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Graphentheorie

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Graph theory is unanimously given a precise birthday: the solution to a then-famous problem concerning the traversability of seven bridges in the town of Königsberg in Eastern Prussia (now Kaliningrad, Russia).

Problem 0.1. *Is there a way to traverse all bridges of Königsberg in a single trip, without doubling back, in such a way that the trip ends in the same place it began?*

The solution to this problem was obtained by L. Euler in 1736. It is in fact the first proof of a graph theoretical theorem. Ninety years later, interesting advances in graph theory were made by G. Kirchhoff. He recognized that electrical circuits can be represented as graphs and introduced methods from linear algebra in order to derive electrodynamical results.

Both the theory of graphs and its applications have been much developed ever since. As one can expect from a field which is almost 300 years old, it is absolutely impossible to give even a mere hint of this rich theory in a lecture course spanning over fifteen weeks. The topic of the *history* of graph theory will be avoided: the interested reader is referred, e.g., to [**11**, § 1.3].

A nice, modern, and reasonably complete treatment can be found in several books, including those mentioned listed in the final references – though not all results, not even all main ones, appear in each of these books. This is in sharp contrast to the case of – say – mathematical analysis, whose introductory expositions typically include standard results.

In fact, even in a course of elementary graph theory, designing a path in accordance with one's taste is possible and necessary. Still this manuscript, which has been developed for the course of Graphenthorie held in the Winter Term 2008/09 at the Universität Ulm, is influenced by some constraints.

It is quite unlikely that this pages will be free of any mistake. If you spot a typo or an error, please do not hesitate and let me know by sending me an e-mail.

Ulm, February 3, 2011.

Contents

CHAPTER 1

Basic notions

Definition 1.1. *A **graph** is a triple* $\mathsf{G} := (\mathsf{V}, \mathsf{E}, g)$*, where* V, E *are sets and* $g : \mathsf{E} \to [\mathsf{V}]_2$ *a mapping. Here* $[\mathsf{V}]_2 \subset \mathcal{P}(\mathsf{V})$ *denotes the set of all either 1-element or 2-element subsets of* V*. The elements of* V *are called **nodes** or **vertices**, the elements of* E ***edges**. Occasionally we will denote* $\mathsf{V} = \mathsf{V}(\mathsf{G})$ *and* $\mathsf{E} = \mathsf{E}(\mathsf{G})$ *if there is ambiguity. (A graph with* $|\mathsf{V}| \leq 1$ *and* $|\mathsf{E}| = 0$ *is called **trivial**).*

The edges $e \in \mathsf{E}$ *such that* $g(e) = \{v\}$ *is a 1-element subset of* V *are called **loops** (**around** v). If* $g(e)$ *is a 2-element subset of* V*, say* $g(e) = \{v, w\}$*, then* v, w *are called **endpoints** of* e*, and* e *is said to be **incident** in* v *and in* w*. In particular,* v, w *are said to be **adjacent**. If* $g(e') = \{v, w'\}$ *for a further edge* e'*, then the edges* e, e' *are called **adjacent**, too. If more than one edge with same endpoints* v, w *exist, they are called **multiple edges** between* v *and* w*.*

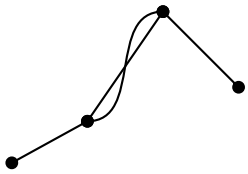
*A graph without loops and multiple edges is called a **simple graph**. In this case,* g *is injective and* E *can in fact be identified with a symmetric subset of* $\mathsf{V} \times \mathsf{V}$*: the edge whose endpoints are* v, w *is usually denoted by* (v, w) *or* vw*.*

The very word "graph" has been first introduced by J.J. Sylvester in 1878 in [**18**]. This and other historical curiosities about graph theory are contained in [**24**].

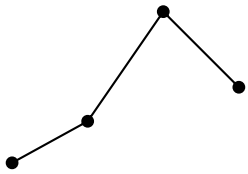
If V' is a node set, $N_G(\mathsf{V}')$ the **neighbourhood** in G of a node set V', i.e., the set of all nodes in $\mathsf{V} \setminus \mathsf{V}'$ that are adjacent to at least one node of V'.

Observe that, by definition of simple graph (and in fact of graph), we do not distinguish between (v, w) and (w, v), resp. vw and wv.

Most usually, in order to represent a graph graphically one draws a dot for each node in V and link two given dots with a curve whenever an edge with these endpoints exist.



A graph.



A simple graph.

Example 1.2. *Also* $(\mathrm{R}, C[0,1], g)$ *is a graph, where*

$$g(f) := \left(\min_{0 \leq x \leq 1} f(x), \max_{0 \leq x \leq 1} f(x) \right).$$

Of course, each two nodes are adjacent.

