The Clamped Plate Equation for the Limaçon

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1 Introduction

Hadamard in [8] states that the clamped plate equation for plates having the shape of a *Limaçon de Pascal* is positivity preserving. Positivity preserving for this (linear) equation on $\Omega \subset \mathbb{R}^2$ means that in the fourth order boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in} \quad \Omega, \\ u = \frac{\partial}{\partial \nu} u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

the sign of f is preserved by u. Here f is the force (density) and u the deflection of the plate of shape Ω. So the statement reads as, say for $f \in L^1(\Omega)$:

$$f \geq 0 \text{ implies } u \geq 0. \tag{2}$$

For a precise citation of Hadamard let $\Gamma_A^B = G_\Omega(A, B)$ be the corresponding Green function, that is, $u(x) = \int_\Omega G_\Omega(x, y) f(y)\, dy$ solves (1). Concerning Γ_A^B Hadamard in [8] writes:

> *M. Boggio, qui a, le premier, noté la signification physique de Γ_A^B, en a déduit l'hypothése que Γ_A^B était toujours positif. Malgré l'absence de démonstration rigoureuse, l'exactitude de cette proposition ne parâit pas doutese pour les aires convexes. Mais il était l'intéressant d'examiner si elle est vraie pour le cas du Limaçon de Pascal, qui est concave. La réponse est affirmative.*

Let us focus on Hadamard's claims separately.

Claim 1 *There is no doubt that Γ_A^B is positive for convex domains.*

This conjecture stood for a long time and only in 1949 a first counterexample, with Ω a long rectangle, was established by Duffin in [3]. This counterexample was soon to be followed by numerous others. A short survey can be found in the introduction of [10]. So by now it is well known that convexity is not a sufficient condition.

Let us remind the reader that around 1905 Boggio [2] did prove that (2) holds in case of a disk. In fact some believed that the disk might be the only domain where (2) holds. However in [4] it is shown that (2) also holds in domains that are small perturbations of the disk. Since smallness of these perturbations is defined by a C^2-norm non-convex domains are not included.

Claim 2 *Γ_A^B is positive for some non-convex domains, namely for the Limaçons of Pascal.*

Hadamard in [8] starts his proof of this claim by observing that:

> *... on constate aisément que, si l'un de ces deux points est très voisin du contour, la partie principale de Γ_A^B est positive.*

Although we are not certain what he meant by '*partie principale*' we know by now that Γ_A^B can be negative when one point is near the boundary. In fact we will show that if the Green function (on a limaçon) is negative somewhere it will be negative for some A and B near the boundary. Hadamard continues his proof by referring to the results in [9].

In [9] he gives an explicit formula for the Green function for (1) in case of a Limaçon. This formula will allow us to show the theorem below. Since there is no explicit proof that his formula indeed gives the Green function we will supply such a proof in the appendix.

The domains under consideration are defined for $a \in \left[0, \frac{1}{2}\right]$ by

$$\Omega_a = \left\{(\rho \cos \varphi, \rho \sin \varphi) \in \mathbb{R}^2; 0 \leq \rho < 1 + 2a \cos \varphi\right\}.$$

For $0 \leq a \leq \frac{1}{2}$ the curve $\rho = 1 + 2a \cos \varphi$ is a non self-intersecting limaçon.

We will show the following:

Theorem 3 *The clamped plate problem on Ω_a with $a \in \left[0, \frac{1}{2}\right]$ is positivity preserving if and only if $a \in \left[0, \frac{1}{6}\sqrt{6}\right]$.*

Remark 4 *The limaçon is convex precisely if $0 \leq a \leq \frac{1}{4}$. Notice that $\frac{1}{4} < \frac{1}{6}\sqrt{6}$. So Hadamard is right in the sense that convexity is not a necessary condition. He is wrong in claiming the positivity preserving property for all limaçons.*

Remark 5 *A related question is if the first eigenfunction has a fixed sign for all limaçons (compare the Boggio-Hadamard-conjecture versus the Szegö-conjecture). Since one cannot expect an explicit formula for the eigenfunction this seems a much harder question. One does know that the number a where positivity of the first eigenfunction breaks down is strictly larger than the number where (2) fails. See [5].*

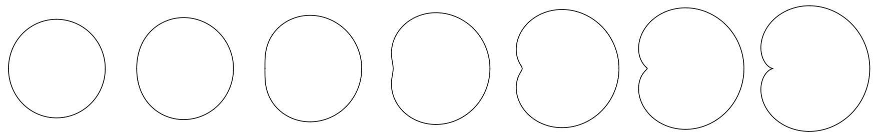

Figure 1: Limaçons for resp. $a = .1, .175, .25, .325, \frac{1}{6}\sqrt{6}, .45, .5$. The fifth one with $a = \frac{1}{6}\sqrt{6}$ is critical for positivity.

2 Proofs

Any limaçon can be seen as the image of a circle through the conformal map $z \to z^2$ combined with two shifts. It will be convenient in the following to use complex notation for the unit disk: $B = \{z \in \mathbb{C}\,; |z| < 1\}$. The appropriate conformal map from $B \subset \mathbb{C}$ to $\Omega_a \subset \mathbb{R}^2$ is now given by

$$\begin{array}{rcl} h_a: B & \to & \Omega_a, \\ \eta & \mapsto & x = \left(\mathrm{Re}\left(\eta + a\eta^2\right), \mathrm{Im}\left(\eta + a\eta^2\right)\right). \end{array} \tag{3}$$

The fact that this conformal map is quadratic seems to allow an explicit Green function which makes the limaçon a special case.

