Estimates for Green function and Poisson kernels of higher order Dirichlet boundary value problems

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Abstract

Pointwise estimates are derived for the kernels associated to the polyharmonic Dirichlet problem on bounded smooth domains. As a consequence one obtains optimal weighted L^p-L^q-regularity estimates for weights involving the distance function.

Key words: Polyharmonic Dirichlet problem, kernel estimates, weighted L^p-L^q-regularity.
1991 MSC: 35J40, 35B45, 31B25

1 Introduction

In this paper we present optimal pointwise estimates for the kernels associated to the following higher order Dirichlet boundary value problem

$$\begin{cases} (-\Delta)^m \, u = \varphi & \text{in } \Omega, \\ u = \psi_0 & \text{on } \partial\Omega, \\ \frac{\partial}{\partial\nu} u = \psi_1 & \text{on } \partial\Omega, \\ \dots & \dots \\ (\frac{\partial}{\partial\nu})^{m-1} u = \psi_{m-1} & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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where $m \in \mathbb{N}^+$ and Ω is an open bounded connected subset of $\mathbb{R}^n$, $n \geq 2$, with for $n = 2$ $\partial\Omega \in C^{6m+4}$ and for $n \geq 3$ $\partial\Omega \in C^{5m+2}$. The Green function G_m and the Poisson kernels K_j are such that the solution of problem (1), for appropriate φ and ψ_i, can be written as

$$u(x) = \int_\Omega G_m(x,y)\varphi(y)\,dy + \sum_{j=0}^{m-1} \int_{\partial\Omega} K_j(x,y)\psi_j(y)\,d\sigma_y.$$

For example when $m = 2$ and $n = 2$ we will prove that there is a constant c_Ω such that

$$|G_2(x,y)| \leq c_\Omega d(x)d(y) \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}, \tag{2}$$

where d is the distance of x to the boundary $\partial\Omega$:

$$d(x) := \inf_{\tilde{x}\in\partial\Omega} |x - \tilde{x}|. \tag{3}$$

For the sake of easy statement we have used $L = (-\Delta)^m$ in system (1) but in fact the estimates that we will derive hold for any uniformly elliptic operator L of order $2m$.

We will focus on the estimates for G_m and K_j. However, we would like to mention that those estimates are the optimal tools for deriving regularity results in spaces that involve the behavior at the boundary. Coming back to the case $m = n = 2$ it follows from (2) that the solution u of

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega \subset \mathbb{R}^2, \\ u = \frac{\partial}{\partial\nu}u = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies for appropriate f

$$\left\|\frac{u}{d^2}\right\|_{L^\infty(\Omega)} \leq c_\Omega \|f\|_{L^1(\Omega)} \text{ and } \|u\|_{L^\infty(\Omega)} \leq c_\Omega \left\|f\ d^2\right\|_{L^1(\Omega)}.$$

These kinds of estimates, for general m and n, and also L^p-L^q estimates will be addressed in Section 4. The estimates are interesting by their own merits. A special case for $m = 1$ appears in [7].

Not only we will derive estimates for those kernels but also for their derivatives. The main tool will be the result of Krasovskiĭ in [12] where he considered general elliptic operators and boundary conditions. The estimates he derived did not involve special growth rates near the boundary. We instead will focus on estimates that contain growth rates near the boundary. These estimates seem to be optimal and indeed, when we consider G_m for $\Omega = B$ a ball in $\mathbb{R}^n$ the growth rates near the boundary are sharp (see e.g. [11]).

For $m = 1$ or $m \geq 2$ and $\Omega = B$ it is known that the Green function is positive and can even be estimated from below by a positive function with the same singular behavior (see [9]). Let us remind the reader that for $m \geq 2$ the Green function in general is not positive. We believe however that for general domains the optimal behavior in absolute values is captured in our estimates. Sharp estimates for K_{m-1} and K_{m-2} in case of a ball can be found in [10].

Instead of using Krasovskiĭ's result one might use appropriate "heat kernel" estimates. Indeed, integrating pointwise estimates for the parabolic kernel $p(t, x, y)$ with respect to t from 0 to ∞, yields pointwise estimates for the Green function. However, only limited results seem to be available. Barbatis [2] considered higher order parabolic problems on domains and derived pointwise estimates for the kernel using a non-Euclidean metric. Classical estimates by Eidel'man (see e.g. [6]) for higher order parabolic systems do not consider domains with boundary.

For a survey on spectral theory of higher order elliptic operators, including some estimates for the corresponding kernels, we refer to [5].

Finally we would like to remark that we do not pretend that our pointwise estimates are completely new. However we have not been able to find any reference to such estimates for the special type of boundary conditions above.

The paper is organized as follows. We will complete Section 1 with the estimates of the Green function and the Poisson kernels. In Section 2 we will state and prove the estimates for the Green function and its derivatives. In section 3 we will do the same for the Poisson kernels. In section 4 we will show applications to regularity estimates in weighted spaces. We will conclude the paper with several appendices.

1.1 Preliminaries and main results

Before stating the main results we fix some notations.

Notation 1 *(See [9]) Let f and g be functions on $\Omega \times \Omega$ with $g \geq 0$. Then we call $f \sim g$ on $\Omega \times \Omega$ if and only if there are $c_1, c_2 > 0$ such that*

$$c_1 f(x, y) \leq g(x, y) \leq c_2 f(x, y) \text{ for all } x, y \in \Omega.$$

We will say $f \preceq g$ on $\Omega \times \Omega$ if and only if there is $c > 0$ such that

$$f(x, y) \leq c\ g(x, y) \text{ for all } x, y \in \Omega.$$

Notation 2 *Let f a function on $\Omega\times\Omega$ and $\alpha,\beta\in\mathbb{N}^n$. Derivatives are denoted*

$$D_x^\alpha D_y^\beta f(x,y)=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}x_2^{\alpha_2}..\;x_n^{\alpha_n}}\frac{\partial^{|\beta|}}{\partial y_1^{\beta_1}y_2^{\beta_2}..\;y_n^{\beta_n}}f(x,y),$$

where $|\alpha|=\sum_{k=1}^n\alpha_k$.

We can now state the main results of the paper.

Theorem 3 *Let $G_m(x,y)$ be the Green function associated to system (1). The following estimates holds for every $x,y\in\Omega$:*

(1) if $2m-n>0$, then

$$|G_m(x,y)|\preceq d(x)^{m-\frac{1}{2}n}\,d(y)^{m-\frac{1}{2}n}\,\min\left\{1,\frac{d(x)d(y)}{|x-y|^2}\right\}^{\frac{1}{2}n},$$

(2) if $2m-n=0$, then

$$|G_m(x,y)|\preceq\log\left(1+\left(\frac{d(x)d(y)}{|x-y|^2}\right)^m\right),$$

(3) if $2m-n<0$, then

$$|G_m(x,y)|\preceq|x-y|^{2m-n}\min\left\{1,\frac{d(x)d(y)}{|x-y|^2}\right\}^m.$$

Theorem 4 *Let $K_j(x,y)$, for $j=0,...,m-1$, be the Poisson kernels associated to system (1). The following estimate holds for every $x\in\Omega$ and $y\in\partial\Omega$*

$$|K_j(x,y)|\preceq\frac{d(x)^m}{|x-y|^{n-j+m-1}}. \tag{4}$$

Remark 5 *If $n-1<j\leq m-1$ inequality (4) gives that on $\Omega\times\partial\Omega$*

$$|K_j(x,y)|\preceq d(x)^{1+j-n}.$$

Remark 6 *The estimates in Theorems 3 and 4 in fact hold for $(-\Delta)^m$ replaced by any uniformly elliptic operator of order $2m$. Indeed, the main ingredients are the Dirichlet boundary condition and the estimates of Krasovskiĭ. In the proof one has to use the Dirichlet boundary condition both for the original and the adjoint problem. Although the adjoint problem is different for general elliptic problems the Dirichlet boundary condition will remain. Notice that Krasovskiĭ's derived the estimates for the general case.*

In [9] the estimates as in Theorem 3 are given for the case that Ω is a ball in $\mathbb{R}^n$. There the authors could use the explicit formula of G_m given by Boggio in [3]. For balls the Green function associated to problem (1) is positive.

