On the lifetime of a conditioned Brownian motion in the ball

Anna Dall'Acqua
Delft Inst. Applied Math., E.W.I.
Delft University of Technology
PObox 5031
2600 GA Delft
The Netherlands
e-mail: a.dallacqua@ewi.tudelft.nl

Abstract

Consider the Brownian motion conditioned to start in x, to converge to y, with $x, y \in \bar{\Omega}$, and to be killed at the boundary $\partial\Omega$. Here Ω is a bounded domain in $\mathbb{R}^n$. For which x and y is the lifetime of this Brownian motion maximal? One would guess for x and y being opposite boundary points and we will show that this holds true for balls in $\mathbb{R}^n$. As a consequence we find the best constant for the positivity preserving property of some elliptic systems and an identity between this constant and a sum of inverse Dirichlet eigenvalues.

1 Introduction

Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and let G_Ω denote the Green function for

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

that is, the solution of (1) is given by $u(x) = \int_\Omega G_\Omega(x, y) f(y) dy$. Let us define

$$H_\Omega(x, y) := \int_\Omega \frac{G_\Omega(x, z) G_\Omega(z, y)}{G_\Omega(x, y)} \text{ for } x, y \in \Omega \times \Omega.$$

The function $H_\Omega(x, y)$ is of some importance in two different areas of mathematics: elliptic partial differential equations and probability.

In p.d.e.'s the function $H_\Omega(x, y)$ appears when studying the positivity preserving property of the following system of second order elliptic equations:

$$\begin{cases} -\Delta u &= f - \lambda v \quad \text{in} \quad \Omega, \\ -\Delta v &= f \quad \text{in} \quad \Omega, \\ u &= v = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

for $\lambda > 0$. One is interested in studying system (2) since this is the model problem for the positivity preserving property of second order elliptic boundary value problems that are coupled in a non-cooperative way (see [11]). In order that for every $f > 0$ the solutions u and v of (2) are also positive one needs that $\lambda \leq \lambda_c(\Omega)$ where

$$\lambda_c^{-1}(\Omega) := \sup_{x,y \in \Omega} H_\Omega(x, y). \tag{3}$$

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The L^∞-bound of the function $H_\Omega(x,y)$ for rather general elliptic operators has been studied in [3] (see also [2], [4] and [5]). In the case of a two-dimensional simply connected domain Ω it has been shown that

$$H_\Omega(x,y) \le \frac{1}{2\pi}|\Omega| \text{ for } x,y \in \bar{\Omega}.$$

In higher dimensions some regularity of the boundary is required in order to prove an L^∞-bound for H_Ω. For a Lipschitz domain $\Omega \subset \mathbb{R}^n$ with $n \ge 3$ it holds that

$$H_\Omega(x,y) \le c|\Omega|^{\frac{2}{n}} \text{ for } x,y \in \bar{\Omega},$$

with c a constant depending on the Lipschitz character of Ω and on the diameter of Ω, see [4].

In probability the function $H_\Omega(x,y)$ represents the lifetime of a conditioned Brownian motion. More precisely, the following relation holds

$$\mathbb{E}_x^y(\tau_\Omega) = H_\Omega(x,y), \tag{4}$$

where $\mathbb{E}_x^y(\tau_\Omega)$ is the expectation of the lifetime of a Brownian motion in Ω starting in x, conditioned to converge to y and to be stopped at y, and to be killed on exiting Ω. Some details for identity (4) can be found in [10] and [7] (see also [9]).

In the present paper we will study where the function $H_\Omega(x,y)$ attains its maximum in $\bar{\Omega} \times \bar{\Omega}$ with Ω the unit ball in $\mathbb{R}^n$, $n \ge 3$. Our aim in studying the problem was to generalize some properties known for the disk to the ball in dimension n.

In literature there are some results concerning the two-dimensional case. In [10] the authors considered the behavior of $x \mapsto H_\Omega(x,y)$ for y fixed at the boundary and Ω a general simply connected domain in $\mathbb{R}^2$. The main result reads as follows. For $y \in \partial\Omega$ the function $x \mapsto H_\Omega(x,y)$ is increasing along "hyperbolic geodesics" in increasing Euclidean distance from y and hence the maximum is attained for $x \in \partial\Omega$. In particular in the case of the unit disk the maximum is attained at opposite boundary points. The main tools are conformal mappings and series expansions. However, for y in the interior there exists almost no results. In [7] the problem has been solved in the case $\Omega = D$ the unit disk. The main result is that $x \mapsto H_\Omega(x,y)$ is increasing along the "hyperbolic geodesic" through y in increasing Euclidean distance, and also it is increasing along the orthogonal trajectories of the "hyperbolic geodesic" through y in increasing Euclidean distance. The proof uses Möbious transformations, the maximum principle and partially the result in [10].

In higher dimensions only the radially symmetric case has been studied. In [6] the authors show that $H_{rad}(r,s)$ attains its maximum for (r,s) being extremal which means $r = 0$ and $s = 1$.

The main result of the paper is that $H_\Omega(x,y)$ with Ω the unit ball in $\mathbb{R}^n$ with $n \ge 3$ attains its supremum at opposite boundary points. This is related to the best constant in (3). The proof consists in studying the direction with which $x \mapsto H_\Omega(x,y)$, for $y \in \bar{\Omega}$ fixed, increases. As a direct application of the localization of the maximum of H_Ω, we will compute explicitly the best constant in (3) when Ω is the unit ball in $\mathbb{R}^n$. We will also prove an identity between $\lambda_c^{-1}(\Omega)$ with Ω the unit ball in $\mathbb{R}^3$ and a sum of Dirichlet eigenvalues. This kind of identities was first observed in [15] and then developed in [11]. It is still an open question if these identities are simply a coincidence or if there is an explanation beyond computation. We are now able to give an explanation to the identity in the case of the unit disk but not in the case of the unit ball in $\mathbb{R}^3$.

The structure of the paper is as follows. First we present some notation and we state the main result. In the second section we study the increasing direction of $x \mapsto H_\Omega(x,y)$ for y fixed in the interior and in the third section we consider y fixed at the boundary. In the last section we discuss some identities involving $\lambda_c^{-1}(\Omega)$ and a sum of inverse Dirichlet eigenvalues. In the appendix we recall some known properties of conformal mappings that will be used in the proof.

1.1 Notation and main result

Let $B = \{x \in \mathbb{R}^n : |x| < 1\}$ denote the unit ball in $\mathbb{R}^n$, $n \geq 3$, and set for $x, y \in B$,

$$G_B(x,y) = \begin{cases} \frac{1}{n(n-2)\omega_n} \left(|x-y|^{2-n} - \left| x\,|y| - \frac{y}{|y|} \right|^{2-n} \right) & \text{for } y \neq 0, \\ \frac{1}{n(n-2)\omega_n} \left(|x|^{2-n} - 1 \right) & \text{for } y = 0, \end{cases}$$

where $\omega_n = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}$ is the volume of B. This function G_B is the Green function for (1) with $\Omega = B$.

Since in the rest of the paper we work in the unit ball we skip the subscript B and write $H(x,y) = H_B(x,y)$. It is convenient to extend the definition of H to all $\bar{B} \times \bar{B}$:

$$H(x,y) = \begin{cases} \displaystyle\int_B \frac{G_B(x,z)G_B(z,y)}{G_B(x,y)} dz & \text{for } x, y \in B,\ x \neq y, \\ 0 & \text{for } x = y \in \bar{B}, \\ \displaystyle\int_B \frac{K_B(x,z)G_B(z,y)}{K_B(x,y)} dz & \text{for } x \in \partial B,\ y \in B, \\ \displaystyle\int_B \frac{K_B(y,z)G_B(z,x)}{K_B(y,x)} dz & \text{for } x \in B,\ y \in \partial B, \\ \frac{n\omega_n}{2} |x-y|^n \displaystyle\int_B K_B(x,z)K_B(y,z)dz & \text{for } x, y \in \partial B,\ x \neq y. \end{cases} \tag{5}$$

One may show that $(x,y) \mapsto H(x,y)$ is continuous on $\bar{B}^2$. Here for $x \in \partial B$ and $y \in B$

$$K_B(x,y) := \frac{1}{n\omega_n} \frac{1 - |y|^2}{|x-y|^n},$$

is the Poisson kernel for

$$\begin{cases} -\Delta u = 0 & \text{in } B, \\ u = g & \text{on } \partial B, \end{cases} \tag{6}$$

that is, the solution of (6) is given by $u(x) = \int_{\partial B} K_B(y,x)g(y)dy$.