2.1 Behaviour of the Green function

In [9, Supplement] one finds the explicit formula of the Green function for (1), which we will denote with G_a. For $x, y \in \Omega_a$ we may rewrite this function as follows

$$G_a(x,y) = \tfrac{1}{2}a^2 s^2 r^2 \left[\log\left(\tfrac{r^2}{r_1^2}\right) + \tfrac{r_1^2}{r^2} - 1 - \tfrac{a^2}{1-2a^2}\tfrac{r^2}{s^2}\left(\tfrac{r_1^2}{r^2} - 1\right)^2\right], \tag{4}$$

where, with $\eta, \xi \in B$ such that $x = h_a(\eta)$ and $y = h_a(\xi)$, the r, r_1 and s are given by

$$r^2 = |\eta - \xi|^2, \quad r_1^2 = \left|1 - \eta\bar{\xi}\right|^2, \quad s^2 = \left|\eta + \xi + \tfrac{1}{a}\right|^2. \tag{5}$$

We want to study when the function G_a is of fixed sign in $\Omega_a \times \Omega_a$. For establishing positivity of the Green function we will need to consider the function

$$F(\beta, q) := \log\left(\frac{1}{q}\right) + q - 1 - \beta\frac{(q-1)^2}{q}. \tag{6}$$

Note that $q = r_1^2/r^2 \geq 1$.

Lemma 6 *Set $I_\beta := \{q \geq 1 : F(\beta, q) \leq 0\}$. It holds that:*

- $I_\beta = \{1\}$ *for* $\beta \in \left[0, \frac{1}{2}\right]$;

- $I_\beta = [1, q_\beta]$ *with* $q_\beta > 1$ *for* $\beta \in \left(\frac{1}{2}, 1\right)$;
- $I_\beta = [1, \infty)$ *for* $\beta \in [1, \infty)$.

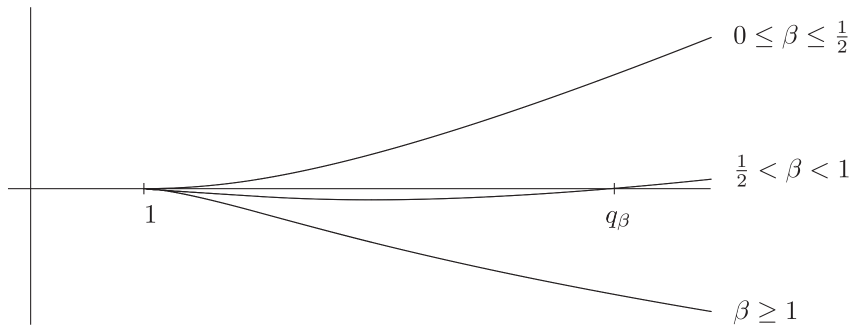

Figure 2: Graphs of $q \mapsto F(\beta, q)$.

Remark 7 *Note that* $\beta \mapsto F(\beta, q)$ *is decreasing and hence that* $\beta \mapsto q_\beta$ *is increasing.*

It will be convenient to work with functions defined in the disk. If f is a function defined on Ω_a, then $\tilde{f}_a$ will denote the function $\tilde{f}_a := f \circ h_a$ defined on the disk.

We define the auxiliary function

$$\tilde{H}_a(\eta, \xi) := \tfrac{a^2}{1-2a^2} \frac{r_1^2}{s^2} = \tfrac{a^2}{1-2a^2} \frac{|1-\eta\bar{\xi}|^2}{|\eta + \xi + \frac{1}{a}|^2}, \tag{7}$$

such that the Green function in (4) becomes

$$\begin{aligned} \tilde{G}_a(\eta, \xi) &:= \tfrac{1}{2} a^2 s^2 r^2 \; F\left(\tilde{H}_a(\eta, \xi), \tfrac{r_1^2}{r^2}\right) \\ &= \tfrac{1}{2} a^2 s^2 r^2 \; F\left(\tilde{H}_a(\eta, \xi), \tfrac{|1-\eta\bar{\xi}|^2}{|\eta-\xi|^2}\right). \end{aligned} \tag{8}$$

The preceding Lemma 6 gives that if

$$\sup_{\eta,\xi \in B} \tilde{H}_a(\eta, \xi) \le \frac{1}{2}, \tag{9}$$

then F and hence G_a are positive. Note that (9) gives a condition on the parameter a and a sufficient condition for the positivity of the function. In the following we will see that this condition is also necessary.

First we will reduce the dimension of the problem. The following lemma states that it is sufficient to study the behaviour of $\tilde{H}_a$ for couples of conjugate points.

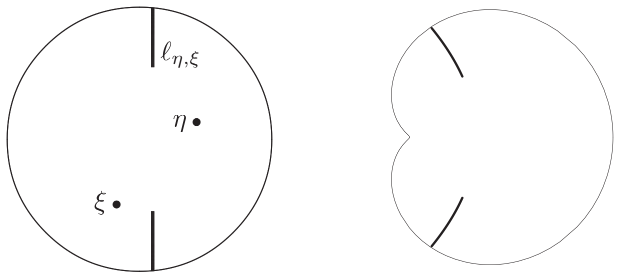

Figure 3: A set $\ell_{\eta,\xi}$ and its image within the limaçon.

Lemma 8 *Let $a < \frac{1}{2}$ and define the sets $\ell_{\eta,\xi}$ for $(\eta, \xi) \in B^2$ by*

$$\ell_{\eta,\xi} = \left\{ \chi = \chi_1 + i\chi_2 \in B : \chi_1 = \tfrac{\eta_1+\xi_1}{2}, |\chi| \geq \max\left\{|\eta|, |\xi|\right\} \right\}, \qquad (10)$$

where $\eta = \eta_1 + i\eta_2$ and $\xi = \xi_1 + i\xi_2$.

If $\tilde{H}_a(\eta, \xi) > \frac{1}{2}$, then for every $\chi \in \ell_{\eta,\xi}$ it holds that $\tilde{H}_a(\chi, \bar{\chi}) > \frac{1}{2}$.