For general domains one cannot expect an explicit formula and instead we will proceed by the estimates of Krasovskiĭ for G_m and K_j given in [12]. For sufficiently regular domains Ω (see Appendix B) he proves that the Green function and the Poisson kernels exist and gives estimates for these functions.

Our aim will be to prove estimates from above of G_m and K_j depending on the distance to the boundary. We will do so by estimating the j-th derivative through an integration of the $(j+1)$-th derivative along a path to the boundary. The dependence on the distance to the boundary $d(x)$ will appear choosing a path which length is proportional do $d(x)$. The path will be constructed explicitly in Lemma 7.

2 Estimates of the Green function

In this section we will prove Theorem 3. First we derive an estimate of the $j-$th derivatives of G_m integrating an estimate of the $(j+1)-$th derivative along an appropriate path. We let the path finish at the boundary to benefit from the boundary condition. Moreover, we have to construct the path such that it stays away from the singularity $x = y$ and such that it has a length of the same magnitude as $d(x)$.

In the following lemma we state the existence of such a path.

Lemma 7 *Let $x \in \Omega$ and $y \in \bar{\Omega}$. There exists a curve $\gamma_x^y : [0,1] \to \bar{\Omega}$ with $\gamma_x^y(0) = x$, $\gamma_x^y(1) \in \partial\Omega$ and such that:*

(1) for every $t \in [0,1] : |\gamma_x^y(t) - y| \geq \frac{1}{2}|x - y|$,
(2) $l \leq (1+\pi)\, d(x)$ where l is the length of γ_x^y.

Moreover, letting $\tilde{\gamma}_x^y : [0,l] \to \bar{\Omega}$ be the parametrization by arclength of γ_x^y, it holds that

(3) $\frac{1}{5}s \leq |x - \tilde{\gamma}_x^y(s)| \leq s$ for $s \in [0,l]$.

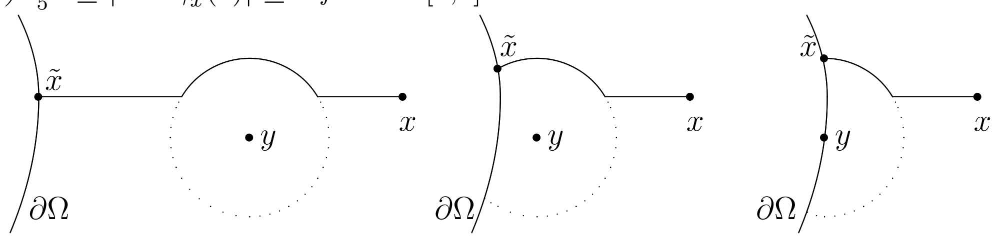

Fig. 1. The path γ_x^y for several positions of y.

PROOF. A rough description on how to define such a path is as follows. One connects x with a straight line to its nearest boundary point $\bar{x}$ until the straight line possibly gets too close to y. To avoid the neighborhood of y we take a circular route on ∂B with $B = B(y, \frac{1}{2}|x-y|)$. In the case that $\bar{x} \in B$ one moves on ∂B to some other point on $\partial\Omega$. We will not give the details of the proof but refer to Figure 1.

We proceed with the proof of Theorem 3 and start from the estimates in [12] of the m-th derivative of G_m. Integrating this function along the path γ_x^y of Lemma 7 we find the estimates of the $(m-1)$-th derivative of G_m in terms of the distance to the boundary. Next starting from the new estimates one repeats the argument. Iterating the procedure m times we find the result as stated in Theorem 3.

There are four cases. Each of the following lemmas will consider one of these cases.

Lemma 8 *Let $\nu_1, \nu_2, k \in \mathbb{N}$, $k \geq 2$. If*

$$|\nabla_x H(x,y)| \preceq |x-y|^{-k} \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2} \quad \textit{for } x, y \in \Omega,$$

and $H(\tilde{x}, y) = 0$ for every $\tilde{x} \in \partial\Omega$ and $y \in \Omega$, then the following inequality holds:

$$|H(x,y)| \preceq |x-y|^{-k+1} \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1+1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2} \quad \textit{for } x, y \in \Omega.$$

PROOF. Let $x, y \in \Omega$ and let γ_x^y the path from x to the boundary from Lemma 7. Let $\tilde{x} := \gamma_x^y(l)$. Since $\tilde{x} \in \partial\Omega$ one has that

$$H(x,y) = H(\tilde{x}, y) + \int_{\gamma_x^y} \nabla_z H(z,y) \cdot dz = \int_0^l \nabla_x H(\tilde{\gamma}_x^y(s), y) \cdot \tau(s) ds, \tag{5}$$

with $\tau(s)$ the unit tangent vector. By the hypothesis and Lemma C.5, i) we obtain from (5) that

$$\begin{aligned}
|H(x,y)| &\preceq \int_0^l |\tilde{\gamma}_x^y(s) - y|^{-k} \min\left\{1, \frac{d(\tilde{\gamma}_x^y(s))^{\nu_1} d(y)^{\nu_2}}{|\tilde{\gamma}_x^y(s) - y|^{\nu_1+\nu_2}}\right\} ds \ \preceq \\
&\preceq \int_0^l \Big(|x-y| + s\Big)^{-k} \min\left\{1, \frac{d(x)^{\nu_1} d(y)^{\nu_2}}{(\,|x-y| + s)^{\nu_1+\nu_2}}\right\} ds \ \preceq \\
&\preceq |x-y|^{-k+1} \int_0^{\frac{l}{|x-y|}} (1+t)^{-k} \min\left\{1, \frac{d(x)^{\nu_1} d(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2} (1+t)^{\nu_1+\nu_2}}\right\} dt. \qquad (6)
\end{aligned}$$

Here we used Lemma 7 and that $d(\tilde{\gamma}_x^y(s)) \preceq d(x)$. It is convenient to separate the following two cases.

