23. (a) Show that $f(z) = \overline{z}$ is nowhere complex differentiable.
(b) Determine all the points $z \in \mathbb{C}$ such that $f(z) = z^3 + |z|^2$ is complex differentiable.

24. Let $D \subset \mathbb{C}$ be an open set and $f : \mathbb{D} \to \mathbb{C}$ holomorphic on $D$. Define for $(x, y)$ satisfying $x + iy \in D$ $u(x, y) = \text{Re}(f(x + iy))$ and $v(x, y) = \text{Im}(f(x + iy))$.

(a) If $u$ and $v$ are twice continuously partially differentiable on $D$, then the identity $u_{xx} + u_{yy} = 0$ is satisfied. Functions that satisfy this identity are called harmonic.
(b) For the sake of simplicity we will assume in this exercise that $D = \mathbb{C}$. Define $g(r, \theta) := u(re^{i\theta})$ and $h(r, \theta) = v(re^{i\theta})$ for $r > 0$ and $\theta \in \mathbb{R}$. Show that the following version of the Cauchy-Riemann-equation holds:
$$g_r = \frac{1}{r} h_\theta, \quad h_r = -\frac{1}{r} g_\theta.$$ 
(c) Let $G \subset \mathbb{C}$ be a domain, such that $|f(z)| = \text{const}$ for each $z \in G$. Show: $f$ is constant on $G$.

25. (a) Compute for $r > 0$
$$\int_{|z|=r} \overline{z} \, dz, \quad \int_{|z|=r} \frac{1}{z^2} \, dz.$$ 
(b) Let $E$ be an ellipse, the semiminor $a$ which has length $a$ and the semimajor of which has length $b$. Let $\gamma_E$ be the curve that runs counterclockwise around $E$. Compute (just using the parametrization)
$$\int_{\gamma_E} \frac{1}{z} \, dz.$$ 
(c) This task justifies the fact that the velocity in which we run through the curve is an invariant of the curve integral. Let $\phi : [0, 1] \to [0, 1]$ be a continuously differentiable bijection fulfilling $\phi'(x) > 0$ for each $x \in (0, 1)$ and $\gamma : [0, 1] \to \mathbb{C}$ a continuously differentiable curve. Define $\tilde{\gamma}(t) := \gamma(\phi(t))$ for $t \in [0, 1]$. Show that for every function $f : \mathbb{C} \to \mathbb{C}$ it holds that
$$\int_{\tilde{\gamma}} f \, dz = \int_{\gamma} f \, dz.$$ 
(d) Show for arbitrary $x_0 \in \mathbb{R}, \ |x_0| < 1$
$$\int_{|z|=1} \frac{z - x_0}{z} \, dz = \int_{|z|=1} \frac{1}{1 - x_0 z} \, dz.$$ 
(e) Let $f : \mathbb{C} \to \mathbb{C}$ be continuous such that the is a complex differentiable $g$ satisfying $g' = f$. Show: In this case we find that
$$\int_{\gamma} f \, dz = 0$$
for each closed curve $\gamma : [a, b] \to \mathbb{C}$. 
(f) For $R > 2$ let $\gamma_R$ be the closed curve, that runs along a straight line from $-R + 0i$ to $R + 0i$, and goes back on a half circle that lies above the $x$- axis. Show that for even $n \in \mathbb{N}, n \geq 2$ we have the approximation
$$\int_{-\infty}^{\infty} \frac{e^{-z}}{1 + x^n} \, dz = \lim_{R \to \infty} \int_{\gamma_R} \frac{e^{-z}}{1 + x^n} \, dz.$$ 
Extra credit: Find a family of closed curves $\gamma_R$ such that
$$\int_{-\infty}^{\infty} \frac{e^{z}}{1 + x^n} \, dz = \lim_{R \to \infty} \int_{\gamma_R} \frac{e^{z}}{1 + x^n} \, dz.$$
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