The main result of the paper is the following.

Theorem 1.1 *For every $y \in \bar{B}$ the function $x \mapsto H(x,y)$, defined in (5), is increasing along the "hyperbolic geodesics" through y in increasing Euclidean distance, and attains its maximum at opposite boundary points.*

Remark 1.1.1 *The hyperbolic geodesics in B are the intersection of B with the Euclidean circles that meet ∂B at right angle (see [17, page 66]). See Figure 1.*

The method used for the proof is similar to the one used in [7] but, to a certain extent, simpler. We look at the differential boundary value problem that the function satisfies and then apply the maximum principle. Compared with [7] the proof here is somewhat simplified since, in some cases, we are able to determine the sign of the functions via a geometrical reasoning. In the present setting we have also to study the case $x \mapsto H_\Omega(x,y)$ for y fixed at the boundary since a result as the one in [10] is not available in dimensions $n \geq 3$.

We remark that although $x \mapsto H_\Omega(x,y)$ is increasing along the "hyperbolic geodesics" through y in increasing Euclidean distance, this is not the 'best' increasing direction. Indeed the gradient of $H_\Omega(\cdot,y)$ has also a non-zero component in a direction orthogonal to the "hyperbolic geodesics" through y (see Remarks 2.5.1 and 3.6.1).

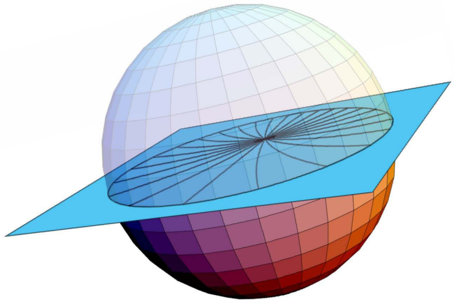

Figure 1: *A generic hyperbolic geodesic through y in $B \subset \mathbb{R}^n$ is obtained in the following way. One considers a generic disk in B to which the origin and y belong. Each hyperbolic geodesic through y in this disk is a hyperbolic geodesic through y in $B \subset \mathbb{R}^n$.*

2 One point fixed in the interior

In the following section we study the function $x \mapsto H(x, y)$ with y fixed in B. Without loss of generality, we can fix $y = -se_1$ with $s \in (0, 1)$ and $e_1 = (1, 0, .., 0) \in \mathbb{R}^n$. The main result of the section is the following.

Theorem 2.1 *Let $s \in (0, 1)$. The function $x \mapsto H(x, -se_1)$ is increasing along the "hyperbolic geodesic" through $-se_1$ in increasing Euclidean distance and attains its maximum at the boundary in the point $x = e_1$.*

2.1 Transformation to the center

Instead of studying directly the function $x \mapsto H(x, -se_1)$ it is convenient to consider a transformation. We consider a (anti-)conformal map h_s from B onto B that maps 0 into $y = -se_1$ and e_1 into e_1 given by

$$\begin{aligned} h_s(x_1, x_2, .., x_n) &= \frac{\left(Id + s^2 Q\right) x}{|sx - e_1|^2} - s\frac{1 + |x|^2}{|sx - e_1|^2} e_1 \\ &= -\frac{1}{s} e_1 - \frac{1 - s^2}{s} \frac{sQx - e_1}{|sx - e_1|^2}, \end{aligned} \tag{7}$$

where $Q_{11} = 1$, $Q_{ii} = -1$ for $i = 2, \ldots, n$ and $Q_{ij} = 0$ for $i, j = 1, \ldots, n$ and $i \neq j$. Notice that h_s is conformal if the dimension n is even, is anti-conformal if the dimension n is odd. One can also see h_s as the combination of the following mappings

$$x \longmapsto Qx - \tfrac{1}{s} e_1 \longmapsto \frac{Qx - \frac{1}{s} e_1}{\left|Qx - \frac{1}{s} e_1\right|^2} \longmapsto -\frac{1-s^2}{s} \frac{sQx - e_1}{|sx - e_1|^2} \longmapsto -\frac{1}{s} e_1 - \frac{1-s^2}{s} \frac{sQx - e_1}{|sx - e_1|^2}.$$

Using the (anti-)conformal transformation h_s, we can write

$$\begin{aligned} H(\tilde{x}, y) &= \int_B \frac{G_B(\tilde{x}, z) G_B(z, y)}{G_B(\tilde{x}, y)} dz \\ &= \int_B \frac{G_B(\tilde{x}, h_s(z')) G_B(h_s(z'), y)}{G_B(\tilde{x}, y)} J_{h_s}(z') dz', \end{aligned}$$

where J_{h_s} is the Jacobian of the transformation h_s. By the definition of the function h_s and Lemma B.2 we find

$$\begin{aligned} H(h_s(x), h_s(0)) &= \int_B \frac{G_B(h_s(x), h_s(z')) G_B(h_s(z'), h_s(0))}{G_B(h_s(x), h_s(0))} J_{h_s}(z') dz' \\ &= \int_B \frac{G_B(x, z') G_B(z', 0)}{G_B(x, 0)} J_{h_s}^{\frac{2}{n}}(z') dz'. \end{aligned} \tag{8}$$

For simplicity of notation we define on B the function H^s given by

$$H^s(x) := \int_B \frac{G_B(x,z)G_B(z,0)}{G_B(x,0)} J_{h_s}^{\frac{2}{n}}(z)dz.$$

Since

$$-\Delta\frac{a}{b} - 2\frac{\nabla b}{b}.\nabla\frac{a}{b} - \frac{a}{b^2}\Delta b = -\frac{\Delta a}{b}, \tag{9}$$

one sees that the function H^s satisfies in $B\backslash\{0\}$ the equation

$$-\Delta_x H^s(x) - 2\frac{\nabla_x G_B(x,0)}{G_B(x,0)} \cdot \nabla_x H^s(x) = J_{h_s}^{\frac{2}{n}}(x). \tag{10}$$

We can rewrite (10) as

$$-\Delta_x H^s(x) = 2(2-n)\frac{|x|^{-n}}{|x|^{2-n}-1} x \cdot \nabla_x H^s(x) + J_{h_s}^{\frac{2}{n}}(x), \tag{11}$$

using the explicit formula of the Green function.

2.2 The radial direction

In the following section we show that the function H^s is increasing in radial direction. The method consists in studying the differential boundary value problem that $\frac{\partial}{\partial r}H^s$ satisfies and then apply the maximum principle.

We first prove that H^s satisfies zero Neumann boundary condition.

Lemma 2.2 *Let $s \in (0,1)$. It holds that $\frac{\partial}{\partial r}H^s(x) = 0$ for every $x \in \partial B$.*

Proof. Let $R^s(x)$ denote the numerator of $H^s(x)$; that is

$$R^s(x) := \int_B G_B(x,z)G_B(z,0)J_{h_s}^{\frac{2}{n}}(z)dz. \tag{12}$$

One has that $R^s(x) = 0$ for $x \in \partial B$ and that it holds

$$-\Delta R^s(x) = G_B(x,0)J_{h_s}^{\frac{2}{n}}(x).$$

Since $-\Delta = -r^{1-n}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right) - r^{-2}\Delta_\Gamma$ where Δ_Γ is the Laplace-Beltrami operator on the surface of the unit ball, we find that at the boundary

$$\begin{aligned} \tfrac{\partial}{\partial r^2}R^s(x) &= -(n-1)\tfrac{\partial}{\partial r}R^s(x), \\ \tfrac{\partial}{\partial r^2}G_B(x,0) &= -(n-1)\tfrac{\partial}{\partial r}G_B(x,0). \end{aligned} \tag{13}$$