Case 1, $\frac{d(x)}{|x-y|} < 1$: Then $\min\left\{1, \frac{d(x)^{\nu_1} d(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2}(1+t)^{\nu_1+\nu_2}}\right\} = \frac{d(x)^{\nu_1} d(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2}(1+t)^{\nu_1+\nu_2}}$ and one finds by Lemma C.4 that

$$
\begin{aligned}
|H(x,y)| &\preceq \frac{d(x)^{\nu_1} d(y)^{\nu_2}}{|x-y|^{k+\nu_1+\nu_2-1}} \int_0^{\frac{l}{|x-y|}} \frac{1}{(1+t)^{k+\nu_1+\nu_2}}\, dt \preceq \\
&\preceq \frac{d(x)^{\nu_1+1} d(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2+k}} \preceq |x-y|^{-k+1} \min\left\{1, \frac{d(x)^{\nu_1+1}}{|x-y|^{\nu_1+1}} \frac{d(y)^{\nu_2}}{|x-y|^{\nu_2}}\right\}.
\end{aligned} \tag{7}
$$

Case 2, $\frac{d(x)}{|x-y|} \geq 1$: Since $k \geq 2$ we get again by Lemma C.4 that

$$
\begin{aligned}
|H(x,y)| &\preceq |x-y|^{-k+1} \int_0^{\frac{l}{|x-y|}} (1+t)^{-k}\, dt \preceq |x-y|^{-k+1} \preceq \\
&\preceq |x-y|^{-k+1} \min\left\{1, \frac{d(x)^{\nu_1+1}}{|x-y|^{\nu_1+1}} \frac{d(y)^{\nu_2}}{|x-y|^{\nu_2}}\right\}.
\end{aligned} \tag{8}
$$

Lemma 9 *Let $\nu_1, \nu_2 \in \mathbb{N}$. If*

$$
|\nabla_x H(x,y)| \preceq |x-y|^{-1} \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2} \quad \text{for } x, y \in \Omega,
$$

and $H(\tilde{x}, y) = 0$ for every $\tilde{x} \in \partial\Omega$ and $y \in \Omega$, then the following inequality holds:

$$
|H(x,y)| \preceq \log\left(2 + \frac{d(x)d(y)}{|x-y|^2}\right) \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1+1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2},
$$

for $x, y \in \Omega$.

PROOF. Similarly as in (6) we find that

$$
|H(x,y)| \preceq \int_0^{\frac{l}{|x-y|}} (1+t)^{-1} \min\left\{1, \frac{d(x)^{\nu_1} d(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2} (1+t)^{\nu_1+\nu_2}}\right\} dt.
$$

Again we will separate the two cases.

Case 1, $\frac{d(x)}{|x-y|} < 1$: As in (7) we obtain

$$
\begin{aligned}
|H(x,y)| &\preceq \frac{d(x)^{\nu_1+1}}{|x-y|^{\nu_1+1}} \frac{d(y)^{\nu_2}}{|x-y|^{\nu_2}} \preceq \\
&\preceq \log\left(2 + \frac{d(x)}{|x-y|}\right) \min\left\{1, \frac{d(x)^{\nu_1+1}}{|x-y|^{\nu_1+1}} \frac{d(y)^{\nu_2}}{|x-y|^{\nu_2}}\right\}.
\end{aligned}
$$

Case 2, $\frac{d(x)}{|x-y|} \geq 1$: As in (8) we get by using Lemma C.5, ii) that

$$
\begin{aligned}
|H(x,y)| &\preceq \int_0^{\frac{l}{|x-y|}} (1+t)^{-1}\, dt \ \preceq \\
\preceq \log &\left(1 + \frac{(1+\pi)\, d(x)}{|x-y|}\right) \\
\sim \log &\left(2 + \frac{d(x)}{|x-y|}\right) \min\left\{1, \frac{d(x)^{\nu_1+1}}{|x-y|^{\nu_1+1}} \frac{d(y)^{\nu_2}}{|x-y|^{\nu_2}}\right\}.
\end{aligned}
$$

The claim follows using Lemma C.5, iii).

Lemma 10 *Let $\nu_1, \nu_2 \in \mathbb{N}$. If*

$$
|\nabla_x H(x,y)| \preceq \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2} \log\left(2 + \frac{d(x)d(y)}{|x-y|^2}\right),
$$

for $x, y \in \Omega$, and $H(\tilde{x}, y) = 0$ for every $\tilde{x} \in \partial\Omega$ and $y \in \Omega$, then the following inequality holds:

$$
|H(x,y)| \preceq d(x) \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2} \quad \textit{for } x, y \in \Omega.
$$

PROOF. Proceeding as before and using Lemma C.5, iii) one obtains that

$$
\begin{aligned}
|H(x,y)| &\preceq \int_0^l \log\left(2 + \frac{d(\tilde{\gamma}_x^y(s))d(y)}{|\tilde{\gamma}_x^y(s) - y|^2}\right) \min\left\{1, \frac{d(\tilde{\gamma}_x^y(s))^{\nu_1} d(y)^{\nu_2}}{|\tilde{\gamma}_x^y(s) - y|^{\nu_1+\nu_2}}\right\} ds \ \preceq \\
&\preceq |x-y| \int_0^{\frac{l}{|x-y|}} \log\left(2 + \tfrac{d(x)}{|x-y|(1+t)}\right) \min\left\{1, \tfrac{d(x)^{\nu_1} d(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2}(1+t)^{\nu_1+\nu_2}}\right\} dt. \qquad (9)
\end{aligned}
$$

Case 1, $\frac{d(x)}{|x-y|} < 1$: From Lemma C.4 it follows that

$$
\begin{aligned}
|H(x,y)| &\preceq \frac{d(x)^{\nu_1} d(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2-1}} \int_0^{\frac{l}{|x-y|}} \frac{1}{(1+t)^{\nu_1+\nu_2}}\, dt \ \preceq \\
&\preceq \frac{d(x)^{\nu_1} d(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2-1}} \frac{d(x)}{|x-y|} \sim d(x) \min\left\{1, \frac{d(x)^{\nu_1} d(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2}}\right\}.
\end{aligned}
$$

Case 2, $\frac{d(x)}{|x-y|} \geq 1$: We first observe that $\frac{6d(x)}{|x-y|(1+t)} > 1$. Indeed since $t \leq \frac{(1+\pi)d(x)}{|x-y|}$

we have that

$$\frac{6d(x)}{|x-y|\,(1+t)} \geq \frac{6d(x)}{|x-y|+(1+\pi)\,d(x)} \geq \tfrac{6}{2+\pi} > 1.$$

Hence from (9) applying Lemma C.4 we obtain

$$\begin{aligned}
&|H(x,y)| \preceq |x-y| \int_0^{\frac{l}{|x-y|}} \log\left(6\frac{d(x)}{|x-y|\,(1+t)}\right) dt \ \sim \\
&\sim |x-y| \left(\left(1+\tfrac{l}{|x-y|}\right)\log\left(\tfrac{6d(x)}{|x-y|\left(1+\frac{l}{|x-y|}\right)}\right) - \log\left(\tfrac{6d(x)}{|x-y|}\right) + \tfrac{l}{|x-y|}\right) \ \preceq \\
&\preceq |x-y| \left(1+\frac{(1+\pi)\,d(x)}{|x-y|}\right)\log\left(\frac{6\frac{d(x)}{|x-y|}}{\frac{(1+\pi)d(x)}{|x-y|}+1}\right) \\
&\quad + d(x)\left(\frac{|x-y|}{d(x)}\log\left(\frac{|x-y|}{6d(x)}\right)+1+\pi\right) \sim \\
&\sim d(x) \sim d(x)\min\left\{1, \frac{d(x)^{\nu_1}d(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2}}\right\}.
\end{aligned}$$

Lemma 11 *Let $\nu_1, \nu_2, \alpha_1, \alpha_2 \in \mathbb{N}$. If*

$$|\nabla_x H(x,y)| \preceq d(x)^{\alpha_1} d(y)^{\alpha_2} \ \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2},$$

for $x, y \in \Omega$, and $H(\tilde{x}, y) = 0$ for every $\tilde{x} \in \partial\Omega$ and $y \in \Omega$, then the following inequality holds:

$$|H(x,y)| \preceq d(x)^{\alpha_1+1} d(y)^{\alpha_2} \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2} \ \text{for } x, y \in \Omega.$$

PROOF. Proceeding as before, one obtains that

$$\begin{aligned}
|H(x,y)| &\preceq \int_0^l d(\tilde{\gamma}_x^y(s))^{\alpha_1} d(y)^{\alpha_2} \ \min\left\{1, \frac{d(\tilde{\gamma}_x^y(s))^{\nu_1} d(y)^{\nu_2}}{|\tilde{\gamma}_x^y(s)-y|^{\nu_1+\nu_2}}\right\} ds \ \preceq \\
&\preceq |x-y|\, d(x)^{\alpha_1} d(y)^{\alpha_2} \int_0^{\frac{l}{|x-y|}} \min\left\{1, \frac{d(x)^{\nu_1} d(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2}\,(1+t)^{\nu_1+\nu_2}}\right\} dt.
\end{aligned}$$

Again we will separate the two cases.