Proof. By hypothesis one has:

$$\tilde{H}(\eta, \xi) = \tfrac{a^2}{1-2a^2} \frac{(1 - \eta_1\xi_1 - \eta_2\xi_2)^2 + (\eta_1\xi_2 - \eta_2\xi_1)^2}{\left(\eta_1 + \xi_1 + \frac{1}{a}\right)^2 + (\eta_2 + \xi_2)^2} > \tfrac{1}{2},$$

which is equivalent to

$$\begin{aligned} &2a^2\left(1 + \eta_1^2\xi_1^2 + \eta_2^2\xi_2^2 - 2\eta_1\xi_1 - 2\eta_2\xi_2 + \eta_1^2\xi_2^2 + \eta_2^2\xi_1^2\right) > \\ &\qquad \left(1 - 2a^2\right)\left(\eta_1^2 + \xi_1^2 + \tfrac{1}{a^2} + 2\eta_1\xi_1 + \tfrac{2}{a}\eta_1 + \tfrac{2}{a}\xi_1 + \eta_2^2 + \xi_2^2 + 2\eta_2\xi_2\right), \end{aligned}$$

or similarly

$$\begin{aligned} &2a^2(1 + |\eta|^2)(1 + |\xi|^2) > \\ &\qquad (\eta_1 + \xi_1)^2 + (\eta_2 + \xi_2)^2 + \tfrac{1}{a^2} + \tfrac{2}{a}(\eta_1 + \xi_1) - 2 - 4a(\eta_1 + \xi_1). \end{aligned} \qquad (11)$$

For $\chi \in \ell_{\eta,\xi}$, we have

$$\begin{aligned} \tilde{H}(\chi, \bar{\chi}) - \tfrac{1}{2} &= \tfrac{a^2}{1-2a^2} \frac{\left(1 - \chi_1^2 + \chi_2^2\right)^2 + 4\chi_1^2\chi_2^2}{\left(2\chi_1 + \frac{1}{a}\right)^2} - \frac{2\chi_1^2 + \frac{1}{2a^2} + \frac{2}{a}\chi_1}{\left(2\chi_1 + \frac{1}{a}\right)^2} \\ &= \tfrac{a^2}{1-2a^2} \frac{1+\chi_1^4+\chi_2^4-2\chi_1^2+2\chi_2^2-2\chi_1^2\chi_2^2+4\chi_1^2\chi_2^2-\frac{2}{a^2}\chi_1^2-\frac{1}{2a^4}-\frac{2}{a^3}\chi_1+4\chi_1^2+\frac{1}{a^2}+\frac{4}{a}\chi_1}{\left(2\chi_1+\frac{1}{a}\right)^2} \\ &= \tfrac{1}{1-2a^2} \tfrac{1}{\left(2\chi_1+\frac{1}{a}\right)^2} \left(a^2(1 + |\chi|^2)^2 - (2\chi_1^2 + \tfrac{1}{2a^2} + \tfrac{2}{a}\chi_1 - 1 - 4a\chi_1)\right). \end{aligned} \qquad (12)$$

By the definition of $\ell_{\eta,\xi}$ and (11) it follows that the last term is positive:

$$(1+|\chi|^2)^2 \geq (1+|\eta|^2)(1+|\xi|^2) > \tfrac{1}{a^2}\left(2\chi_1^2 + \tfrac{1}{2a^2} + \tfrac{2}{a}\chi_1 - 1 - 4a\chi_1\right).$$

■

Remark 9 *Note that (12) implies:* $\tilde{H}_a(\chi,\bar{\chi})$ *is increasing in* $|\chi_2|$.

We are now able to prove that (9) also gives a necessary condition for the positivity of F and hence of G_a.

Lemma 10 *Let* $a < \frac{1}{2}$.

i. If $\tilde{H}_a(v,\bar{v}) > \frac{1}{2}$ *then there is* $\chi \in \ell_{v,\bar{v}}$ *such that* $F\left(\tilde{H}_a(\chi,\bar{\chi}), \frac{|1-\chi^2|^2}{|\chi-\bar{\chi}|^2}\right) < 0$.

ii. If $F\left(\tilde{H}_a(\chi,\bar{\chi}), \frac{|1-\chi^2|^2}{|\chi-\bar{\chi}|^2}\right) < 0$, *then* $F\left(\tilde{H}_a(z,\bar{z}), \frac{|1-z^2|^2}{|z-\bar{z}|^2}\right) < 0$ *for every* $z \in \ell_{\chi,\bar{\chi}}$.

Proof. First claim: Since the function $\beta \mapsto F(\beta,q)$ is decreasing, see (6), and the function $\tilde{H}_a(z,\bar{z})$ is increasing in $|z_2|$, by Remark 9, one gets that

$$F\left(\tilde{H}_a(z,\bar{z}), \frac{|1-z^2|^2}{|z-\bar{z}|^2}\right) \leq F\left(\tilde{H}_a(v,\bar{v}), \frac{|1-z^2|^2}{|z-\bar{z}|^2}\right) \text{ for every } z \in \ell_{v,\bar{v}}. \quad (13)$$

In $F\left(\tilde{H}_a(v,\bar{v}), \frac{|1-z^2|^2}{|z-\bar{z}|^2}\right)$ the first argument does not depend on z; it is a fixed coefficient which is larger then $1/2$ by hypothesis. Hence, applying Lemma 6, one has that there exists a $q_{\tilde{H}_a(v,\bar{v})} > 1$ such that

$$F\left(\tilde{H}_a(v,\bar{v}), \frac{|1-z^2|^2}{|z-\bar{z}|^2}\right) < 0,\ \forall z \in \ell_{v,\bar{v}} \text{ with } \frac{|1-z^2|^2}{|z-\bar{z}|^2} < q_{\tilde{H}_a(v,\bar{v})}. \quad (14)$$