Remark 1.3. *It should be said that in the literature several different definitions appear. In particular, some authors call* graph *what we have called a* simple graph*. In order to*

help consulting manuals, we list in the following the basic convention for the books in the bibliography.

graphs are canonically simple	*graphs may have loops and multiple edges*
[**2, 7, 8, 13**]	[**20, 22**]

If we focus on the *connectivity* of $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$, i.e., if the relevant issue is *which pairs of nodes are connected by an edge*, then the structure of G is accurately described by the $|\mathsf{V}| \times |\mathsf{V}|$ **adjacency matrix** $\mathcal{A} = (\alpha_{ij})$, where each entry $\alpha_{ij} \in \mathrm{N}_0$ gives the number of edges whose endpoints are v_i, v_j (by convention, $a_{ii} = 2$ if there is a loop around v_i).

Remark 1.4. *Observe that, by definition, the square matrix $\mathcal{A}$ is symmetric and that it consists of 0s and 1s only (all 0s on the diagonal) if the graph is simple. In particular, it has only real eigenvalues. We will come back to this issue in Chapter 12.*

If we are rather interested in the relation between edges $e_1, \ldots, e_{|\mathsf{E}|}$ and nodes $v_1, \ldots, v_{|\mathsf{V}|}$, then a more appropriate description is provided by the $|\mathsf{V}| \times |\mathsf{E}|$ **incidence matrix** $\mathcal{I} = (\iota_{ij})$, where

$$\iota_{ij} := \begin{cases} 1 & \text{if } v_i \text{ is endpoint of } e_j, \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.5. *Molecules are perhaps the most ubiquitous natural structures that can be modelled as a graphs. Typically, chemical bonds are responsible for the attractive interactions between atoms. Thus, we can see bonds as edges and atoms as nodes of a graph. This has certainly been observed long time ago: already in the 1930s Nobel laureate Linus Pauling discussed interesting relations between the chemical properties of aromatic hydrocarbon molecules and the eigenvalues of the adjacency matrix of the associated graph, see* [**7**, Chapt. 8] *for an account of graph theoretical applications to chemistry.*

Example 1.6. *An elementary example of a (simple, infinite) graph is given by $\mathsf{V} = \mathrm{N}$ and $\mathsf{E} := \{(n, n+1) : n \in \mathrm{N}\}$. Its adjacency matrix is*

$$\begin{pmatrix} 0 & 1 & 0 & \ldots \\ 1 & 0 & 1 & \ddots \\ 0 & 1 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

and its incidence matrix is

$$\begin{pmatrix} 1 & 0 & \ldots & \\ 1 & 1 & \ddots & \\ 0 & 1 & \ddots & \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Remark 1.7. *It is possible to generalize the notion of a (multi)graph in order to take into account some constraints that may be, e.g., of geometric or economic nature. More precisely, a* ***weighted graph*** *is a pair (G, ρ), where $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ is a graph and $\rho : \mathsf{E} \to \mathrm{R}$ a mapping. For example, if V, V' are identified with points of R^3, then ρ may describe the euclidean distance of two nodes.*

For the sake of simplicity, in the following we will usually (that is, unless we say otherwise) consider finite sets of nodes, and hence finite graphs.

Definition 1.8. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *and* $\mathsf{G}' = (\mathsf{V}', \mathsf{E}', g')$ *be graphs. If there exist two mappings* $\beta : \mathsf{V} \to \mathsf{V}'$ *and* $B : \mathsf{E} \to \mathsf{E}'$ *such that*

$$g(e) = \{v, w\} \text{ implies } g'(B(e)) = \{\beta(v), \beta(w)\},$$

then G, G' *are called* ***homomorphic****. If both* β *and* B *are bijective, the graphs are called* ***isomorphic****.*

In other words, adjacent nodes are mapped by β into adjacent nodes, and an edge e incident in a node v is mapped into an edge $B(e)$ incident in $\beta(v)$.

Remark 1.9. *If in particular we are considering a simple graph, so that an edge is completely determined by its endpoints, then the mapping* B *is completely determined by* β*. In fact, two simple graphs* G, G' *are isomorphic if and only if there is a bijection* $\beta : \mathsf{V} \to \mathsf{V}'$ *such that* $(v, w) \in \mathsf{E}$ *implies* $(\beta(v), \beta(w)) \in \mathsf{E}$*. In particular, a* ***graph automorphism****, i.e., an isomorphism of a graph onto itself, is fully determined by a bijection of* V *onto* V *(i.e., by a permutation of the nodes) respecting adjacence. Clearly, graph automorphisms define a group, the so-called* ***graph automorphism group*** $\mathrm{Aut}(\mathsf{G})$*.*

Example 1.10. *Following Cantor's proof of countability of the rational numbers, one can construct a graph with node set* $\mathbb{Q}$ *that is isomorphic to that introduced in Example 1.6.*

Observe that graph isomorphy defines an equivalence relation.