Case 1, $\frac{d(x)}{|x-y|} < 1$: As before it follows that

$$|H(x,y)| \preceq \frac{d(x)^{\nu_1+\alpha_1} d(y)^{\nu_2+\alpha_2}}{|x-y|^{\nu_1+\nu_2-1}} \int_0^{\frac{l}{|x-y|}} \frac{1}{(1+t)^{\nu_1+\nu_2}} \, dt \preceq$$
$$\preceq d(x) \frac{d(x)^{\nu_1+\alpha_1} d(y)^{\nu_2+\alpha_2}}{|x-y|^{\nu_1+\nu_2}} \sim d(x)^{\alpha_1+1} d(y)^{\alpha_2} \min \left\{ 1, \frac{d(x)^{\nu_1} d(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2}} \right\}.$$

Case 2, $\frac{d(x)}{|x-y|} \geq 1$: We obtain

$$|H(x,y)| \preceq |x-y| \, d(x)^{\alpha_1} d(y)^{\alpha_2} \int_0^{\frac{l}{|x-y|}} 1 \, dt \sim$$
$$\sim d(x)^{\alpha_1+1} d(y)^{\alpha_2} \sim d(x)^{\alpha_1+1} d(y)^{\alpha_2} \min \left\{ 1, \frac{d(x)^{\nu_1} d(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2}} \right\}.$$

The four lemmas above allow us to prove the following theorem of which Theorem 3 is a special case.

Theorem 12 *Let $G_m(x,y)$ be the Green function associated to system (1). Let $k \in \mathbb{N}^n$. The following estimates hold for every $x, y \in \Omega$:*

(1) For $|k| \geq m$:
(a) if $2m - n - |k| < 0$, then

$$\left| D_x^k G_m(x,y) \right| \preceq |x-y|^{2m-n-|k|} \min \left\{ 1, \frac{d(y)}{|x-y|} \right\}^m,$$

(b) if $2m - n - |k| = 0$, then

$$\left| D_x^k G_m(x,y) \right| \preceq \log \left(1 + \frac{d(y)^m}{|x-y|^m} \right) \sim$$
$$\sim \log \left(2 + \frac{d(y)}{|x-y|} \right) \min \left\{ 1, \frac{d(y)}{|x-y|} \right\}^m,$$

(c) if $2m - n - |k| > 0$, then

$$\left| D_x^k G_m(x,y) \right| \preceq d(y)^{2m-n-|k|} \min \left\{ 1, \frac{d(y)}{|x-y|} \right\}^{n+|k|-m},$$

(2) For $|k| < m$:
(a) if $2m - n - |k| < 0$, then

$$\left| D_x^k G_m(x,y) \right| \preceq |x-y|^{2m-n-|k|} \min \left\{ 1, \frac{d(x)}{|x-y|} \right\}^{m-|k|} \min \left\{ 1, \frac{d(y)}{|x-y|} \right\}^m,$$

(b) *if* $2m-n-|k|=0$, *then*

$$\left|D_x^k G_m(x,y)\right| \preceq \log\left(1+\frac{d(y)^m d(x)^{m-|k|}}{|x-y|^{2m-|k|}}\right) \sim$$

$$\sim \log\left(2+\frac{d(y)}{|x-y|}\right) \min\left\{1,\frac{d(y)}{|x-y|}\right\}^m \min\left\{1,\frac{d(x)}{|x-y|}\right\}^{m-|k|},$$

(c) *if* $2m-n-|k|>0$, *and moreover*

(i) $m-\frac{1}{2}n\le|k|$, *then*

$$\left|D_x^k G_m(x,y)\right| \preceq d\,(y)^{2m-n-|k|} \min\left\{1,\tfrac{d(x)}{|x-y|}\right\}^{m-|k|} \min\left\{1,\tfrac{d(y)}{|x-y|}\right\}^{n-m+|k|},$$

(ii) $|k|<m-\frac{1}{2}n$, *then*

$$\left|D_x^k G_m(x,y)\right| \preceq d(y)^{m-\frac{n}{2}} d(x)^{m-\frac{n}{2}-|k|} \min\left\{1,\frac{d(x)d(y)}{|x-y|^2}\right\}^{\frac{n}{2}}.$$

PROOF. Let $x,y\in\Omega$. We will start from the estimates of Krasovskiĭ for the higher order derivatives of G_m which are stated in Theorem B.2. The estimates for the lower order derivatives of G_m will be obtained by integrating the higher order estimates along the path γ_x^y from Lemma 7. Each of the four lemmas above corresponds to one such integration step. Indeed, with $\alpha,\beta\in\mathbb{N}^n$ and $\tilde{x}\in\partial\Omega$ the end point of γ_x^y, we find

$$D_x^\alpha D_y^\beta G_m(x,y) = D_x^\alpha D_y^\beta G_m(\tilde{x},y) + \int_{\gamma_x^y} \nabla_z D_z^\alpha D_y^\beta G_m(z,y)\cdot dz. \tag{10}$$

If $|\alpha|\le m-1$ then the first term on the right hand side of (10) equals 0 and we get

$$\left|D_x^\alpha D_y^\beta G_m(x,y)\right| \le \int_0^l \left|\nabla_x D_x^\alpha D_y^\beta G_m(\tilde{\gamma}_x^y(s),y)\right| ds. \tag{11}$$

If $|\beta|\le m-1$, then similarly by integrating with respect to y we find

$$\left|D_x^\alpha D_y^\beta G_m(x,y)\right| \le \int_0^l \left|\nabla_y D_y^\beta D_x^\alpha G_m(x,\tilde{\gamma}_y^x(s))\right| ds. \tag{12}$$

The explicit estimate coming out of one of such steps depends on which of the four lemmas above we have to use. We take $H(x,y)=D_x^\alpha D_y^\beta G_m(x,y)$ and depending on $|k|=r$ we have to make an appropriate choice for α and β.

We distinguish the cases as in the statement of the theorem.

Case 1, $r\ge m$: Let $\beta\in\mathbb{N}^n$ with $|\beta|=m-1$. Then proceeding from (12) with $k=\alpha$ and using the estimate in Theorem B.2, namely $\left|D_x^\alpha D_y^\beta G_m(x,y)\right|\le |x-y|^{m-n-r}$, three different cases have to be considered.

Case 1(a), $2m-n-r<0$: The claim follows applying m times Lemma 8.

Case 1(b), $2m-n-r=0$: One gets the estimates by using Lemma 8 $m-1$ times and Lemma 9 once.