Note that the function $|z_2| \mapsto \frac{|1-z^2|^2}{|z-\bar{z}|^2}$ is decreasing, since

$$\frac{\partial}{\partial z_2}\frac{|1-z^2|^2}{|z-\bar{z}|^2} = -\tfrac{1}{2z_2^3}(1-|z|^2+2z_2^2)(1-|z|^2). \quad (15)$$

Hence, since $\frac{|1-z^2|^2}{|z-\bar{z}|^2}$ is equal to 1 at the boundary, it follows that there exists $\chi \in \ell_{v,\bar{v}}$ such that

$$\frac{|1-\chi^2|^2}{|\chi-\bar{\chi}|^2} < q_{\tilde{H}_a(v,\bar{v})}. \quad (16)$$

Combining (13), (14) and (16) the first claim follows.

Second claim: If $F\left(\tilde{H}_a(\chi,\bar{\chi}), \frac{|1-\chi^2|^2}{|\chi-\bar{\chi}|^2}\right) < 0$ we can deduce from Lemma 6 that

$$\tilde{H}_a(\chi,\bar{\chi}) > \tfrac{1}{2} \text{ and } \frac{|1-\chi^2|^2}{|\chi-\bar{\chi}|^2} < q_{\tilde{H}_a(\chi,\bar{\chi})}. \quad (17)$$

Since $\tilde{H}_a(z,\bar{z})$ is increasing in $|z_2|$ (Remark 9) and the function $|z_2| \mapsto \frac{|1-z^2|^2}{|z-\bar{z}|^2}$ is decreasing, see (15), from (17) one gets that

$$\tilde{H}_a(z,\bar{z}) > \tfrac{1}{2} \text{ and } \tfrac{|1-z^2|^2}{|z-\bar{z}|^2} < q_{\tilde{H}_a(\chi,\bar{\chi})} \text{ for every } z \in \ell_{\chi,\bar{\chi}}. \tag{18}$$

Since $\beta \mapsto q_\beta$ is increasing (Remark 7), from (18) we have that

$$\tfrac{|1-z^2|^2}{|z-\bar{z}|^2} < q_{\tilde{H}_a(z,\bar{z})} \text{ for every } z \in \ell_{\chi,\bar{\chi}}. \tag{19}$$

By (18), (19) and Lemma 6 it follows that $F\left(\tilde{H}_a(z,\bar{z}), \frac{|1-z^2|^2}{|z-\bar{z}|^2}\right) < 0$ for every $z \in \ell_{\chi,\bar{\chi}}$. ■

The previous results give the following:

Corollary 11 *If the function $G_a(x,y)$ is negative for some $x,y \in \Omega_a$ then G_a will be negative somewhere near opposite boundary points. To be precise: for all $\varepsilon > 0$ there is $x^\varepsilon \in \Omega_a$ with $d(x^\varepsilon, \partial\Omega_a) < \varepsilon$ such that it holds $G((x_1^\varepsilon, x_2^\varepsilon),(x_1^\varepsilon, -x_2^\varepsilon)) < 0$.*

Proof. If $\tilde{G}_a(\eta,\xi) < 0$, Lemma 6 gives that necessarily $\tilde{H}_a(\eta,\xi) > \frac{1}{2}$. Hence, one has from Lemma 8 that $\tilde{H}_a(z,\bar{z}) > \frac{1}{2}$ for every $z \in \ell_{\eta,\xi}$. The claim follows directly from Lemma 10. ■

2.2 Positivity of the Green function

Using the results of the previous section, we have seen that the function $\tilde{H}_a$ in (7) plays a crucial role for the positivity of the Green function. Let us collect this result.

Corollary 12 *The Green function for the clamped plate equation on a limaçon is positive if and only if*

$$\sup_{\eta,\xi \in B} \tilde{H}_a(\eta,\xi) = \tfrac{a^2}{1-2a^2} \sup_{\eta,\xi \in B} \frac{|1-\eta\bar{\xi}|^2}{|\eta+\xi+1|^2} \le \frac{1}{2}. \tag{20}$$

Condition (20) gives an upper bound for the parameter a. In the following Lemma we give the explicit value of this upper bound.

Lemma 13 *Inequality (20) is satisfied if and only if $a \le \frac{1}{6}\sqrt{6}$.*

Proof. Lemma 8 implies that it is sufficient to verify (20) for couples of conjugate points, that is:

$$\sup_{\chi \in B} \tilde{H}_a(\chi,\bar{\chi}) = \tfrac{a^2}{1-2a^2} \sup_{\chi \in B} \frac{|1-\chi^2|^2}{|\chi+\bar{\chi}+1|^2} \le \frac{1}{2}.$$

By (12) we find

$$\tilde{H}_a(\chi,\bar{\chi})-\tfrac{1}{2}=\tfrac{1}{1-2a^2}\frac{a^2(1+|\chi|^2)^2+1+4a\chi_1}{\left(2\chi_1+\frac{1}{a}\right)^2}-\tfrac{1}{1-2a^2}\tfrac{1}{2},$$

which gives

$$\sup_{\chi\in B}\tilde{H}_a(\chi,\bar{\chi})-\frac{1}{2}=\tfrac{1}{1-2a^2}\sup_{\chi\in B}\frac{4a^2+1+4a\chi_1}{\left(2\chi_1+\frac{1}{a}\right)^2}-\tfrac{1}{1-2a^2}\tfrac{1}{2}. \tag{21}$$