Hence from the series expansion near the boundary of $R^s(\cdot)$ and $G_B(\cdot,0)$ one gets for $x \in \partial B$

$$\lim_{B\ni\xi\to x}\frac{\partial}{\partial r}H^s(\xi) = \lim_{B\ni\xi\to x}\frac{\frac{\partial}{\partial r}G_B(\xi,0)}{G_B(\xi,0)}\left(\frac{\frac{\partial}{\partial r}R^s(\xi)}{\frac{\partial}{\partial r}G_B(\xi,0)} - \frac{R^s(\xi)}{G_B(\xi,0)}\right)$$

$$= \lim_{B\ni\xi\to x}\frac{(2-n)}{|\xi|^{2-n}-1}\left(\frac{\frac{\partial}{\partial r}R^s(x) + (|\xi|-1)\frac{\partial^2}{\partial r^2}R^s(x) + ..}{\frac{\partial}{\partial r}G_B(x,0) + (|\xi|-1)\frac{\partial^2}{\partial r^2}G_B(x,0) + ..} - \frac{\frac{\partial}{\partial r}R^s(x) + \frac{|\xi|-1}{2}\frac{\partial^2}{\partial r^2}R^s(x) + ..}{\frac{\partial}{\partial r}G_B(x,0) + \frac{|\xi|-1}{2}\frac{\partial^2}{\partial r^2}G_B(x,0) + ..}\right)$$

$$= \frac{1}{2}\frac{\frac{\partial^2}{\partial r^2}R^s(x)\frac{\partial}{\partial r}G_B(x,0) - \frac{\partial}{\partial r}R^s(x)\frac{\partial^2}{\partial r^2}G_B(x,0)}{\left(\frac{\partial}{\partial r}G_B(x,0)\right)^2}. \tag{14}$$

The claim follows from (14) using (13). ∎

We now show that $r\frac{\partial}{\partial r}H^s(x)$ is well defined in 0.

Lemma 2.3 *Let $s \in (0,1)$. Then* $\lim_{x\to 0} r\frac{\partial}{\partial r}H^s(x) = 0$.

Proof. With R^s defined as in (12) one finds

$$r\frac{\partial}{\partial r}H^s(x) = x.\nabla H^s(x) = \frac{x}{G_B(x,0)}.\nabla R^s(x) - \frac{R^s(x)}{G_B(x,0)}\frac{x}{G_B(x,0)}.\nabla G_B(x,0). \tag{15}$$

Since

$$\frac{x}{G_B(x,0)}.\nabla G_B(x,0) = \frac{(2-n)\,|x|^{2-n}}{|x|^{2-n}-1} = \frac{2-n}{1-|x|^{n-2}},$$

and since from Lemma A.1 and [15, Sec.5] (see Remark 2.3.1) it follows that

$$\frac{R^s(x)}{G_B(x,0)} \le \frac{(1-s^2)^2}{(s-1)^4}\frac{1}{G_B(x,0)}\int_B G_B(x,z)G_B(z,0)dz \le \frac{(1-s^2)^2}{(s-1)^4}c_\Omega\,|x|\,,$$

we get

$$\lim_{x\to 0}\left(\frac{R^s(x)}{G_B(x,0)}\frac{x}{G_B(x,0)}.\nabla G_B(x,0)\right) = 0.$$

The other term in (15) is given by

$$\frac{x}{G_B(x,0)}.\nabla R^s(x) = -\frac{1}{n\omega_n}\frac{|x|^{n-2}}{1-|x|^{n-2}}\cdot$$

$$\cdot x.\int_B\left(|x-z|^{-n}(x-z) - \left|x\,|z| - \tfrac{z}{|z|}\right|^{-n}(x\,|z| - \tfrac{z}{|z|})\,|z|\right)\left(|z|^{2-n}-1\right)J_{h_s}^{\frac{2}{n}}(z)dz.$$

One sees directly that

$$\lim_{x\to 0}\frac{|x|^{n-2}}{1-|x|^{n-2}}\;x.\int_B\left|x\,|z| - \tfrac{z}{|z|}\right|^{-n}(x\,|z| - \tfrac{z}{|z|})\,|z|\,(|z|^{2-n}-1)J_{h_s}^{\frac{2}{n}}(z)dz = 0.$$

Hence to show that $\lim_{x\to 0}\frac{x}{G_B(x,0)}.\nabla R^s(x) = 0$ it is sufficient to prove that the limit for x going to 0 of the modulus of

$$\frac{|x|^{n-2}}{1-|x|^{n-2}}\;x.\int_B|x-z|^{-n}(x-z)(|z|^{2-n}-1)J_{h_s}^{\frac{2}{n}}(z)dz,$$

is zero. One has

$$\begin{aligned}&\lim_{x\to 0}\frac{|x|^{n-2}}{1-|x|^{n-2}}\left|x.\int_B|x-z|^{-n}(x-z)(|z|^{2-n}-1)J_{h_s}^{\frac{2}{n}}(z)dz\right|\\ \le\;& 4\frac{(1-s^2)^2}{(1-s)^4}\lim_{x\to 0}\left(|x|^{n-1}\int_B|x-z|^{1-n}|z|^{2-n}\,dz\right).\end{aligned}$$

We study separately the integral term. Writing

$$\begin{aligned}|x|^{n-1}\int_B|x-z|^{1-n}|z|^{2-n}\,dz \;=\;& |x|^{n-1}\int_{|z|<\frac{|x|}{2}}|x-z|^{1-n}|z|^{2-n}\,dz\\ &+|x|^{n-1}\int_{B\setminus\left\{|z|<\frac{|x|}{2}\right\}}|x-z|^{1-n}|z|^{2-n}\,dz = \ldots,\end{aligned}$$

since $|x-z| \ge \frac{|x|}{2}$ for $|z| < \frac{|x|}{2}$, one finds

$$\cdots \le 2^{n-1}\int_{|z|<\frac{|x|}{2}}|z|^{2-n}\,dz + 2^{n-2}\,|x|\int_{B\setminus\left\{|z|<\frac{|x|}{2}\right\}}|x-z|^{1-n}\,dz,$$

that goes to zero for x going to 0. ■

Remark 2.3.1 *In [15] it is proved that for $x, y \in \Omega$ it holds*

$$\begin{aligned} H_\Omega(x,y) &\le c_\Omega \left(\ln \frac{C_\Omega}{|x-y|}\right)^{-1} \text{ for } n=2, \\ H_\Omega(x,y) &\le c_\Omega |x-y| \text{ for } n \ge 3, \\ H_\Omega(x,y) &\le c_{\Omega,\varepsilon} |x-y|^{2-\varepsilon} \text{ for } n \ge 4 \text{ and } \varepsilon > 0. \end{aligned}$$

Notice that there is a different behavior for $n = 2$ and $n \ge 3$ but also between the case $n = 3$ and $n \ge 4$.

Proposition 2.4 *For every $x \in B$ it holds that $r\frac{\partial}{\partial r} H^s(x) \ge 0$.*

Proof. Let Σ denote $r\frac{\partial}{\partial r} H^s(x)$(which is equal to $x.\nabla H^s(x)$). By definition of Σ and (11) one has that

$$\begin{aligned} -\Delta\Sigma(x) &= -2\Delta_x H^s(x) - x.\nabla\Delta_x H^s(x) \\ &= 4(2-n)\frac{|x|^{-n}}{|x|^{2-n}-1}\Sigma(x) + 2J_{h_s}^{\frac{2}{n}}(x) + 2(2-n)x.\nabla\left(\frac{|x|^{-n}}{|x|^{2-n}-1}\Sigma(x)\right) + x.\nabla J_{h_s}^{\frac{2}{n}}(x). \end{aligned}$$

Hence Σ satisfies

$$-\Delta\Sigma(x) - 2(2-n)\frac{|x|^{-n}}{|x|^{2-n}-1}x.\nabla\Sigma(x) + 2(n-2)^2\frac{|x|^{-n}}{(|x|^{2-n}-1)^2}\Sigma(x) = 2J_{h_s}^{\frac{2}{n}}(x) + x.\nabla J_{h_s}^{\frac{2}{n}}(x), \quad (16)$$

and the right hand side in (16) is positive. Indeed from Lemma A.1 and since $s \in (0,1)$ it holds for $x \in B$

$$\begin{aligned} 2J_{h_s}^{\frac{2}{n}}(x) + x.\nabla J_{h_s}^{\frac{2}{n}}(x) &= (1-s^2)^2\left(\frac{2}{|sx-e_1|^4} + x.\nabla\frac{1}{|sx-e_1|^4}\right) \\ &= 2(1-s^2)^2\left(\frac{1}{|sx-e_1|^4} - \frac{(sx-e_1).2sx}{|sx-e_1|^6}\right) \\ &= -2(1-s^2)^2\frac{(sx-e_1).(sx+e_1)}{|sx-e_1|^6} \\ &= 2(1-s^2)^2\frac{1-s^2|x|^2}{|sx-e_1|^6} > 0. \end{aligned}$$

Using the result of Lemmas 2.2 and 2.3 one finds

$$\begin{cases} -\Delta\Sigma(x) - 2(2-n)\frac{|x|^{-n}}{|x|^{2-n}-1}x.\nabla\Sigma(x) + 2(n-2)^2\frac{|x|^{-n}}{(|x|^{2-n}-1)^2}\Sigma(x) \ge 0 & \text{in } B\setminus\{0\}, \\ \Sigma(x) = 0 & \text{on } \partial B \cup \{0\}. \end{cases}$$

The claim follows by the maximum principle. ■

2.3 Behavior at the boundary

In the previous section we have shown that $x \mapsto H^s(x)$ is radially increasing. Hence it remains to study the behavior at the boundary of this function.