Case 1(c), $2m-n-r>0$: By first applying Lemma 8 $n+r-m-1$ times and then Lemma 9 once we find

$$\left|D_y^{\tilde{\beta}} D_x^k G_m(x,y)\right| \preceq \log\left(1+\frac{d(y)^{n+r-m}}{|x-y|^{n+r-m}}\right),$$

with $\tilde{\beta}\in\mathbb{N}^n$, $\tilde{\beta}\le\beta$ and $\left|\tilde{\beta}\right| = 2m-n-r$. Next one uses Lemma 10 once and Lemma 11 $2m-n-r-1$ times.

Case 2, $r<m$: Let $\alpha,\beta\in\mathbb{N}^n$ with $|\alpha|=m-r$ and $|\beta|=m$. One starts from the Krasovskiĭ estimates for $|D_y^{\beta} D_x^{\alpha} D_x^k G_m(x,y)|$ and then integrates m times with respect of y and $m-r$ times with respect to x.

Case 2(a), $2m-n-r<0$: The claim follows by applying Lemma 8 first m times with respect to y and then $m-r$ times with respect to x.

Case 2(b), $2m-n-r=0$: One proves the estimates by using Lemma 8 m times with respect to y, $m-r-1$ times with respect to x and then Lemma 9 once with respect to x.

Case 2(c), $2m-n-r>0$: One has to separate the cases $m-r\le n-1$ and $m-r>n-1$.

Case $m-r\le n-1$: Applying Lemma 8 $n-1$ times and Lemma 9 once we get

$$\left|D_y^{\tilde{\beta}} D_x^k G_m(x,y)\right| \preceq \log\left(1+\frac{d(x)^{m-r}d(y)^{n-m+r}}{|x-y|^{n}}\right),$$

with $\tilde{\beta}\in\mathbb{N}^n$, $\tilde{\beta}\le\beta$ with $\left|\tilde{\beta}\right| = 2m-n-r$. Then using Lemma 10 once and Lemma 11 $2m-n-r-1$ times we obtain

$$\left|D_x^k G_m\left(x,y\right)\right| \preceq d\left(y\right)^{2m-n-r} \min\left\{1,\tfrac{d(x)}{|x-y|}\right\}^{m-r} \min\left\{1,\tfrac{d(y)}{|x-y|}\right\}^{n-m+r}. \tag{13}$$

The claim follows from (13) when $m-\frac{1}{2}n\le r$. Otherwise when $r<m-\frac{1}{2}n$ we rewrite (13) as

$$\begin{aligned}\left|D_x^k G_m(x,y)\right| &\preceq d(y)^{2m-n-r}\left(\frac{d(y)}{d(x)}\right)^{\frac{n}{2}-m+r} \min\left\{1,\frac{d(x)d(y)}{|x-y|^2}\right\}^{\frac{n}{2}} \sim \\ &\sim d(y)^{m-\frac{n}{2}} d(x)^{m-\frac{n}{2}-r} \min\left\{1,\frac{d(x)d(y)}{|x-y|^2}\right\}^{\frac{n}{2}}.\end{aligned}$$

Here we use Lemma C.5, iii).

Case $m-r>n-1$: Let $\tilde{\alpha}\in\mathbb{N}^n$, $\tilde{\alpha}\leq\alpha$ with $|\tilde{\alpha}|=m-n-r$. Using Lemma 8 $n-1$ times and 9 once we get

$$\left|D_y^{\beta}D_x^{\tilde{\alpha}}D_x^kG_m\left(x,y\right)\right|\preceq\log\left(1+\frac{d(x)^n}{|x-y|^n}\right).$$

Then applying Lemma 11 m times with respect to y and $m-r-n$ times with respect to x, one obtains

$$\begin{aligned}\left|D_x^kG_m\left(x,y\right)\right|&\preceq d\left(y\right)^m d\left(x\right)^{m-r-n}\min\left\{1,\frac{d(x)}{|x-y|}\right\}^n\sim\\&\sim d\left(y\right)^{m-\frac{1}{2}n}d\left(x\right)^{m-r-\frac{n}{2}}\min\left\{1,\frac{d(x)d\left(y\right)}{|x-y|^2}\right\}^{\frac{n}{2}},\end{aligned}$$

using again Lemma C.5, iii). Observe that $m-r>n-1$ implies $r<m-\frac{1}{2}n$ for $n\geq 2$.

3 Estimates of the Poisson kernels

In this section we prove Theorem 4. The method is similar to the one used for Theorem 3. A difference is that in this case there is no symmetry between x and y.

We proceed with the proof of Theorem 4. The lemma that corresponds to one integration step is as follows.

Lemma 13 *Let $\nu_1,k\in\mathbb{N}$ with $k\geq 2$. If*

$$\left|\nabla_xH(x,y)\right|\preceq|x-y|^{-k}\ d(x)^{\nu_1}\ \text{for}\ x\in\Omega,\ y\in\partial\Omega,$$

and $H(\tilde{x},y)=0$ for every $\tilde{x}\in\partial\Omega$ with $\tilde{x}\neq y$, then the following inequality holds

$$\left|H(x,y)\right|\preceq|x-y|^{-k}\ d(x)^{\nu_1+1}\ \text{for}\ x\in\Omega,\ y\in\partial\Omega.$$

PROOF. Let $x\in\Omega$ and $y\in\partial\Omega$. Let γ_x^y the path from x to the boundary from Lemma 7 and let $\tilde{x}:=\gamma_x^y(1)$. Since $\tilde{x}\in\partial\Omega$ and $\tilde{x}\neq y$ it holds that

$$H(x,y)=H(\tilde{x},y)+\int_{\gamma_x^y}\nabla_zH(z,y)\cdot dz.$$

By the hypothesis we get that

$$|H(x,y)| \preceq \int_0^l |\nabla_x H(\tilde{\gamma}_x^y(s),y)|\; ds \preceq \int_0^l |\tilde{\gamma}_x^y(s)-y|^{-k}\; d(\tilde{\gamma}_x^y(s))^{\nu_1}\, ds.$$

Since $d(\tilde{\gamma}_x^y(s)) \preceq d(x)$, from Lemma 7 it follows that

$$\begin{aligned}|H(x,y)| &\preceq d(x)^{\nu_1} \int_0^l (|x-y|+s)^{-k}\;\; ds \;\;\preceq \\ &\preceq d(x)^{\nu_1}\;|x-y|^{-k+1} \int_0^{\frac{l}{|x-y|}} (1+t)^{-k}\, dt \preceq \frac{d(x)^{\nu_1+1}}{|x-y|^k}.\end{aligned}$$

The lemma above allow us to prove the following theorem of which Theorem 4 is a special case.