A straightforward computation shows that the maximum in (21) is attained for $\chi_1=-2a$ (and $|\chi|=1$). We obtain

$$\sup_{\chi\in B}\tilde{H}_a(\chi,\bar{\chi})-\tfrac{1}{2}=\tfrac{a^2}{1-2a^2}\frac{1}{-4a^2+1}-\tfrac{1}{1-2a^2}\tfrac{1}{2}=\tfrac{1}{1-2a^2}\frac{6a^2-1}{2(1-4a^2)},$$

which is non-negative for $a>\frac{1}{6}\sqrt{6}$. ■

2.3 Sharp estimates for the Green function

The Green function for the biharmonic problem in two dimensions does not have a singularity in L^∞-sense: $(x,y)\mapsto G(x,y)$ is uniformly bounded. However, a natural solution space concerning the Dirichlet boundary condition ($u=\frac{\partial}{\partial\nu}u=0$), see [1], is the Banach lattice (with the natural ordering):

$$C_e(\bar{\Omega})=\{u\in C(\bar{\Omega});\|u\|_e:=\sup_{x\in\Omega}\left|\frac{u(x)}{d_\Omega^2(x)}\right|<\infty\}.$$

As an operator $(x\mapsto G(x,.))\in L\left(\bar{\Omega};C_e(\bar{\Omega})'\right)$ the Green function G does show 'a singularity'. Precise information for the singularity of polyharmonic Dirichlet Green functions on balls in $\mathbb{R}^n$, where the Green function is known to be positive, can be found in [7].

In the next theorem one finds how the estimate of G_a from below changes depending on a. It is interesting to see that although the Green function becomes negative, no 'boundary-singularity' from below appears.

Note that Theorem 3 is a direct consequence of Theorem 14.

Theorem 14 *Let $d_B(\cdot)$ denotes the distance function from the boundary of B. For every $(\eta,\xi)\in B\times B$, the following estimates hold:*

i. for all $a\in\left(0,\frac{1}{2}\right)$ there exists $c_1>0$ such that

$$\tilde{G}_a(\eta,\xi)\le c_1\ d_B(\eta)\,d_B(\xi)\min\left\{1,\frac{d_B(\eta)\,d_B(\xi)}{|\eta-\xi|^2}\right\}, \tag{22}$$

ii. for all $a \in \left(0, \frac{1}{6}\sqrt{6}\right)$*, there exists* $c_2 > 0$ *such that*

$$\tilde{G}_a(\eta, \xi) \geq c_2 \ \left(\tfrac{1}{6}\sqrt{6} - a\right) d_B(\eta)\, d_B(\xi) \min\left\{1, \frac{d_B(\eta)\, d_B(\xi)}{|\eta - \xi|^2}\right\}, \tag{23}$$

iii. for $a \in \left(\frac{1}{6}\sqrt{6}, \frac{1}{2}\right)$ *there exists* $(\eta^*, \xi^*) \in B \times B$ *such that*

$$\tilde{G}_a(\eta^*, \xi^*) < 0.$$

iv. for all $a \in \left(\frac{1}{6}\sqrt{6}, \frac{1}{2}\right)$*, it holds that*

$$\tilde{G}_a(\eta, \xi) \geq -6 \ \left(a - \tfrac{1}{6}\sqrt{6}\right) d_B(\eta)^2\, d_B(\xi)^2, \tag{24}$$

where the constants c_1 *and* c_2 *do not depend on* a*, and the distance function is defined as* $d_B(\eta) = \inf\{|\eta - \chi| : \chi \in \partial B\}$*.*

Remark 15 *Let us observe that for every* $\varepsilon > 0$ *there exists two constants* m_ε*,* M *such that for every* $\eta, \xi \in B$ *and* $a \in [0, \frac{1}{2} - \varepsilon]$ *it holds:*

$$\begin{array}{rcl} m_\varepsilon \,.\, |\eta - \xi| & \leq & |h_a(\eta) - h_a(\xi)| \leq M \,.\, |\eta - \xi|, \\ m_\varepsilon \,.\, d(\eta, \partial B) & \leq & d(h_a(\eta), \partial\Omega_a) \leq M \,.\, d(\eta, \partial B). \end{array} \tag{25}$$

Using (25) one can prove estimates for G_a *similar to the one proven for* $\tilde{G}_a$ *in Theorem 14. Near the cusp (when* $a \to \frac{1}{2}$*) the estimate from below breaks down.*

Remark 16 *One can derive that for* $a \in \left[0, \frac{1}{6}\sqrt{6}\right]$ *there exist constants* c_4*,* c_5*, independent on* a*, such that*

$$c_4 \ \left(\tfrac{1}{6}\sqrt{6} - a\right) D(x, y) \leq G_a(x, y) \leq c_5 \ D(x, y),$$

where $D(x, y) = d(x)\, d(y) \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}$ *and* $d(\cdot)$ *denotes the distance function from the boundary of the limaçon.*

Proof. We will prove the several statements separately.

i. One has from (4) that

$$\begin{aligned} \tilde{G}_a(\eta, \xi) & \leq \tfrac{1}{2} a^2 s^2 \left[-r^2 \log\left(\frac{r_1^2}{r^2}\right) + r_1^2 - r^2\right] \\ & \leq 2\left[-r^2 \log\left(\frac{r_1^2}{r^2}\right) + r_1^2 - r^2\right]. \end{aligned} \tag{26}$$

The term inside the brackets in the right hand side of (26) is the Green function for the clamped plate equation on the disk. Inequality (22) follows using the estimate in [6, Prop.2.3(iii)].