For $x \in \partial B$ one finds

$$\begin{aligned} H^s(x) &= \int_B \frac{K_B(x,z)G_B(z,0)}{K_B(x,0)} J_{h_s}^{\frac{2}{n}}(z)dz \\ &= \tfrac{(1-s^2)^2}{n(n-2)\omega_n}\int_B \frac{1-|z|^2}{|x-z|^n}\frac{|z|^{2-n}-1}{|sz-e_1|^4}dz. \end{aligned} \quad (17)$$

Lemma 2.5 *It holds that* $\max_{x\in\partial B} H^s(x) = H^s(e_1)$.

Proof. We first notice that by symmetry it is sufficient to consider $x = (x_1, x_2, \vec{0})$ with $\vec{0} \in \mathbb{R}^{n-2}$ and $x_1^2 + x_2^2 = 1$. Then in order to see how the function $H^s(x)$ varies when x belongs to this circumference we consider

$$\frac{\partial}{\partial\theta} H^s(x) = -x_2 \frac{\partial}{\partial x_1} H^s(x) + x_1 \frac{\partial}{\partial x_2} H^s(x).$$

From (17) one finds

$$\begin{aligned} \frac{\partial}{\partial\theta} H^s(x) &= \tfrac{-(1-s^2)^2}{(n-2)\omega_n} \int_B (1-|z|^2) \frac{|z|^{2-n}-1}{|sz-e_1|^4} \frac{-x_2(x_1-z_1)+x_1(x_2-z_2)}{|x-z|^{n+2}} dz \\ &= \tfrac{(1-s^2)^2}{(n-2)\omega_n} \int_B (1-|z|^2) \frac{|z|^{2-n}-1}{|sz-e_1|^4} \frac{x_1z_2-x_2z_1}{|x-z|^{n+2}} dz. \end{aligned}$$

We now study the sign of the integral. Let

$$B_p := \{z \in B : x_1z_2 - x_2z_1 > 0\} \text{ and } B_n := \{z \in B : x_1z_2 - x_2z_1 < 0\}.$$

One sees that if $\xi \in B_p$ then $-\xi \in B_n$ and that the intersection of the closure of B_p and B_n is a hyperplane in $\mathbb{R}^n$ going through x and the origin.

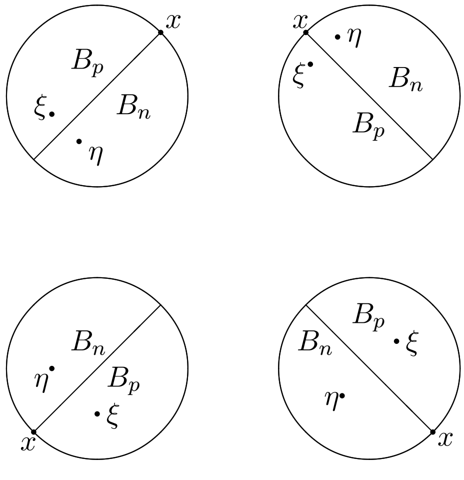

Figure 2: *The sets B_p and B_n for different positions of x.*

Let $\xi \in B_p$ and let η the unique element in B_n such that: $|\xi| = |\eta|$, $\xi_i = \eta_i$ for every $i \geq 3$ and $|x-\xi| = |x-\eta|$. By the choice it follows that

$$(1-|\xi|^2)(|\xi|^{2-n}-1)\frac{x_1\xi_2 - x_2\xi_1}{|x-\xi|^{n+2}} = -(1-|\eta|^2)(|\eta|^{2-n}-1)\frac{x_1\eta_2 - x_2\eta_1}{|x-\eta|^{n+2}}.$$

We notice that the term

$$(1-|\xi|^2)\frac{|\xi|^{2-n}-1}{|s\xi-e_1|^4}\frac{x_1\xi_2 - x_2\xi_1}{|x-\xi|^{n+2}} + (1-|\eta|^2)\frac{|\eta|^{2-n}-1}{|s\eta-e_1|^4}\frac{x_1\eta_2 - x_2\eta_1}{|x-\eta|^{n+2}},$$

is positive if $x_2 < 0$, is negative if $x_2 > 0$ and is zero if $x_2 = 0$. This follows from the observation that

$$\begin{aligned} s\left|\xi - \tfrac{1}{s}e_1\right| &< s\left|\eta - \tfrac{1}{s}e_1\right| \text{ if } x_2 < 0, \\ s\left|\xi - \tfrac{1}{s}e_1\right| &= s\left|\eta - \tfrac{1}{s}e_1\right| \text{ if } x_2 = 0, \\ s\left|\xi - \tfrac{1}{s}e_1\right| &> s\left|\eta - \tfrac{1}{s}e_1\right| \text{ if } x_2 > 0, \end{aligned}$$

(See Figure 3).

Repeating the same reasoning for every $\xi \in B_p$ we get that $x_2 \frac{\partial}{\partial\theta} H^s(x) \leq 0$ for every $x \in \partial B$ with $x = (x_1, x_2, \vec{0})$. Hence, by symmetry it follows that $\sup_{x\in\partial B} H^s(x) = H^s(e_1)$. ■

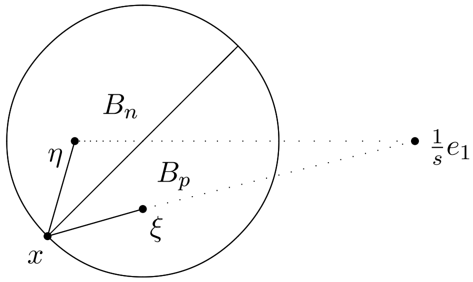

Figure 3: *The distances* $|\eta - x|$, $|\xi - x|$, $|\xi - \frac{1}{s}e_1|$ *and* $|\eta - \frac{1}{s}e_1|$.

Remark 2.5.1 *With the same method used in the proof of Proposition 2.5 one can prove that*

$$x_2 \frac{\partial}{\partial\theta} H^s(x) \leq 0 \text{ for } \left\{x \in B : x_i = 0 \text{ for } i \geq 3 \text{ and } x_1^2 + x_2^2 \leq 1\right\},$$

wriring $x_1 = r\cos(\theta)$ *and* $x_2 = r\sin(\theta)$. *This inequality gives that* $\nabla H^s(x)$ *has a non-zero component in the tangential direction, implying that* $\nabla H(x,y)$ *has not the direction of the hyperbolic geodesic through* y.

Corollary 2.6 *Let* $s \in (0,1)$. *The function* $H^s(x)$ *is radially increasing in* B *and*

$$\max_{x\in\bar{B}} H^s(x) = H^s(e_1).$$

Theorem 2.1 is a consequence of the previous corollary.

3 One point fixed at the boundary

In this section we study the function $x \mapsto H(x,y)$ with $y \in \partial B$. Without loss of generality, we can fix $y = e_1$. The main result is the following.

Theorem 3.1 *The function* $x \to H(x, e_1)$ *is increasing along the "hyperbolic geodesic" through* e_1 *in increasing Euclidean distance, and attains its maximum at the boundary at* $x = -e_1$.

Theorem 1.1 will follow from Theorems 2.1 and 3.1.

3.1 Transformation to the half n-space

Instead of studying the problem in the ball it is convenient to consider a transformation to the half n-space. We consider a (anti-)conformal map φ from $S := \mathbb{R}^+ \times \mathbb{R}^{n-1}$, the half n-space, onto B that maps 0 into $-e_1$ and e_1 into 0 given by

$$\varphi(X_1, X_2, .., X_n) = e_1 - 2\frac{QX + e_1}{|X + e_1|^2}, \tag{18}$$

where $Q_{11} = 1$, $Q_{ii} = -1$ for $i = 2, \ldots, n$ and $Q_{ij} = 0$ for $i,j = 1,\ldots,n$ and $i \neq j$. The map φ is conformal if the dimension n is even, is anti-conformal if the dimension n is odd.