Theorem 14 *Let $K_j\,(x,y)\,$, for $j=0,...,m-1$, be the Poisson kernels associated to system (1). Let $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq m-1$. The following estimate holds for $x \in \Omega$, $y \in \partial\Omega$*

$$|D_x^\alpha K_j(x,y)| \preceq \frac{d(x)^{m-|\alpha|}}{|x-y|^{n-j+m-1}}.$$

Remark 15 *The estimates of $D_x^\alpha K_j(x,y)$ for $|\alpha| \geq m$ can be found in the paper of Krasovskiĭ [12]: for $x \in \Omega$ and $y \in \partial\Omega$*

$$|D_x^\alpha K_j(x,y)| \preceq |x-y|^{-n+j-|\alpha|+1}\,.$$

PROOF. Let $x \in \Omega$, $y \in \partial\Omega$, $j \in \{0,\ldots,m-1\}$ and $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq m-1$. We will start from the estimates of Krasovskiĭ for the derivative of order m of K_j which are stated in Theorem B.2. The estimates for the lower order derivatives of K_j will be obtained by integrating the higher order estimates along the path γ_x^y from Lemma 7. Indeed, with $\beta \in \mathbb{N}^n$, $\beta \geq \alpha$ and $|\beta| = m-1$ we find

$$D_x^\beta K_j(x,y) = D_x^\beta K_j(\gamma_x^y(1),y) + \int_{\gamma_x^y} \nabla_z D_z^\beta K_j(z,y)\cdot dz = \int_{\gamma_x^y} \nabla_z D_z^\beta K_j(z,y)\cdot dz.$$

Applying Lemma 8 with $H(x,y) = D_x^\beta K_j(x,y)$ we get

$$\left|D_x^\beta K_j(x,y)\right| \preceq |x-y|^{j-n+1-m}\; d(x).$$

The claim follows iterating the procedure $m-|\alpha|-1$ times.

4 Estimates for the solution with zero boundary conditions

In this section we will derive regularity estimates for

$$\begin{cases} (-\Delta)^m \; u = f & \text{in } \Omega, \\ \quad (\frac{\partial}{\partial\nu})^k u = 0 \text{ on } \partial\Omega \text{ with } 0 \le k \le m-1, \end{cases} \tag{14}$$

where $\Omega \subset \mathbb{R}^n$ is bounded and has the boundary regularity as before. First we recall an estimate involving the Riesz potential (see [8]). Defining $K_\gamma(x) = |x|^{-\gamma}$ and

$$\left(K_\gamma * f\right)(x) := \int_\Omega |x-y|^{-\gamma} f(y) dy,$$

one has:

Lemma 16 *Let $\Omega \subset \mathbb{R}^n$ be bounded, $\gamma < n$ and $1 \le p \le q \le \infty$. If $\frac{\gamma}{n} < \frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p}$ then there is $C_{n-\gamma r,\Omega} > 0$ such that for all $f \in L^p(\Omega)$:*

$$\|K_\gamma * f\|_{L^q(\Omega)} \le C_{n-\gamma r,\Omega} \|f\|_{L^p(\Omega)} . \tag{15}$$

PROOF. This proof is standard, let us recall it for easy reference. Let

$$\frac{\gamma}{n} < \frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p} = 1 - \delta.$$

Denote σ_n the surface area of the unit ball in $\mathbb{R}^n$. For $1 < p \le q < \infty$ one finds by Hölder, setting $c_{n-\gamma r,\Omega} = \frac{1}{n-\gamma r}\sigma_n \left(\mathrm{diam}_\Omega\right)^{n-\gamma r}$,

$$\begin{aligned} \left(K_\gamma * f\right)(x) &= \int_\Omega \frac{1}{|x-y|^{\gamma \frac{r}{q}}} |f(y)|^{\frac{p}{q}} \frac{1}{|x-y|^{\gamma\left(\frac{p-1}{p}\right)r}} |f(y)|^{p\delta} \, dy \le \\ &\le \left(\int_\Omega \frac{1}{|x-y|^{\gamma r}} |f(y)|^p \, dy\right)^{\frac{1}{q}} \left(\int_\Omega \frac{1}{|x-y|^{\gamma r}} dy\right)^{\frac{p-1}{p}} \left(\int_\Omega |f(y)|^p \, dy\right)^{\delta} \\ &\le \left(c_{n-\gamma r,\Omega}\right)^{\frac{p-1}{p}} \left(\int_\Omega \frac{1}{|x-y|^{\gamma r}} |f(y)|^p \, dy\right)^{\frac{1}{q}} \left(\int_\Omega |f(y)|^p \, dy\right)^{\delta}. \end{aligned}$$

Hence, with a change in the order of integration,

$$\int_\Omega \left|\left(K_\gamma * f\right)(x)\right|^q dx \le \left(c_{n-\gamma r,\Omega}\right)^{\frac{p-1}{p}q+1} \left(\int_\Omega |f(y)|^p \, dy\right)^{1+\delta q}$$

implying (15) since $\left(c_{n-\gamma r,\Omega}\right)^{1-\frac{1}{p}+\frac{1}{q}} \le C_{n-\gamma r,\Omega} := c_{n-\gamma r,\Omega} + 1$.

For $p = 1$ one may skip the middle term in the Hölder estimate; for $q = \infty$ the first term.

As a consequence of the pointwise estimates and using the lemma above, we next state the optimal L^p-L^q-regularity results mentioned before. Let us recall that $d(.)$ is the distance function defined in (3).

Proposition 17 *Let $u \in C^{2m}\left(\bar{\Omega}\right)$ and $f \in C\left(\bar{\Omega}\right)$ satisfy (14).*

- *If $2m > n$, then there exists $C^1_{\Omega,m} > 0$ such that for all $\theta \in [0,1]$*
$$\left\| d(.)^{-m+\theta n} u \right\|_{L^\infty(\Omega)} \leq C^1_{\Omega,m} \left\| d(.)^{m-(1-\theta)n} f \right\|_{L^1(\Omega)}. \tag{16}$$
- *Let $1 \leq p \leq q \leq \infty$. If $\frac{1}{p} - \frac{1}{q} < \min\left\{\frac{2m}{n}, 1\right\}$, then taking*
$$\alpha \in \left(\frac{1}{p} - \frac{1}{q}, \min\left\{1, \frac{2m}{n}\right\}\right]$$
there exists $C^2_{\Omega,m,\alpha} > 0$ such that for all $\theta \in [0,1]$
$$\left\| d(.)^{-m+\theta n\alpha} u \right\|_{L^q(\Omega)} \leq C^2_{\Omega,m,\alpha} \left\| d(.)^{m-(1-\theta)n\alpha} f \right\|_{L^p(\Omega)}. \tag{17}$$

Remark 18 *Notice that the shift in the exponent of $d(.)$ between the right and the left hand side of (17) is $2m - n\alpha$. Hence the shift increases when α goes to $\frac{1}{p} - \frac{1}{q}$.*

Remark 19 *The conditions $u \in C^{2m}\left(\bar{\Omega}\right)$ and $f \in C\left(\bar{\Omega}\right)$ may be considerable relaxed for each of the estimates by using a density argument.*

Remark 20 *The estimate in (16) is sharp and does not seem to follow through imbedding results. The estimates in (17) do need an application of Hölder's inequality. As a consequence the condition $\frac{1}{p} - \frac{1}{q} < \min\left\{\frac{2m}{n}, 1\right\}$ appears with a strict inequality. Such estimates will also follow through regularity results in L^p, Poincaré estimates, Sobolev imbeddings and dual Sobolev imbeddings. See [4].*

Remark 21 *In a similar way one may also derive estimates for combinations of boundary behavior and derivatives. For example if $n = m = 2$ one finds with $\theta \in [0,1]$:*
$$\left\| d(.)^{-1+2\theta} D_x u \right\|_{L^\infty(\Omega)} \leq C^3_{\Omega,m} \left\| d(.)^{2\theta} f \right\|_{L^1(\Omega)}.$$

Remark 22 *Fila, Souplet and Weissler in [7, Proposition 2.2] obtained for the case $m = 1$, the following estimate. Assume that $1 \leq p \leq q \leq \infty$ satisfy $\frac{1}{p} - \frac{1}{q} < \frac{2}{n+1}$, then any $u \in W^{1,2}_0(\Omega)$ with $d(.)^{\frac{1}{p}} \Delta u \in L^p(\Omega)$ satisfies*
$$\left\| d(.)^{\frac{1}{q}} u \right\|_{L^q(\Omega)} \leq C^4_{\Omega} \left\| d(.)^{\frac{1}{p}} \Delta u \right\|_{L^p(\Omega)}.$$
This is a special case of (17). The proof in [7] uses heat kernel estimates.