ii. Let $a_0 = \frac{1}{6}\sqrt{6}$. We have that

$$\begin{aligned} \tilde{G}_a(\eta,\xi) &= \tfrac{a^4}{1-2a^2}\tfrac{1-2a_0^2}{a_0^4}\tilde{G}_{a_0}(\eta,\xi) \\ &\quad +\tfrac{1}{2}a^2s^2\left(1-\tfrac{a^2}{1-2a^2}\tfrac{1-2a_0^2}{a_0^2}\right)\left[-r^2\log\left(\frac{r_1^2}{r^2}\right)+r_1^2-r^2\right] \\ &\geq \tfrac{1}{2}a^2s^2\left(1-4\tfrac{a^2}{1-2a^2}\right)\left[-r^2\log\left(\frac{r_1^2}{r^2}\right)+r_1^2-r^2\right], \end{aligned}$$

since $\tilde{G}_{a_0}(\eta,\xi) \geq 0$, see Corollary 12 and Lemma 13. For $a \in \left[0, \frac{1}{6}\sqrt{6}\right]$ one has $\frac{1}{2}a^2s^2(1-4\frac{a^2}{1-2a^2}) \geq \frac{1}{10}(\frac{1}{6}\sqrt{6}-a)$, hence by using [6, Prop.2.3(iii)] one gets

$$\tilde{G}_a(\eta,\xi) \geq c_2\ \left(\tfrac{1}{6}\sqrt{6}-a\right)d_B(\eta)\,d_B(\xi)\min\left\{1, \frac{d_B(\eta)\,d_B(\xi)}{|\eta-\xi|^2}\right\}.$$

iii. It follows from Corollary 12 and Lemma 13.

iv. Let $a_0 = \frac{1}{6}\sqrt{6}$. Using that $\tilde{G}_{a_0}$ is positive in the entire domain and that $\frac{1}{8}a^2\frac{6a^2-1}{1-2a^2}(1+|\eta|)^2(1+|\xi|)^2 \leq 6(a-\frac{1}{6}\sqrt{6})$ for $a \in (\frac{1}{6}\sqrt{6}, \frac{1}{2})$ we get

$$\begin{aligned} \tilde{G}_a(\eta,\xi) &= \tilde{G}_{a_0}(\eta,\xi) - \tfrac{1}{2}a^2\left(\tfrac{a^2}{1-2a^2}-\tfrac{a_0^2}{1-2a_0^2}\right)\left(r_1^2-r^2\right)^2 \\ &\geq -\tfrac{1}{2}a^2\left(\tfrac{a^2}{1-2a^2}-\tfrac{1}{4}\right)\left(r_1^2-r^2\right)^2 \\ &= -\tfrac{1}{8}a^2\tfrac{6a^2-1}{1-2a^2}\left(r_1^2-r^2\right)^2 \\ &\geq -6\ \left(a-\tfrac{1}{6}\sqrt{6}\right)d_B(\eta)^2\,d_B(\xi)^2, \end{aligned}$$

which gives the estimate in (24).

■

A The Green function for the limaçon

As promised in the introduction this appendix will contain a proof that the function supplied by Hadamard is indeed the Green function for the limaçons. For $x, y \in \mathbb{R}^2$ let $R = |x-y|$. The function $U = R^2\log(R)$ satisfies $\Delta^2 U(\cdot) = \delta_y(\cdot)$ in $\mathbb{R}^2$. Then writing

$$G_a(x,y) = R^2\log(R) + J_a(x,y), \tag{27}$$

the function

$$J_a(x,y) := -R^2\log(ar_1s) + \tfrac{a^2}{2}s^2\left(r_1^2-r^2\right) - \tfrac{a^4}{2(1-2a^2)}\left(r_1^2-r^2\right)^2, \tag{28}$$

should be biharmonic and such that G_a satisfies the boundary condition.

Note that (27) follows from (4) using that $ars = R$. In fact one has

$$\begin{aligned} R &= \left|a\eta^2 + \eta - a\xi^2 - \xi\right| = a\left|\eta^2 + \tfrac{\eta}{a} - \xi^2 - \tfrac{\xi}{a}\right| = \\ &= a\ |\eta - \xi|\left|\eta + \xi + \tfrac{1}{a}\right| = ars. \end{aligned}$$

Boundary condition: Let us rewrite (27) as

$$G_a(x,y) = \tfrac{1}{2}a^2 s^2\left[r^2 \log\left(\tfrac{r^2}{r_1^2}\right) + r_1^2 - r^2\right] - \tfrac{a^4}{2(1-2a^2)}\left(r_1^2 - r^2\right)^2. \tag{29}$$

When $x \in \partial\Omega_a$, then $\eta \in \partial D$ and it holds $r_1 = r$. It follows from (29) that $G_a(x,y) = 0$ at the boundary. Now we are interested in $\frac{\partial}{\partial\nu}G_a(x,y)$ on $\partial\Omega_a$. One observes that the term $\left(r_1^2 - r^2\right)^2$ gives no contribution because it is a zero of order two at the boundary. The remaining term is a product of two factors: one that is non-zero at the boundary and the other that is identically zero. Hence, when we look at the normal derivative at the boundary the only relevant term will be

$$\tfrac{\partial}{\partial\nu}\left[r^2 \log\left(\tfrac{r^2}{r_1^2}\right) + r_1^2 - r^2\right]. \tag{30}$$

Using that the term inside the brackets in (30) is the Green function for the disk, see [2], one gets that also the second Dirichlet boundary condition is satisfied.

The function $J_a(x,y)$ is biharmonic on Ω_a. To prove the biharmonicity of J_a it is convenient to consider separately the term with the logarithm and the remaining part.

We first observe that $\log(ar_1 s)$ is a harmonic function on Ω_a. From this, the identity $\Delta^2\left(R^2 \log(ar_1 s)\right) = 0$ follows using that if u is an harmonic function then $R^2 u$ is biharmonic.