In the following, to avoid ambiguity in the notation, we denote with capital letters the coordinates on the half n-space.

Using the (anti-)conformal transformation φ, we can write

$$\begin{aligned} H(x, e_1) &= \int_B \frac{K_B(e_1, z) G_B(z, x)}{K_B(e_1, x)} dz \\ &= \int_S \frac{K_B(e_1, \varphi(Z)) G_B(\varphi(Z), x)}{K_B(e_1, x)} J_\varphi(Z) dZ, \end{aligned}$$

where J_φ is the Jacobian of the transformation φ. By the definition of the function φ and Lemma B.2 we find

$$\begin{aligned} H(\varphi(X), e_1) &= \int_S \frac{K_B(e_1, \varphi(Z)) G_B(\varphi(Z), \varphi(X))}{K_B(e_1, \varphi(X))} J_\varphi(Z) dZ \\ &= \int_S \frac{K_B(e_1, \varphi(Z)) G_S(Z, X)}{K_B(e_1, \varphi(X))} (J_\varphi(Z) J_\varphi(X))^{\frac{1}{n}-\frac{1}{2}} J_\varphi(Z) dZ. \end{aligned}$$

Since

$$K_B(e_1, \varphi(Z)) = \tfrac{1}{n\omega_n} \frac{|Z+e_1|^{n-2}}{2^{n-2}} Z_1 = \tfrac{1}{n\omega_n} \frac{1}{2^{\frac{n}{2}-1}} J_\varphi(Z)^{\frac{1}{n}-\frac{1}{2}} Z_1,$$

one has

$$\begin{aligned} H(\varphi(X), e_1) &= \int_S \frac{J_\varphi(Z)^{\frac{1}{n}-\frac{1}{2}}}{J_\varphi(X)^{\frac{1}{n}-\frac{1}{2}}} \frac{Z_1}{X_1} G_S(Z, X) (J_\varphi(Z) J_\varphi(X))^{\frac{1}{n}-\frac{1}{2}} J_\varphi(Z) dZ \\ &= \tfrac{1}{n(n-2)\omega_n} \int_S \frac{Z_1}{X_1} \left(|X-Z|^{2-n} - |X+QZ|^{2-n} \right) J_\varphi(Z)^{\frac{2}{n}} dZ. \end{aligned}$$

For simplicity of notation we define the function $\tilde{H}$ given by

$$\tilde{H}(X) := \tfrac{1}{n(n-2)\omega_n} \int_S \frac{Z_1}{X_1} \left(|X-Z|^{2-n} - |X+QZ|^{2-n} \right) J_\varphi(Z)^{\frac{2}{n}} dZ.$$

3.2 Increasing along the "hyperbolic geodesics" through e_1

In the following section we show that the function $x \mapsto H(x, e_1)$ is increasing along the "hyperbolic geodesics" through e_1. That's equivalent to prove that the function $\tilde{H}(X)$ is decreasing in the X_1 direction. Indeed, the pre-image through the mapping φ, defined in (18), of the hyperbolic geodesics in B through e_1 are the straight lines in S that intersect the hyperplane $\{X_1 = 0\}$ orthogonally.

Let $\tilde{H}_{X_1}$ denote $\frac{\partial}{\partial X_1} \tilde{H}(X)$. We proceed studying the differential boundary value problem that $\tilde{H}_{X_1}$ satisfies in order to apply the maximum principle.

Since ∂S is composed of two parts, $\partial S = \{Z \in \mathbb{R}^n : Z_1 = 0\} \cup \{\infty\}$, we treat those separately. In the following $\{Z_1 = 0\}$ denotes the hyperplane $\{Z \in \mathbb{R}^n : Z_1 = 0\}$.

Lemma 3.2 *It holds that* $\tilde{H}_{X_1}(X) = 0$ *for* $X \in \{X_1 = 0\}$.

Proof. Writing $\tilde{H}(X) = \frac{1}{X_1} \tilde{R}(X)$ with

$$\tilde{R}(X) := \tfrac{1}{n(n-2)\omega_n} \int_S Z_1 \left(|X-Z|^{2-n} - |X+QZ|^{2-n} \right) J_\varphi(Z)^{\frac{2}{n}} dZ,$$

one finds

$$\tilde{H}_{X_1}(X) = \frac{1}{X_1} \left(\frac{\partial}{\partial X_1} \tilde{R}(X) - \frac{\tilde{R}(X)}{X_1} \right).$$

We first notice that since $\tilde{R}(X) = 0$ for $X \in \{X_1 = 0\}$ and $-\Delta \tilde{R}(X) = X_1 J_\varphi(X)^{\frac{2}{n}}$, one finds that $\frac{\partial^2}{\partial X_1^2} \tilde{R}(X) = 0$ holds on $\{X_1 = 0\}$. Hence using the series expansion near $X \in \{X_1 = 0\}$

$$\begin{aligned} \lim_{S \ni Y \to X} \tilde{H}_{X_1}(Y) &= \lim_{S \ni Y \to X} \frac{1}{Y_1} \left(\frac{\partial}{\partial X_1} \tilde{R}(X) + Y_1 \frac{\partial^2}{\partial X_1^2} \tilde{R}(X) + .. - \frac{\partial}{\partial X_1} \tilde{R}(X) - \frac{1}{2} Y_1 \frac{\partial^2}{\partial X_1^2} \tilde{R}(X) + .. \right) \\ &= \frac{1}{2} \frac{\partial^2}{\partial X_1^2} \tilde{R}(X) = 0. \end{aligned}$$

The claim follows. ∎

Lemma 3.3 *It holds that* $\lim_{|X|\to\infty} \tilde{H}_{X_1}(X) = 0.$

Proof. Since

$$\begin{aligned}\tilde{H}_{X_1}(X) &= -\tfrac{1}{n\omega_n}\int_S \frac{Z_1}{X_1}\left(|X-Z|^{-n}(X_1-Z_1) - |X+QZ|^{-n}(X_1+Z_1)\right)\frac{2^2}{|Z+e_1|^4}dZ \\ &\quad -\tfrac{1}{n(n-2)\omega_n}\int_S \frac{Z_1}{X_1^2}\left(|X-Z|^{2-n} - |X+QZ|^{2-n}\right)\frac{2^2}{|Z+e_1|^4}dZ,\end{aligned}$$

and it holds $|X-Z| < |X+QZ|$, one has

$$\begin{aligned}\left|\tilde{H}_{X_1}(X)\right| &\le \tfrac{2^3}{n\omega_n}\int_S \frac{Z_1}{X_1}\frac{|X_1-Z_1|}{|X-Z|^n}\frac{1}{|Z+e_1|^4}dZ + \tfrac{2^3}{n(n-2)\omega_n}\int_S \frac{Z_1}{X_1^2}\frac{1}{|X-Z|^{n-2}}\frac{1}{|Z+e_1|^4}dZ \\ &\le \tfrac{2^3}{n\omega_n}\frac{1}{X_1}\int_S \frac{1}{|X-Z|^{n-1}}\frac{1}{|Z+e_1|^3}dZ + \tfrac{2^3}{n(n-2)\omega_n}\frac{1}{X_1^2}\int_S \frac{1}{|X-Z|^{n-2}}\frac{1}{|Z+e_1|^3}dZ. \qquad (19)\end{aligned}$$

We now proceed studying separately the terms in the right hand side of (19). For the first term one finds

$$\begin{aligned}\int_S \frac{1}{|X-Z|^{n-1}}\frac{1}{|Z+e_1|^3}dZ &= \int_{S\cap B_{\frac{|X|}{2}}(X)} \frac{1}{|X-Z|^{n-1}}\frac{1}{|Z+e_1|^3}dZ + \\ &+ \int_{\substack{S\setminus B_{\frac{|X|}{2}}(X),\\ |Z|<2|X|}} \frac{1}{|X-Z|^{n-1}}\frac{1}{|Z+e_1|^3}dZ + \int_{\substack{S\setminus B_{\frac{|X|}{2}}(X),\\ |Z|>2|X|}} \frac{1}{|X-Z|^{n-1}}\frac{1}{|Z+e_1|^3}dZ = \dots.\end{aligned}$$