PROOF. In order to consider all the possible splitting between the boundary behavior and the internal regularity we use Lemma C.5, v) to find for all $\beta \in [0,1]$ and $\sigma \in [-1,1]$ that

$$\min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\} \leq \left(\frac{d(x)d(y)}{|x-y|^2}\right)^{1-\beta} \left(\frac{d(x)}{d(y)}\right)^{\beta\sigma}.$$

Hence, for $2m - n > 0$, we may use Theorem 3.1 to obtain that there exists $C_{\Omega,m} > 0$ such that for $\sigma \in [-1,1]$,

$$\begin{aligned} G_m(x,y) &\leq C_{\Omega,m}\ d(x)^{m-\frac{1}{2}n\alpha} d(y)^{m-\frac{1}{2}n\alpha} \frac{1}{|x-y|^{n(1-\alpha)}} \left(\frac{d(y)}{d(x)}\right)^{\frac{1}{2}n\alpha\sigma} \\ &\leq C_{\Omega,m}\ d(x)^{m-\frac{1}{2}n\alpha(1+\sigma)} d(y)^{m-\frac{1}{2}n\alpha(1-\sigma)} \frac{1}{|x-y|^{n(1-\alpha)}} \text{ for all } x,y \in \Omega. \quad (18) \end{aligned}$$

For $2m - n < 0$ and since $\frac{n}{2m}\alpha \in [0,1]$ we find with $\sigma \in [-1,1]$

$$\begin{aligned} G_m(x,y) &\leq C_{\Omega,m}\ |x-y|^{2m-n} \left(\frac{d(x)d(y)}{|x-y|^2}\right)^{m(1-\frac{n}{2m}\alpha)} \left(\frac{d(y)}{d(x)}\right)^{m\frac{n}{2m}\alpha\sigma} \\ &\leq C_{\Omega,m}\ d(x)^{m-\frac{1}{2}n\alpha(1+\sigma)} d(y)^{m-\frac{1}{2}n\alpha(1-\sigma)} \frac{1}{|x-y|^{n(1-\alpha)}} \text{ for all } x,y \in \Omega. \quad (19) \end{aligned}$$

So we may use Lemma 16 and

$$d(x)^{-m+\frac{1}{2}n\alpha(1+\sigma)} |u(x)| \leq C_{\Omega,m} \int_\Omega \frac{1}{|x-y|^{n(1-\alpha)}} d(y)^{m-\frac{1}{2}n\alpha(1-\sigma)} |f(y)|\, dy,$$

to find, since $\alpha \in \left(\frac{1}{p} - \frac{1}{q}, \min\left\{1, \frac{2m}{n}\right\}\right]$, that:

$$\left\| d^{-m+\frac{1}{2}n\alpha(1+\sigma)} u \right\|_{L^q(\Omega)} \leq c \left\| d^{m-\frac{1}{2}n\alpha(1-\sigma)} f \right\|_{L^p(\Omega)}.$$

So with $\theta = \frac{1}{2}(1+\sigma)$ we obtain the estimate in (16).

In the case that $2m - n = 0$ we may proceed as for (19) except for a logarithmic term. This term can be taken care of through

$$\log\left(2 + \frac{d(x)}{|x-y|}\right) \leq C_{\Omega,\varepsilon} \frac{1}{|x-y|^\varepsilon},$$

where we take $\varepsilon = \frac{1}{2}n\left(\alpha - \frac{1}{p} + \frac{1}{q}\right)$.

Appendix A Green function and Poisson kernels

In this section we recall some of the well known properties of the Green function and the Poisson kernels.

The Green function for (1)

This function $G_m : \Omega \times \Omega \to \mathbb{R}$ is such that for every $y \in \Omega$ the mapping $x \mapsto G(x, y)$ satisfies (in the sense of distribution)

$$\begin{cases} (-\Delta)^m G_m(\cdot, y) = \delta_y(\cdot) \text{ in } \Omega, \\ \left(\frac{\partial}{\partial \nu}\right)^j G_m(\cdot, y) \ = 0 \qquad \text{on } \partial\Omega,\ j = 0, ..., m-1. \end{cases} \tag{Appendix A.1}$$

Since $(-\Delta)^m$ is selfadjoint on $W^{2m,2}(\Omega) \cap W_0^{m,2}(\Omega) \subset L^2(\Omega)$, the Green function is symmetric. Observe that for $y \in \Omega$ identity (Appendix A.1) gives for $|s| \le m-1$

$$D_x^s G(x, y) = 0 \text{ for } x \in \partial\Omega. \tag{Appendix A.2}$$

In fact, taking $j = 0$ in (Appendix A.1) one finds that $x \mapsto G_m(x, y)$ for $y \in \Omega$ is zero at the boundary. Hence the tangential derivatives of $x \mapsto G_m(x, y)$ of any order, for $y \in \Omega$, are identically zero on $\partial\Omega$. Since the normal derivatives up to order $m-1$ are zero at the boundary, (Appendix A.2) follows.

The function G_m has a singular behavior on $D_\Omega := \left\{(x, x) : x \in \bar{\Omega}\right\}$. Assuming that $\partial\Omega$ is C^∞ one finds that G_m belongs to $C^\infty\left(\left(\bar{\Omega} \times \bar{\Omega}\right) \setminus D_\Omega\right)$.

The Poisson kernels for (1)

For $j = 0, ..., m-1$, and $y \in \partial\Omega$ the functions $x \mapsto K_j(x, y)$ satisfy (in the sense of distribution)

$$\begin{cases} (-\Delta)^m K_j(\cdot, y) = 0 \qquad \text{in } \Omega, \\ \left(\frac{\partial}{\partial \nu}\right)^k K_j(\cdot, y) \ = 0 \qquad \text{on } \partial\Omega, \text{ for } k \neq j,\ 0 \le k \le m-1, \\ \left(\frac{\partial}{\partial \nu}\right)^j K_j(\cdot, y) \ = \delta_{y,\partial\Omega}(\cdot) \text{ on } \partial\Omega, \end{cases} \tag{Appendix A.3}$$

where $\delta_{y,\partial\Omega}$ is the delta-function defined on $\partial\Omega$ (that is, the delta-function on an $(n-1)$-dimensional manifold). Moreover, the kernels satisfy for $|s| \le m-1$ and $j = 0, ..., m-1$

$$D_x^s K_j(x, y) = 0, \text{ for } x, y \in \partial\Omega, x \neq y. \tag{Appendix A.4}$$

In fact, the mappings $x \mapsto K_j(x, y)$ on $\bar{\Omega} \setminus \{y\}$ with $j = 0, ..., m-1$ are zero on $\partial\Omega \setminus \{y\}$. Hence the tangential derivatives of any order are zero on $\partial\Omega \setminus \{y\}$.

Since (Appendix A.3) implies that the normal derivatives up to order $m-1$ are zero, we find (Appendix A.4).