Lemma 17 *It holds that*

$$\Delta_x^2\left(s^2\left(r_1^2 - r^2\right) - \tfrac{a^2}{1-2a^2}\left(r_1^2 - r^2\right)^2\right) = 0.$$

Proof. Next to $h_a : B \subset \mathbb{C} \to \mathbb{R}^2$ we will use $\mathbf{h}_a(\eta) : \mathbb{C} \to \mathbb{C}$ defined by $\mathbf{h}_a(\eta) = \eta + a\eta^2$, and $\eta = \eta_1 + i\eta_2$
Let us consider

$$\begin{aligned} K(x,y) &:= \left|h_a^{-1}(x) + h_a^{-1}(y) + \tfrac{1}{a}\right|^2 (1 - \left|h_a^{-1}(x)\right|^2) \\ &\quad - \tfrac{a^2}{1-2a^2}(1 - \left|h_a^{-1}(x)\right|^2)^2(1 - \left|h_a^{-1}(y)\right|^2), \end{aligned}$$

and then $s^2\left(r_1^2 - r^2\right) - \frac{a^2}{1-2a^2}\left(r_1^2 - r^2\right)^2 = (1 - |h_a^{-1}(y)|^2)K(x,y)$, and

$$\begin{aligned} Y(\eta,\xi) := K(h_a(\eta), h_a(\xi)) &= \left|\eta + \xi + \tfrac{1}{a}\right|^2\left(1 - |\eta|^2\right) \\ &\quad - \tfrac{a^2}{1-2a^2}\left(1 - |\eta|^2\right)^2\left(1 - |\xi|^2\right). \end{aligned}$$

Since h is a conformal map, it holds that:

$$\begin{aligned}
\Delta_\eta Y(\eta,\xi) &= \left|\mathbf{h}_a'(\eta)\right|^2 (\Delta_x K)(h_a(\eta), h_a(\xi)), \qquad (31)\\
\Delta_\eta^2 Y(\eta,\xi) &= \Delta_\eta \left|\mathbf{h}_a'(\eta)\right|^2 \Delta_x K(h_a(\eta), h_a(\xi))\\
&\quad + 2\sum_{i=1}^{2} \tfrac{\partial}{\partial \eta_i} \left|\mathbf{h}_a'(\eta)\right|^2 \tfrac{\partial}{\partial \eta_i} (\Delta_x K)(h_a(\eta), h_a(\xi))\\
&\quad + \left|\mathbf{h}_a'(\eta)\right|^4 \left(\Delta_x^2 K\right)(h_a(\eta), h_a(\xi)). \qquad (32)
\end{aligned}$$

The idea is to use (32) in order to calculate $\left(\Delta_x^2 K\right)(h_a(\eta), h_a(\xi))$ in terms of $\Delta_\eta^2 Y(\eta,\xi)$. Since $\Delta_\eta = 4\frac{\partial}{\partial\bar\eta}\frac{\partial}{\partial\eta}$, one has

$$\begin{aligned}
\tfrac{\partial}{\partial\eta} Y(\eta,\xi) &= \left(\bar\eta + \bar\xi + \tfrac{1}{a}\right)(1-|\eta|^2) - \bar\eta\left|\eta+\xi+\tfrac{1}{a}\right|^2\\
&\quad + \tfrac{2a^2}{1-2a^2}\bar\eta(1-|\eta|^2)(1-|\xi|^2),\\
\tfrac{\partial^2}{\partial\bar\eta\partial\eta} Y(\eta,\xi) &= (1-|\eta|^2) - \eta\left(\bar\eta+\bar\xi+\tfrac{1}{a}\right) - \left|\eta+\xi+\tfrac{1}{a}\right|^2 - \bar\eta\left(\eta+\xi+\tfrac{1}{a}\right)\\
&\quad + \tfrac{2a^2}{1-2a^2}(1-|\eta|^2)(1-|\xi|^2) - \tfrac{2a^2}{1-2a^2}\bar\eta\eta(1-|\xi|^2),\\
\tfrac{\partial^3}{\partial\eta\partial\bar\eta\partial\eta} Y(\eta,\xi) &= -2\bar\eta - 2\left(\bar\eta+\bar\xi+\tfrac{1}{a}\right) - \tfrac{4a^2}{1-2a^2}\bar\eta(1-|\xi|^2),\\
\tfrac{\partial^4}{\partial\bar\eta\partial\eta\partial\bar\eta\partial\eta} Y(\eta,\xi) &= -4 - \tfrac{4a^2}{1-2a^2}(1-|\xi|^2),
\end{aligned}$$

that gives

$$\begin{aligned}
\Delta_\eta Y(\eta,\xi) &= 4(1-3|\eta|^2) - 4\eta\left(\bar\xi+\tfrac{1}{a}\right) - 4\left|\eta+\xi+\tfrac{1}{a}\right|^2 - 4\bar\eta\left(\xi+\tfrac{1}{a}\right)\\
&\quad + \tfrac{8a^2}{1-2a^2}(1-2|\eta|^2)(1-|\xi|^2),\\
\Delta_\eta^2 Y(\eta,\xi) &= -64 - \tfrac{64a^2}{1-2a^2}(1-|\xi|^2).
\end{aligned}$$

By the definition of the conformal map h_a in (3) and from (31) we obtain that $\left|\mathbf{h}_a'(\eta)\right|^2 = |2a\eta+1|^2$, $\Delta_\eta\left|\mathbf{h}_a'(\eta)\right|^2 = 16a^2$ and

$$\begin{aligned}
(\Delta_x K)(h_a(\eta), h_a(\xi)) &= \tfrac{4}{|2a\eta+1|^2}\left((1-3|\eta|^2) - \eta\left(\bar\xi+\tfrac{1}{a}\right) - \left|\eta+\xi+\tfrac{1}{a}\right|^2\right)\\
&+ \tfrac{4}{|2a\eta+1|^2}\left(-\bar\eta\left(\xi+\tfrac{1}{a}\right) + \tfrac{2a^2}{1-2a^2}(1-2|\eta|^2)(1-|\xi|^2)\right).
\end{aligned}$$