One observes that $|Z+e_1| > |Z| \ge \frac{|X|}{2}$ if $Z \in B_{\frac{|X|}{2}}(X)$. While if $Z \notin B_{\frac{|X|}{2}}(X)$ it holds $|X-Z| > \frac{|X|}{2}$ and even more $|X-Z| > \frac{|Z|}{2}$ if also $|Z| > 2|X|$. Hence we get

$$\begin{aligned}\dots &\le \frac{2^3}{|X|^3}\int_{S\cap B_{\frac{|X|}{2}}(X)} \frac{1}{|X-Z|^{n-1}}dZ + \frac{2^{n-1}}{|X|^{n-1}}\int_{S,|Z|<2|X|} \frac{1}{|Z+e_1|^3}dZ \\ &\quad +2^{n-1}\int_{S,|Z|>2|X|} \frac{1}{|Z|^{n-1}}\frac{1}{|Z+e_1|^3}dZ \\ &\le \frac{C_1}{|X|^2} + \frac{2^{n-1}}{|X|^{n-1}}\int_{|Z|<2|X|}\frac{1}{|Z|^2}dZ + 2^{n-1}\int_{|Z|>2|X|}\frac{1}{|Z|^{n+2}}dZ \\ &\le \frac{C_1}{|X|^2} + \frac{C_2}{|X|^{n-1}}|X|^{n-2} + \frac{C_3}{|X|^2},\end{aligned}$$

that goes to zero when $|X|$ goes to infinity. Proceeding similarly one finds also that

$$\lim_{|X|\to\infty}\int_S \frac{1}{|X-Z|^{n-2}}\frac{1}{|Z+e_1|^3}dZ = 0.$$

The claim follows. ∎

Proposition 3.4 *The function* $\tilde{H}(X)$ *is decreasing in the* X_1 *direction.*

Proof. Since it holds

$$-\Delta\tilde{H}(X) = J_\varphi(X)^{\frac{2}{n}} + \frac{2}{X_1}\frac{\partial}{\partial X_1}\tilde{H}(X),$$

one gets

$$-\Delta\tilde{H}_{X_1}(X) - \frac{2}{X_1}\frac{\partial}{\partial X_1}\tilde{H}_{X_1}(X) + \frac{2}{X_1^2}\tilde{H}_{X_1}(X) = \frac{\partial}{\partial X_1}J_\varphi(X)^{\frac{2}{n}} = -2^4\frac{X_1+1}{|X+e_1|^6} \le 0.$$

Hence the function $\tilde{H}_{X_1}$ satisfies

$$\begin{cases} -\Delta\tilde{H}_{X_1}(X) - \frac{2}{X_1}\frac{\partial}{\partial X_1}\tilde{H}_{X_1}(X) + \frac{2}{X_1^2}\tilde{H}_{X_1}(X) & \leq & 0 & \text{in } S, \\ \tilde{H}_{X_1} & = & 0 & \text{on } \partial S. \end{cases}$$

Applying the maximum principle we find that $\tilde{H}_{X_1} \leq 0$ on S. ■

By the result in the previous proposition and using that the hyperbolic geodesics are transformed onto hyperbolic geodesics by Möbious transformations, we get the following.

Corollary 3.5 *The function $x \mapsto H(x, e_1)$ is increasing along the "hyperbolic geodesics" through e_1 in increasing Euclidean distance.*

3.3 Behavior at the boundary

In this section we study the behavior of $x \mapsto H(x, e_1)$ on ∂B. Indeed, since by the result of the previous section we already know that

$$\max_{x\in\bar{B}} H(x, e_1) = \max_{x\in\partial B} H(x, e_1),$$

it only remains to find the location on ∂B of this maximum. Also in this case it is convenient to use the transformation φ, defined in (18), and to work in the half n-space.

Proposition 3.6 *For any $i \in \{2, \ldots, n\}$ it holds that $X_i\frac{\partial}{\partial X_i}\tilde{H}(X) \leq 0$ on $\{X_1 = 0\}$.*

Proof. We find that for $X \in \{X_1 = 0\}$

$$\tilde{H}(X) = \frac{2}{n\omega_n}\int_S \frac{Z_1^2}{|X - Z|^n} J_\varphi(Z)^{\frac{2}{n}} dZ.$$

Fix $i \in \{2, \ldots, n\}$ and $X \in \{X_1 = 0\}$. We have

$$\frac{\partial}{\partial X_i}\tilde{H}(X) = \frac{2^3}{\omega_n}\int_S \frac{Z_1^2}{|X - Z|^{n+2}}\frac{Z_i - X_i}{|Z + e_1|^4} dZ. \tag{20}$$

We will now determine the sign of the integral in (20). Let

$$S_{p,i} := \{Z \in S : Z_i - X_i > 0\} \text{ and } S_{n,i} := \{Z \in S : Z_i - X_i < 0\}.$$

Let $P \in S_{p,i}$ and let P' the unique element in $S_{n,i}$ such that: $P_j = P'_j$ for $j \in \{1, \ldots, n\}$ with $j \neq i$, and $|X - P| = |X - P'|$. By the choice it follows that

$$\frac{P_1^2}{|X - P|^{n+2}}(P_i - X_i) = -\frac{P_1'^2}{|X - P'|^{n+2}}(P'_i - X_i).$$

We notice that the term

$$\frac{P_1^2}{|X - P|^{n+2}}\frac{P_i - X_i}{|P + e_1|^4} + \frac{P_1'^2}{|X - P'|^{n+2}}\frac{P'_i - X_i}{|P' + e_1|^4},$$

is positive if $X_2 < 0$, is negative if $X_2 > 0$ and is zero if $X_2 = 0$. This follows from the observation that

$$\begin{aligned} |P' + e_1| &> |P + e_1| \text{ if } X_2 < 0, \\ |P' + e_1| &= |P + e_1| \text{ if } X_2 = 0, \\ |P' + e_1| &< |P + e_1| \text{ if } X_2 > 0, \end{aligned}$$

(see Figure 4).

The claim follows repeating the same reasoning for every $P \in S_{p,i}$ and $i \in \{2, \ldots, n\}$. ■

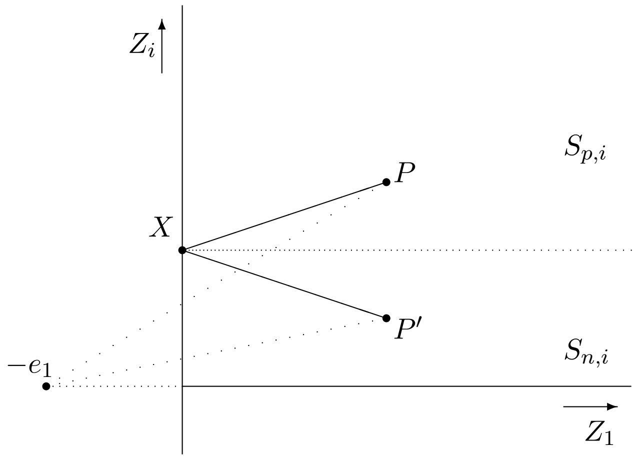

Figure 4: *The sets $S_{p,i}$, $S_{n,i}$ and the distance to $-e_1$.*

Remark 3.6.1 *With the same method used in the proof of Proposition 3.6 one can prove that*

$$X_i \frac{\partial}{\partial X_i} \tilde{H}(X) \leq 0 \text{ for every } X \in S \text{ and } i \in \{2, \ldots, n\}\,. \tag{21}$$

Notice that from (21) it follows that $\nabla \tilde{H}(X)$ is not in the X_1 direction. For the function $H(\cdot, e_1)$ this reads as $\nabla H(\cdot, e_1)$ is not tangent to the hyperbolic geodesics through e_1.

Corollary 3.7 *The function $X \mapsto \tilde{H}(X)$ for $X \in \bar{S}$ attains its maximum in $X = \vec{0}$.*

Theorem 3.1 follows directly from the previous Corollary.