By an integration by part and by using the explicit order of the singularities of the Green function (for instance from the result of Krasovskiĭ in [12]), one can explicitly write the relation between the Poisson kernels and the Green function. Namely for $j \in \{0, ..., m-1\}$ and y in $\partial\Omega$ the following relation holds in Ω

$$K_j(x,y) = \begin{cases} \frac{\partial}{\partial \nu_y} \left(-\Delta_y\right)^{m-\left(\frac{j}{2}+1\right)} G(x,y) & \text{for } j \text{ even,} \\ \left(-\Delta_y\right)^{m-\frac{j+1}{2}} G(x,y) & \text{for } j \text{ odd,} \end{cases}$$

where ν_y denotes the external normal to $\partial\Omega$ in y.

The kernels K_j have a singular behavior on $D_{\partial\Omega} = \{(x,x) : x \in \partial\Omega\}$. Assuming that $\partial\Omega$ is C^∞ one finds that K_j belong to $C^\infty\left(\left(\bar{\Omega} \times \bar{\Omega}\right) \setminus D_{\partial\Omega}\right)$.

Appendix B The estimates of Krasovskiĭ

We will recall the theorem in [12] which gives the estimates of the Green function and the Poisson kernels. We first recall the main assumption.

Consider the boundary value problem

$$\begin{cases} \mathcal{L}u = \varphi_0 & \text{in } \Omega, \\ B_j u = \psi_j & \text{on } \partial\Omega \text{ for } j = 0, ..., m-1. \end{cases} \qquad \text{(Appendix B.1)}$$

The following hypothesis are assumed.

(1) The operator
$$\mathcal{L} := \sum_{|\beta| \le 2m} a_\beta\left(x\right) D^\beta,$$
is uniformly elliptic (see the condition for $\mathcal{L}$ on page 663 of [1]).

(2) The boundary operators
$$B_j = \sum_{|\beta| \le m_j} b_{j\beta}\left(x\right) D^\beta, \text{ for } j = 0, ..., m-1,$$
satisfy the complementing condition relative to $\mathcal{L}$ (see the complementing condition on page 663 of [1]).

(3) Let $l_1 > \max_j\left(2m - m_j\right)$ and $l_0 = \max_j\left(2m - m_j\right)$. The coefficients a_β belong to $C^{l_1+1}\left(\bar{\Omega}\right)$ and $b_{j\beta}$ belong to $C^{l_1+1}\left(\partial\Omega\right)$;

(4) The boundary $\partial\Omega$ is C^{l_1+2m+1}.

Theorem B.1 *Let the condition above be satisfied and let l_1 be such that $l_1 > 2(l_0 + 1)$ for $n = 2$ and $l_1 > \frac{3}{2} l_0$ for $n \geq 3$. If problem (Appendix B.1) is uniquely solvable then the Green function G_m and the Poisson kernels K_j, with $j = 0, ..., m-1$, for (Appendix B.1) exist.*

Theorem B.2 *Assume that the conditions of Theorem B.1 are satisfied. Moreover let $\alpha, \beta, \gamma \in \mathbb{N}^n$ with $|\alpha| \leq 2m + l_1 - l_0$, $|\beta| \leq l_1$ and $|\gamma| \leq l_1 - 2m + m_j + 1$.*

Then wherever they are defined, the derivatives of the Green function G_m satisfy:

(1) if $|\alpha| + |\beta| < 2m - n$ then

$$\left| D_x^\alpha D_y^\beta G_m(x, y) \right| \preceq 1,$$

(2) if $|\alpha| + |\beta| = 2m - n$ then

$$\left| D_x^\alpha D_y^\beta G_m(x, y) \right| \preceq \log \left(\frac{2 \operatorname{diam}_\Omega}{|x - y|} \right),$$

(3) if $|\alpha| + |\beta| > 2m - n$ then

$$\left| D_x^\alpha D_y^\beta G_m(x, y) \right| \preceq |x - y|^{2m - n - |\alpha| - |\beta|},$$

and the derivatives of K_j satisfy:

(1) if $|\alpha| + |\gamma| < m_j - n + 1$ then

$$\left| D_x^\alpha D_y^\gamma K_j(x, y) \right| \preceq 1,$$

(2) if $|\alpha| + |\gamma| = m_j - n + 1$ then

$$\left| D_x^\alpha D_y^\gamma K_j(x, y) \right| \preceq \log \left(\frac{2 \operatorname{diam}_\Omega}{|x - y|} \right),$$

(3) if $|\alpha| + |\gamma| > m_j - n + 1$ then

$$\left| D_x^\alpha D_y^\gamma K_j(x, y) \right| \preceq |x - y|^{m_j - n + 1 - |\alpha| - |\gamma|}.$$

Here $\operatorname{diam}_\Omega$ *denotes the diameter of* Ω.

Remark B.3 *In case of Dirichlet boundary conditions, hence $l_0 = 2m$, the conditions on l_1 are:*

for $n \geq 3$: $l_1 > 3m$,

for $n = 2$: $l_1 > 4m + 2$.

Hence one needs $\partial\Omega \in C^{6m+4}$ for $n = 2$ and $\partial\Omega \in C^{5m+2}$ for $n \geq 3$.

Appendix C Some technical lemmas

The following lemmas can be found in [9]. For the sake of convenience we recall these.

Lemma C.4 *If* $|x-y| \leq \frac{1}{2}\max\{d(x), d(y)\}$ *then it holds*

$$\tfrac{1}{2}d(x) \leq d(y) \leq 2d(x) \ \ \textit{and}\ 1 \leq \frac{d(x)\,d(y)}{|x-y|^2}.$$

Otherwise if $|x-y| \geq \frac{1}{2}\max\{d(x), d(y)\}$ *then it holds*

$$\frac{d(x)}{|x-y|} \leq 2,\ \frac{d(y)}{|x-y|} \leq 2 \ \textit{and}\ \frac{d(x)\,d(y)}{|x-y|^2} \leq 4.$$

Lemma C.5 *Let* $p, q \geq 0$*. The following relations hold on* $\Omega \times \Omega$*:*

$$\begin{array}{ll}
\textit{i:} & \min\left\{1, \frac{d(x)^p d(y)^q}{|x-y|^{p+q}}\right\} \sim \min\left\{1, \frac{d(x)^p}{|x-y|^p}\right\}\min\left\{1, \frac{d(y)^q}{|x-y|^q}\right\}, \\
\textit{ii:} & \log\left(1+\frac{d(x)^p d(y)^q}{|x-y|^{p+q}}\right) \sim \log\left(2+\frac{d(x)}{|x-y|}\right)\min\left\{1, \frac{d(x)^p d(y)^q}{|x-y|^{p+q}}\right\}, \\
\textit{iii:} & \log\left(2+\frac{d(x)}{|x-y|}\right) \sim \log\left(2+\frac{d(x)d(y)}{|x-y|^2}\right), \\
\textit{iv:} & \min\left\{1, \frac{d(x)^p d(y)^q}{|x-y|^{p+q}}\right\} \sim \left(\frac{d(y)}{d(x)}\right)^{\frac{1}{2}(q-p)}\min\left\{1, \frac{d(x)^{\frac{1}{2}(p+q)} d(y)^{\frac{1}{2}(p+q)}}{|x-y|^{p+q}}\right\}, \\
\textit{v:} & \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\} \sim \min\left\{\frac{d(y)}{d(x)}, \frac{d(x)}{d(y)}, \frac{d(x)d(y)}{|x-y|^2}\right\}.
\end{array}$$

Acknowledgement. We would like to thank V.A. Kozlov for the reference to [12].

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