We can compute

$$\begin{aligned}
&\sum_{i=1}^{2} \tfrac{\partial}{\partial\eta_i}\left|\mathbf{h}_a'(\eta)\right|^2 \tfrac{\partial}{\partial\eta_i}(\Delta_x K)(h_a(\eta), h_a(\xi))\\
= &-\tfrac{64a^2}{|2a\eta+1|^2}\left(-4\eta_1^2 - 4\eta_2^2 - 4\eta_1\xi_1 - \tfrac{4}{a}\eta_1 - 4\eta_2\xi_2 - \tfrac{1}{a^2} - \tfrac{4}{a}\xi_1 + 1 - |\xi|^2\right)\\
&-\tfrac{64a^2}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}(1-2|\eta|^2)(1-|\xi|^2) + \tfrac{32a^2}{|2a\eta+1|^2}\left(-8\eta_2^2 - 4\xi_2\eta_2\right)\\
&+\tfrac{16a}{|2a\eta+1|^2}\left(-8\eta_1 - 4\xi_1 - \tfrac{4}{a} - 16a\eta_1^2 - 8a\eta_1\xi_1 - 8\eta_1\right)\\
&+\tfrac{16a}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}\left(-8a\eta_1^2 - 4\eta_1 - 8a\eta_2^2\right)(1-|\xi|^2)
\end{aligned}$$

$$
\begin{aligned}
= \quad & -\tfrac{64a^2}{|2a\eta+1|^2}\left(-2\eta_1\xi_1 - 2\eta_2\xi_2 - \tfrac{1}{a}\xi_1 + 1 - |\xi|^2\right) \\
& -\tfrac{64a^2}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}\left(1+\tfrac{\eta_1}{a}\right)(1-|\xi|^2).
\end{aligned}
$$

Hence from (32) we get

$$
\begin{aligned}
& -1 - \tfrac{a^2}{1-2a^2}(1-|\xi|^2) \\
= \quad & \tfrac{a^2}{|2a\eta+1|^2}\left(-4\eta_1^2 - 4\eta_2^2 - 4\eta_1\xi_1 - \tfrac{4}{a}\eta_1 - 4\eta_2\xi_2 - \tfrac{1}{a^2} - \tfrac{4}{a}\xi_1 + 1 - |\xi|^2\right) \\
& + \tfrac{a^2}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}(1-2|\eta|^2)(1-|\xi|^2) - \tfrac{2a^2}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}\left(1+\tfrac{\eta_1}{a}\right)(1-|\xi|^2) \\
& - \tfrac{2a^2}{|2a\eta+1|^2}\left(-2\eta_1\xi_1 - 2\eta_2\xi_2 - \tfrac{1}{a}\xi_1 + 1 - |\xi|^2\right) \\
& + |\mathbf{h}_a'(\eta)|^4\left(\Delta_x^2 K\right)(h_a(\eta), h_a(\xi)),
\end{aligned}
$$

$$
\begin{aligned}
& -1 - \tfrac{a^2}{1-2a^2}(1-|\xi|^2) \\
= \quad & -\tfrac{1}{|2a\eta+1|^2}|2a\eta+1|^2 - \tfrac{4a^2}{|2a\eta+1|^2}\left(\eta_1\xi_1 + \eta_2\xi_2 + \tfrac{1}{a}\xi_1\right) + \tfrac{a^2}{|2a\eta+1|^2}(1-|\xi|^2) \\
& + \tfrac{a^2}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}(1-|\xi|^2) - \tfrac{a^2}{|2a\eta+1|^2}\tfrac{4a^2}{1-2a^2}|\eta|^2(1-|\xi|^2) \\
& - \tfrac{2a^2}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}(1-|\xi|^2) - \tfrac{2a^2}{|2a\eta+1|^2}\tfrac{2a}{1-2a^2}\eta_1(1-|\xi|^2) \\
& - \tfrac{2a^2}{|2a\eta+1|^2}(1-|\xi|^2) - \tfrac{2a^2}{|2a\eta+1|^2}\left(-2\eta_1\xi_1 - 2\eta_2\xi_2 - \tfrac{1}{a}\xi_1\right) \\
& + |\mathbf{h}_a'(\eta)|^4\left(\Delta_x^2 K\right)(h_a(\eta), h_a(\xi)),
\end{aligned}
$$

$$
\begin{aligned}
& -\tfrac{a^2}{1-2a^2}(1-|\xi|^2) \\
= \quad & -\tfrac{a^2}{|2a\eta+1|^2}(1-|\xi|^2) - \tfrac{a^2}{|2a\eta+1|^2}\tfrac{1}{1-2a^2}(1-|\xi|^2)|2a\eta+1|^2 \\
& + \tfrac{a^2}{|2a\eta+1|^2}\tfrac{1}{1-2a^2}(1-|\xi|^2) - \tfrac{a^2}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}(1-|\xi|^2) \\
& + |\mathbf{h}_a'(\eta)|^4\left(\Delta_x^2 K\right)(h_a(\eta), h_a(\xi)),
\end{aligned}
$$

$$
\begin{aligned}
0 = \quad + \quad & \tfrac{a^2}{|2a\eta+1|^2}(1-|\xi|^2)\left(-1 + \tfrac{1}{1-2a^2} - \tfrac{2a^2}{1-2a^2}\right) \\
+ \quad & |\mathbf{h}_a'(\eta)|^4\left(\Delta_x^2 K\right)(h_a(\eta), h_a(\xi)),
\end{aligned}
$$

which gives the claim. ■

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