4 Relation with the eigenvalues

4.1 Previous results

In [11] the authors show that there exists a relation between the inverse of $\lambda_c(\Omega)$, defined in (3), and the Dirichlet eigenvalues for two choices of Ω : $\Omega = [0, 1] \subset \mathbb{R}$ (see also [16]) and Ω the unit disk. In an interval $I = [0, 1] \subset \mathbb{R}$ the following identities hold

$$\frac{1}{\lambda_c(I)} = \sum_{m=1}^{\infty} \frac{1}{\lambda_m} = 2 \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\lambda_m},$$

with $\lambda_m = \pi^2 m^2$. For the disk D it holds

$$\frac{1}{\lambda_c(D)} = 4 \sum_{m=0}^{\infty} (-1)^{m-1} \sum_{i=1}^{\infty} \frac{\nu_{m,i}}{\lambda_{m,i}}, \tag{22}$$

where $\nu_{0,i} = 1$ and $\nu_{m,i} = 2$ for $m \geq 1$. The eigenvalue $\lambda_{m,i}$ corresponds to the eigenfunctions with $i - 1$ circular nodal lines and m radial nodal lines.

We are now able to give an explanation to identity (22). A complete orthonormal set of eigenfunctions for (1) on the disk is given by, writing $x = re^{i\varphi}$:

$$\begin{aligned}
\varphi_{0,i}(x) &= \frac{1}{\sqrt{2\pi}} \frac{J_0(j_{0,i}r)}{\frac{1}{\sqrt{2}}\,|J_0'(j_{0,i})|} \text{ for } i \in \mathbb{N}, \\
\varphi_{e,m,i}(x) &= \frac{\cos(m\varphi)}{\sqrt{\pi}} \frac{J_m(j_{m,i}r)}{\frac{1}{\sqrt{2}}\,|J_m'(j_{m,i})|} \text{ for } m, i \in \mathbb{N}, \\
\varphi_{o,m,i}(x) &= \frac{\sin(m\varphi)}{\sqrt{\pi}} \frac{J_m(j_{m,i}r)}{\frac{1}{\sqrt{2}}\,|J_m'(j_{m,i})|} \text{ for } m, i \in \mathbb{N},
\end{aligned}$$

with eigenvalues

$$\lambda_{0,i} = j_{0,i}^2 \text{ and } \lambda_{e,m,i} = \lambda_{o,m,i} = j_{m,i}^2 \text{ for } i, m \in \mathbb{N}.$$

Here, as usual, J_m denotes the m-th Bessel function of the first kind and $j_{m,i}$ denotes the i-th zero of J_m. For the normalization of the Bessel function see [18, 5.11 (11)]. By orthonormality one finds

$$\begin{aligned}\frac{1}{\lambda_c(D)} &= \sup_{x,y\in D} \frac{1}{G_D(x,y)} \left[\sum_{i=1}^{\infty} \frac{1}{j_{0,i}^4} \frac{J_0(j_{0,i}r)J_0(j_{0,i}\rho)}{\pi J_0'^2(j_{0,i})} + \right. \\ &\left. + \sum_{m=1}^{\infty} \frac{1}{\pi} (\cos(m\varphi)\cos(m\varphi') + \sin(m\varphi)\sin(m\varphi')) \sum_{i=1}^{\infty} \frac{2}{j_{m,i}^4} \frac{J_m(j_{m,i}r)J_m(j_{m,i}\rho)}{J_m'^2(j_{m,i})} \right] \\ &= \lim_{\substack{x\to e_1,\\ y\to -e_1}} \frac{1}{\pi G_D(x,y)} \left[\sum_{i=1}^{\infty} \frac{1}{j_{0,i}^4} \frac{J_0(j_{0,i}r)J_0(j_{0,i}\rho)}{J_0'^2(j_{0,i})} + \right. \\ &\left. + 2\sum_{m=1}^{\infty} \frac{1}{\pi} (\cos(m\varphi)\cos(m\varphi') + \sin(m\varphi)\sin(m\varphi')) \sum_{i=1}^{\infty} \frac{1}{j_{m,i}^4} \frac{J_m(j_{m,i}r)J_m(j_{m,i}\rho)}{J_m'^2(j_{m,i})} \right] = \dots.\end{aligned}$$

Differentiating with respect to ρ and computing for $y = -e_1$, we get

$$\dots = \lim_{x\to e_1} \frac{1}{\pi K_D(x,-e_1)} \left[\sum_{i=1}^{\infty} \frac{1}{j_{0,i}^3} \frac{J_0(j_{0,i}r)}{J_0'(j_{0,i})} + 2\sum_{m=1}^{\infty} (-1)^m \cos(m\varphi) \sum_{i=1}^{\infty} \frac{1}{j_{m,i}^3} \frac{J_m(j_{m,i}r)}{J_m'(j_{m,i})} \right] = \dots,$$

and differentiating with respect to r and computing for $x = e_1$

$$\dots = \frac{1}{\pi} \frac{\sum_{i=1}^{\infty} \frac{1}{j_{0,i}^2} + 2\sum_{m=1}^{\infty} (-1)^m \sum_{i=1}^{\infty} \frac{1}{j_{m,i}^2}}{\frac{1}{4\pi}} = 4\left(\sum_{i=1}^{\infty} \frac{1}{j_{0,i}^2} + 2\sum_{m=1}^{\infty} (-1)^m \sum_{i=1}^{\infty} \frac{1}{j_{m,i}^2} \right).$$

In [11] the numbers $\nu_{m,i}$ in (22) were interpreted as the multiplicity of the eigenvalue $\lambda_{m,i}$, since it holds $\nu_{0,i} = 1$ and $\nu_{m,i} = 2$ for $m \geq 1$. Instead from the derivation of the formula it seems that what plays a role is the different normalization of the eigenfunctions.

4.2 The identity for $\lambda_c^{-1}(B)$

We will now show that an identity holds also between $\lambda_c^{-1}(B)$ and a sum of Dirichlet eigenvalues when B is the unit ball in $\mathbb{R}^3$. We first compute the value of $\lambda_c^{-1}(B)$ for $B \subset \mathbb{R}^n$ with $n \geq 3$. By Theorem 1.1 and (5) one has that it holds

$$\frac{1}{\lambda_c(B)} = H(-e_n, e_n) = \frac{2^{n-1}}{n\omega_n} \int_B \frac{(1-|z|^2)^2}{|z-e_n|^n\,|z+e_n|^n} dz.$$

Via a C.A.S. (computer algebra system) one finds the following

$$\frac{1}{\lambda_c(B)} = \frac{\sqrt{\pi}\left(2\Gamma(\frac{n}{2}) - (2+n)\Gamma(1+n)\ {}_2F_1(2+\frac{1}{2}n, n; 3+\frac{1}{2}n; -1)\right)}{4(n-1)\Gamma(\frac{1}{2}(n-1))},$$

where $\Gamma(\cdot)$ denotes the Gamma function and ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ denotes the Gauss hypergeometric function (see [1, Chap.6 and 15]). In the following table we collect the values of $\lambda_c^{-1}(B)$ with $B \subset \mathbb{R}^n$ for $n \leq 5$.

n	$\lambda_c^{-1}(B)$
1	$\frac{2}{3} \simeq 0.6666$
2	$2\log 2 - 1 \simeq 0.3862$
3	$2(\pi - 3) \simeq 0.2831$
4	$3 - 2\log(4) \simeq 0.2274$
5	$\frac{1}{3}(10 - 3\pi) \simeq 0.1917$

On the unit ball in $\mathbb{R}^3$ a complete orthonormal set of eigenfunctions is given in polar coordinates (r,φ,θ) by

$$u_{0,k,i}(r,\varphi,\theta)=\sqrt{\tfrac{2k+1}{4\pi}}P_k(\cos(\theta))\frac{j_k(j_{\frac{1}{2}+k,i}r)}{\frac{1}{\sqrt{2}}j_k'(j_{\frac{1}{2}+k,i})} \text{ with } k\in\mathbb{N}_0 \text{ and } i\in\mathbb{N},$$

and with $m,k,i\in\mathbb{N}$ and $k\geq m$,

$$\begin{aligned} u_{e,m,k,i}(r,\varphi,\theta) &= \sqrt{\tfrac{2k+1}{2\pi}}\sqrt{\tfrac{(k-|m|)!}{(k+|m|)!}}\cos(m\varphi)\,P_k^m(\cos(\theta))\frac{j_k(j_{k+\frac{1}{2},i}r)}{\frac{1}{\sqrt{2}}j_k'(j_{k+\frac{1}{2},i})},\\ u_{o,m,k,i}(r,\varphi,\theta) &= \sqrt{\tfrac{2k+1}{2\pi}}\sqrt{\tfrac{(k-|m|)!}{(k+|m|)!}}\sin(m\varphi)\,P_k^m(\cos(\theta))\frac{j_k(j_{k+\frac{1}{2},i}r)}{\frac{1}{\sqrt{2}}j_k'(j_{k+\frac{1}{2},i})},\end{aligned}$$

(see [14, App. A]). We use the usual convention: $0\leq r\leq 1$, $0\leq\varphi<2\pi$ and $0\leq\theta\leq\pi$. Here $P_k^m(\cdot)$ denotes the Legendre function, j_k denotes the fractional Bessel function of first kind and $j_{k+\frac{1}{2},i}$ denotes the i-th zero of j_k (see [1, Chap.8 and 10] and [18]). We choose this notation for the i-th zero of j_k since it coincides with the i-th zero of $J_{k+\frac{1}{2}}$. Notice that $j_k(z)=\frac{1}{\sqrt{z}}J_{k+\frac{1}{2}}(z)$.

