Exercise Sheet 3 – Analysis III

(Homework solutions will be handed in and discussed at 10:00, 05.12, H12)

1. Show that

(a) \( M = \{ (3x, -x^2) , x \in \mathbb{R} \} \) is a 1-dimensional \( C^\infty \)-submanifold (smooth submanifold) of \( \mathbb{R}^2 \).

(b) if \( M \subset \mathbb{R}^n \) is open, then \( M \) is an \( n \)-dimensional smooth submanifold of \( \mathbb{R}^n \).

(c) if \( M = M_1 \times M_2 \) where \( M_i \) is a \( k_i \)-dimensional submanifold of \( \mathbb{R}^{n_i} \), \( n_1 + n_2 = n \), then \( M \) is \( (k_1 + k_2) \)-dimensional submanifold of \( \mathbb{R}^n \).

(e) The set \( SL(n, \mathbb{R}) := \{ A \in M_{n \times n}(\mathbb{R}) \cong \mathbb{R}^{n^2} : \det A = 1 \} \) is a \( (n^2 - 1) \)-dimensional smooth submanifold of \( \mathbb{R}^{n^2} \).

2. Let \( M \) be a \( m \)-dimensional \( C^l \)-submanifold of \( \mathbb{R}^n \) and \( a \in M \). Suppose that \( F : U \to V \cap M \) is a \( C^l \)-homeomorphism such that \( F(0) = a \) and \( \text{rank } DF(0) = m \) where \( V \) is an open neighborhood of \( a \) and \( U \) is an open neighborhood of \( 0 \) in \( \mathbb{R}^m \). Prove that there is an open neighborhood \( a \in V' \subset V \cap M \) s.t. \( \text{rank } DF(u) = m \) for all \( u \in U' = F^{-1}(V') \).

3. Consider a torus \( T \subset \mathbb{R}^3 \) centered at \( 0 \) with radii \( r \) and \( R \), \( 0 < r < R \) (resp. the radius of the tube and the distance from the center of the tube to the center of the torus):

\[
T = \left\{ (x, y, z) \in \mathbb{R}^3 : \left( R - \sqrt{x^2 + y^2} \right)^2 + z^2 = r^2 \right\}
\]

(a) Show that \( T \) is a 2-dimensional smooth submanifold of \( \mathbb{R}^3 \).

(b) Compute the tangent space and the normal space of \( T \) at \( p = (R, 0, r) \).

4. Let \( \mathbb{S}^n \) be a unit sphere in \( \mathbb{R}^{n+1} \).

(a) Find tangent space at any point \( a \in \mathbb{S}^n \).

Let \( N, S \) be the north and resp. south poles of \( S^n \), i.e. \( N = (0, ..., 0, 1) \), \( S = (0, ..., 0, -1) \). Consider the function \( \pi_N \), called the stereographic projection from \( N \), defined by

\[
\pi_N : \mathbb{S}^n \setminus \{ N \} \to \mathbb{R}^n
\]

which maps the point \( P \in \mathbb{S}^n \setminus \{ N \} \) to the intersection of the line \( NP \) with the hyperplane \( \{ x \in \mathbb{R}^{n+1} : x_{n+1} = 0 \} \cong \mathbb{R}^n \). The stereographic projection \( \pi_S \) from \( S \) is defined similarly.

(b) Determine \( \pi_N(x) \) for \( x \in \mathbb{S}^n \setminus \{ N \} \) and \( \pi_S(x) \) for \( x \in \mathbb{S}^n \setminus \{ S \} \)

(c) Show that \( \pi_N, \pi_S \) are bijective and the inverse maps \( \pi_N^{-1}, \pi_S^{-1} \) are smooth charts of \( \mathbb{S}^n \).

5. Let \( m \leq n \), \( \Omega \subset \mathbb{R}^m \) be open and \( \varphi : \Omega \to \mathbb{R}^n \) be an immersion of class \( C^l \) (i.e. \( \varphi \) is \( C^l \)-differentiable and \( \text{rank } D\varphi(x) = m \) for all \( x \in \Omega \)). Show that for every \( a \in \Omega \), there exists an open neighborhood \( a \in U \subset \Omega \) such that \( \varphi(U) \) is an \( m \)-dimensional \( C^l \)-submanifold and \( \varphi : U \to \varphi(U) \) is a \( C^l \)-homeomorphism.

http://www.uni-ulm.de/mawi/analysis/lehre/veranstaltungen/ws20160/analysis-3/