frac{4}{|x|^2 \log|x|^2} x \cdot \nabla\Xi\,(x) + \frac{8\Xi(x)}{|x|^2\left(\log^2|x|^2\right)} \geq 0 & \text{in } B \setminus \{0\}, \\ \Xi\,(x) = 0 & \text{on } \partial B \cup \{0\}. \end{cases}$$

The maximum principle stated in Theorem A finally yields $\Xi \geq 0$ in B. ■

Appendix A: A version of the Maximum Principle

The maximum principle had to be repeatedly applied to differential operators of which the coefficients become singular on the boundary. We prefer to give the precise formulation of a maximum principle which is appropriate for this situation. For a proof we refer to [8, Sect. 3.1].

Theorem A *Suppose that $\Omega \subset \mathbb{R}^n$ is open, bounded and connected, and that $b \in C(\Omega; \mathbb{R}^n)$ and $c \in C(\Omega; \mathbb{R})$ with $c \geq 0$. Set $L = -\Delta + b \cdot \nabla + c$. If $u \in C^2(\Omega)$ satisfies*

$$\begin{cases} Lu(x) \geq 0 & \text{for } x \in \Omega, \\ \liminf\limits_{\Omega \ni x \to x_\partial} u(x) \geq 0 & \text{for } x_\partial \in \partial\Omega, \end{cases}$$

then $u \geq 0$ in Ω.

References

[1] H. Brezis, Analyse fonctionnelle, Masson, Paris, 1983.

[2] G. Caristi, E. Mitidieri, Maximum principles for a class of noncooperative elliptic systems, Delft Progr. Rep. 14 (1990), 33–56.

[3] J. Conway, Functions of one complex variable, Springer–Verlag, New York etc., 1973.

[4] M. Cranston, E. Fabes and Zh. Zhao, Potential theory for the Schrödinger equation, Bull. Amer. Math. Soc., New Ser. **15** (1986), 213–216.

[5] M. Cranston, T.R. McConnell, The lifetime of conditioned Brownian motion, Z. Wahrsch. Verw. Gebiete **65** (1983), 1–11.

[6] M. Cranston, Lifetime of conditioned Brownian motion in Lipschitz domains, Z. Wahrsch. Verw. Gebiete **70** (1985), 335–340.

[7] J.L. Doob, Classical potential theory and its probabilistic counterpart, Springer-Verlag, Berlin etc., 1984.

[8] D. Gilbarg, N.S. Trudinger, Elliptic partial differential equations of second order, Second edition. Springer–Verlag, Berlin etc., 1983.

[9] P.S. Griffin, T.R. McConnell and G. Verchota, Conditioned Brownian motion in simply connected planar domains, Ann. Inst. H. Poincaré Probab. Statist. **29** (1993), 229–249.

[10] B. Kawohl and G. Sweers, Among all two-dimensional convex domains the disk is not optimal for the lifetime of a conditioned Brownian motion, Journal d'Analyse Mathématiques **86** (2002), 335-357.

[11] B. Kawohl and G. Sweers, On 'anti'-eigenvalues for elliptic systems and a question of McKenna and Walter, Indiana U. Math. J. **51** (2002).

[12] E. Mitidieri, G. Sweers, Weakly coupled elliptic systems and positivity, Math. Nachrichten **173** (1995), 259-286.

[13] S.H. Schot, The Green's function method for the supported plate boundary value problem, Z. Anal. Anwendungen **11** (1992), 359-370.

[14] J. Schröder, Operator Inequalities, Mathematics in Science and Engineering **147**, Academic Press, New York–London, 1980.

[15] G. Sweers, Positivity for a strongly coupled elliptic system by Green function estimates. J. Geom. Anal. 4 (1994), 121–142.

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