Exercise Sheet 5 – Analysis III

(Homework solutions will be handed in and discussed at 10:00, 16.01, H12)

0. (i) In this sheet, \( G \) denotes a nonempty bounded domain with \( C^1 \)-boundary, \( G \subset \mathbb{R}^n \), and \( \nu = (\nu^1, ..., \nu^n) \) denotes the outward unit normal vector, \( \nu = (\nu^1, ..., \nu^n) : \partial G \to \mathbb{S}^{n-1} \).

(ii) The Laplace operator \( \triangle \) is defined by the map

\[
\triangle : C^2(G) \to C(G), \quad f \mapsto \triangle f := \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}.
\]

1. Show that

(a) For \( f, g \in C^1(\overline{G}) \), we have

\[
\int_G f(x) \frac{\partial g}{\partial x_i}(x) \, dx = \int_{\partial G} f(x) g(x) \nu^i(x) \, dx - \int_G g(x) \frac{\partial f}{\partial x_i}(x) \, dx, \quad i = 1, ..., n \quad (1)
\]

which can be written as follows:

\[
\int_G f \nabla g \, dx = \int_{\partial G} fg \nu \, ds - \int_G g \nabla f \, dx.
\]

(b) For \( f \in C^1(\overline{G}) \) and \( g \in C^2(\overline{G}) \), we have

\[
\int_G (f(x) \triangle g(x) + \langle \nabla f(x), \nabla g(x) \rangle) \, dx = \int_{\partial G} f(x) \frac{\partial g}{\partial \nu}(x) \, ds(x), \quad (2)
\]

where \( \frac{\partial g}{\partial \nu} \) is the directional derivative of \( g \) in the direction of the vector \( \nu \), that is

\[
\frac{\partial g}{\partial \nu} = \sum_{i=1}^{n} \nu^i \frac{\partial g}{\partial x_i}.
\]

The equality (2) is called Green’s first identity.

(c) For \( f, g \in C^2(\overline{G}) \), we have

\[
\int_G (g(x) \triangle f(x) - f(x) \triangle g(x)) \, dx = \int_{\partial G} \left( g(x) \frac{\partial f}{\partial \nu}(x) - f(x) \frac{\partial g}{\partial \nu}(x) \right) \, ds(x).
\]

This equality is called Green’s second identity.

2. Prove the Poincaré inequality:

There exists a positive constant \( C \) depending only on \( n \) and \( \text{diam}(G) := \sup \{|x - y| : x, y \in G\} \) such that for any function \( u \in C^1(\overline{G}) \) satisfying \( u = 0 \) on \( \partial G \), we have

\[
\|u\|_{L^2(G)} := \left( \int_G |u|^2 \, dx \right)^{1/2} \leq C \left( \int_G |\nabla u|^2 \, dx \right)^{1/2} =: C \|\nabla u\|_{L^2(G)}.
\]

\( \text{Hint: Using (1) and the Cauchy-Schwarz inequality with the remark that} \)

\[
\int_G |u|^2 \, dx = \frac{1}{n} \int_G u^2 \text{div} \varphi \, dx \text{ where } \varphi \text{ is the identity on } \overline{G}.
\]

3. Let \( \lambda \in \mathbb{R} \).

(a) Prove that if the following partial differential equation

\[
\begin{cases}
-\triangle u = \lambda u & \text{in } G, \\
u = 0 & \text{on } \partial G,
\end{cases}
\]  

admits a nontrivial solution \( u \in C^2(\overline{G}) \), then \( \lambda > 0 \).

\( \text{Hint: Consider } \lambda \int_G u^2 \, dx \text{ and use (2).} \)
(b) Is the above statement in (a) still true for the following problem?
\[
\begin{align*}
-\Delta u &= \lambda u \quad \text{in } G, \\
\frac{\partial u}{\partial v} &= 0 \quad \text{on } \partial G.
\end{align*}
\]

(c) Let \( f : G \rightarrow \mathbb{R} \) and \( g : \partial G \rightarrow \mathbb{R} \). Show that if the problem
\[
\begin{align*}
-\Delta u &= f \quad \text{in } G, \\
u &= g \quad \text{on } \partial G,
\end{align*}
\]
has a solution \( u \in C^2(G) \), then \( u \) is unique.

(d*) Set \( \sigma (-\Delta) := \{ \lambda \in \mathbb{R} : \text{the problem (3) has a nontrivial solution } u \in C^2(G) \} \). Prove that \( \inf \sigma (-\Delta) > 0 \).

4. Let \( a, b \in \mathbb{R} \) and \( 0 < a < b \). On the space \( C([a, b] \), \( \mathbb{C}) \) of continuous complex-valued functions on \([a, b] \), define the map \( \langle \cdot, \cdot \rangle \) by
\[
\langle f, g \rangle := \int_a^b f(x) \overline{g(x)} \, dx \quad \text{for } f, g \in C([a, b] \), \( \mathbb{C}) \).
\]
Prove that

(a) \( \langle \cdot, \cdot \rangle \) is a scalar product on \( C([a, b] \), \( \mathbb{C}) \).

(b) \( \{ x \mapsto \varphi_k(x) = c_k e^{ikx} \}_{k \in \mathbb{Z}} \) is an orthonormal sequence on the space \( C([0, 2\pi] \), \( \mathbb{C}) \), \( \langle \cdot, \cdot \rangle \) for some suitable constant \( c_k \in \mathbb{C} \). Determine \( c_k \).

(c) If \( f = \sum_{k=-n}^{n} \alpha_k \varphi_k \) for \( \alpha_k \in \mathbb{C} \) \( \forall k \), then there exist complex numbers \( a_0, a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) such that for all \( x \in [0, 2\pi] \),
\[
f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos (kx) + b_k \sin (kx).
\]

Frohe Festtage!

und

Alles Gute zum neuen Jahr!