The associated eigenvalues are

$$\lambda_{0,0,i}=\frac{1}{j^2_{\frac{1}{2},i}} \text{ and } \lambda_{0,k,i}=\lambda_{e,m,k,i}=\lambda_{o,m,k,i}=\frac{1}{j^2_{k+\frac{1}{2},i}} \text{ with } m,k,i\in\mathbb{N} \text{ and } k\geq m.$$

Notice that each eigenvalue has multiplicity $2k+1$. For simplicity of notation we fix

$$\mu_{k,i}=\frac{1}{j^2_{k+\frac{1}{2},i}} \text{ for } k\in\mathbb{N}_0 \text{ and } i\in\mathbb{N}. \tag{23}$$

Hence, $\mu_{k,i}$ for $k\in\mathbb{N}_0$ and $i\in\mathbb{N}$ are the eigenvalues for problem (1) on B the unit ball in $\mathbb{R}^3$ counted without multiplicity.

Lemma 4.1 *For $k\in\mathbb{N}_0$ and $i\in\mathbb{N}$ let $\mu_{k,i}$ as defined in (23). Then it holds that*

$$\frac{1}{\lambda_c(B)}=4\sum_{k=0}^{\infty}(-1)^{k+1}\nu_k\sum_{i=1}^{\infty}\frac{1}{\mu_{k,i}}, \tag{24}$$

with $\nu_0=1$ and $\nu_k=4$ for $k\geq 1$.

Proof. By [18, 15.51] one gets for $k\in\mathbb{N}_0$

$$\sum_{i=1}^{\infty}\frac{1}{\mu_{k,i}}=\sum_{i=1}^{\infty}\frac{1}{j^2_{k+\frac{1}{2},i}}=\frac{1}{4(k+\frac{3}{2})}.$$

Hence it holds

$$\begin{aligned} 4\sum_{k=0}^{\infty}(-1)^{k+1}\nu_k\sum_{i=1}^{\infty}\frac{1}{\mu_{k,i}} &= -4\sum_{i=1}^{\infty}\frac{1}{\mu_{0,i}}+16\sum_{k=1}^{\infty}(-1)^{k+1}\sum_{i=1}^{\infty}\frac{1}{\mu_{k,i}}\\ &= -\frac{1}{\frac{3}{2}}+4\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k+\frac{3}{2}}\\ &= -\frac{2}{3}+\frac{4}{6}(3\pi-8)=2(\pi-3).\end{aligned}$$

The claim follows. ■

Appendices

A The Jacobian

In the present section we compute the Jacobians of the transformations h_s and φ defined in (7) and (18) respectively.

Lemma A.1 *Let h_s the (anti-)conformal map defined in (7). For any $n \geq 3$ it holds that*

$$J_{h_s}(x) = \frac{(1-s^2)^n}{|sx - e_1|^{2n}}.$$

Proof. By the definition of h_s in (7) it follows

$$\frac{\partial}{\partial x_j} h_{s,i}(x) = -(1-s^2)\frac{\delta_i^j}{|sx-e_1|^2} Q\ e_j + 2(1-s^2)\frac{(sQx-e_1)_i(sx-e_1)_j}{|sx-e_1|^4},$$

that gives

$$(\partial_j h_{s,i}(x))_{i,j} = -\frac{(1-s^2)}{|sx-e_1|^2} Q \left(Id - 2\frac{sx-e_1}{|sx-e_1|}\left(\frac{sx-e_1}{|sx-e_1|}\right)^T \right),$$

using column notation for $sx - e_1$. The claim follows directly since the matrix $Id - 2\frac{sx-e_1}{|sx-e_1|}\left(\frac{sx-e_1}{|sx-e_1|}\right)^T$ defines the reflection in the hyperplane through 0 perpendicular to $sx - e_1$. ■

Lemma A.2 *Let φ be the (anti-)conformal map defined in (18). For any $n \geq 3$ it holds that*

$$J_\varphi(X) = \frac{2^n}{|X + e_1|^{2n}}.$$

Proof. The proof is similar to the one of Lemma A.1. One uses that by the definition of φ in (18) it holds

$$(\partial_j \varphi_i(x))_{i,j} = -\frac{2}{|X+e_1|^2} Q \left(Id - 2\frac{X+e_1}{|X+e_1|}\left(\frac{X+e_1}{|X+e_1|}\right)^T \right),$$

using column notation for $X + e_1$. ■

B Conformal transformation

In the following section, for completeness, we recall some known properties of conformal maps. The situation is different in $\mathbb{R}^n$ for $n = 2$ and $n \geq 3$.

Conformal maps are a very useful tool for problems in the plane. The first reason is that there are many conformal maps: every simply connected domain $D \subsetneq \mathbb{R}^2$ can be mapped conformally onto the ball (Riemann Mapping Theorem, [13]). A second important property of conformal maps is the 'invariance' of the Green function. The precise result is stated in the following lemma.

Lemma B.1 *Let A, $D \subsetneq \mathbb{R}^2$ simply connected and let $\varphi : A \to D$ a conformal map. Let G_A denote the Green function for the Laplace problem with Dirichlet boundary condition in A.*

Then it holds $G_D(\varphi(x), \varphi(y)) = G_A(x, y)$.

In higher dimension the situation is different. The only conformal mappings are the Möbious transforms. Liouville's Theorem, [12], states that every conformal transformation in $\mathbb{R}^n$ with $n \geq 3$ must necessarily reduce to a translation, a magnification, an orthogonal transformation, a reflection through reciprocal radii, or a combination of these elementary transformations. Moreover there is no 'invariance' of the Green function via conformal mappings. However a relation still holds. We write the result in the following lemma.

Lemma B.2 *Let A, $D \subsetneq \mathbb{R}^n$, $n \geq 3$, simply connected and let $\varphi : A \to D$ be a conformal map. Let J_φ denote the Jacobian of φ. Then it holds that*

$$G_D(\varphi(x), \varphi(y)) = (J_\varphi(x) J_\varphi(y))^{\frac{1}{n} - \frac{1}{2}} G_A(x, y).$$

Remark B.2.1 *The result stated in Lemma B.2 holds also if φ is an anti-conformal map since there is only a change in the orientation.*

Proof. In [8, Cor. 2] it is proved that for any Möbious transformation ψ in $\mathbb{R}^n$ and $k \in \mathbb{N}$ it holds

$$\Delta^k (J_\psi^{\frac{1}{2} - \frac{k}{n}} \, u \circ \psi) = J_\psi^{\frac{1}{2} + \frac{k}{n}} \left(\Delta^k u \right) \circ \psi. \tag{25}$$

In our setting using (25) with $k = 1$, we get that for any $x \in B$

$$\begin{aligned} u(\varphi(x)) &= J_\varphi^{\frac{1}{n} - \frac{1}{2}}(x) \int_A G_A(x, y) J_\varphi^{\frac{1}{2} + \frac{1}{n}}(y) (\Delta u)(\varphi(y)) dy \\ &= \int_A G_A(x, y) (J_\varphi(x) J_\varphi(y))^{\frac{1}{n} - \frac{1}{2}} (\Delta u)(\varphi(y)) J_\varphi(y) dy. \end{aligned} \tag{26}$$

We can also write

$$\begin{aligned} u(\varphi(x)) &= \int_D G_D(\varphi(x), z) \Delta u(z) dz \\ &= \int_A G_D(\varphi(x), \varphi(y)) (\Delta u)(\varphi(y)) J_\varphi(y) dy. \end{aligned} \tag{27}$$

The claim follows from (26) and (27). ■

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