Definition 1.11. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph. Then the number of edges incident in a given node* v *is called* ***degree*** *(or sometimes* ***node degree****) of* v *and is denoted by* $d(v)$*, or sometimes by* $d(v, \mathsf{G})$*. (Here we adopt the convention that a loop around* v *incides in* v *twice). Moreover, the* ***total degree*** *of* G *is* $d(\mathsf{G}) := \sum_{v \in \mathsf{V}} d(v, \mathsf{G})$*. We also introduce the* ***average degree*** $d_a(\mathsf{G}) := \frac{d(\mathsf{G})}{|\mathsf{V}|}$ *of* G*, where* $|\mathsf{V}|$ *denotes as usual the cardinality of* V*.*

We further denote by $\delta(\mathsf{G}) := \inf_{v \in \mathsf{V}} d(v, \mathsf{G})$ *and* $\Delta(\mathsf{G}) := \sup_{v \in \mathsf{V}} d(v, \mathsf{G})$ *the* ***minimal*** *and* ***maximal degree*** *of* G*, respectively.*

Exercise 1.12. *Show that node degree is invariant under isomorphism of graphs, but not under an arbitrary homomorphism.*

Lemma 1.13. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}.g)$ *be a graph. Then the total degree* $d(\mathsf{G})$ *is given by* $2|\mathsf{E}|$*. Furthermore, the number of nodes with odd degree is even.*

Proof. Each edge e has two endpoints, hence the contribution of e to $d(\mathsf{G})$ is 2. In other words, $2|\mathsf{E}| = d(\mathsf{G})$. Accordingly

$$2|\mathsf{E}| - \sum_{\{v : d(v,\mathsf{G}) \text{ even}\}} d(v, \mathsf{G}) = \sum_{\{v : d(v,\mathsf{G}) \text{ odd}\}} d(v, \mathsf{G})$$

is an even number, which yields the claim. □

In the following we denote by $\mathsf{V}_{(i)}$ the set of all nodes of degree i of a graph G.

Exercise 1.14. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph. Prove the formula*

$$2|\mathsf{V}_{(0)}| + |\mathsf{V}_{(1)}| + 2(|\mathsf{E}| - |\mathsf{V}|) = \sum_{i=3}^{\Delta(\mathsf{G})} (i-2)|\mathsf{V}_{(i)}|.$$

Remark 1.15. *Consider the adjacency and incidence matrix of a graph* $\mathsf{G} = (\{v_1, \ldots, v_n\}, \{e_1, \ldots, e_m\}, g)$. *Then*

(i) $\sum_{j=1}^{n} \alpha_{ij} = \sum_{j=1}^{n} \alpha_{ji} = d(v_i, \mathsf{G})$,
(ii) $\sum_{j=1}^{m} \iota_{ij} = d(v_i, \mathsf{G})$, *and*
(iii) $\sum_{i=1}^{m} \iota_{ij} = 2$ *for all* $j = 1, \ldots, n$.

Remark 1.16. *For any graph* G *there holds* $\delta(\mathsf{G}) \leq d_a(\mathsf{G}) \leq \Delta(\mathsf{G})$.

Definition 1.17. *A graph* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *is called* r-***regular*** *if* $\delta(\mathsf{G}) = \Delta(\mathsf{G}) = r$. *A simple graph* G *is called* ***complete*** *if any two of its nodes are adjacent – or equivalently, if it is* $(|\mathsf{V}| - 1)$*-regular. In this case, it is denoted by* $\mathsf{K}^{|\mathsf{V}|}$.

Remark 1.18. *A* 1*-regular graph is a graph consisting of* $2n$ *nodes, each of which is only adjacent to one further node – i.e., of* n *pairwise non-adjacent edges. A* 2*-regular graph is called a* ***polygon*** *(depending on* $|\mathsf{V}|$*, a* ***triangle****, a* ***square****, a* ***pentagon****...). A* 3*-regular graph is called* ***cubic****.*

Corollary 1.19. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a* r*-regular graph. Then* r *or* $|\mathsf{V}|$ *has to be an even number.*

Proof. By Lemma 1.13, the average degree is given by $\frac{2|\mathsf{E}|}{|\mathsf{V}|}$, hence if G is r-regular there holds $r = \delta(\mathsf{G}) = \Delta(\mathsf{G}) = d_a(\mathsf{G})$, i.e., the product $|\mathsf{V}|r$ has to be even because $|\mathsf{V}|r = 2|\mathsf{E}|$. □

Remark 1.20. *Since each of the* n *nodes is linked to* $n-1$ *further nodes, the complete graph* K^n *has* $\frac{n(n-1)}{2} = \binom{n}{2}$ *edges, where the factor* $\frac{1}{2}$ *is necessary in order to avoid to count the edge* $vw = wv$ *twice. Accordingly, since each simple graph with* n *nodes has at most as many edges as* K^n*, the upper bound on the edge set cardinality of a simple graph with* n *nodes is* $\binom{n}{2}$.

Another way to see this is to observe that since the k*-element subsets of a* n*-element set are exactly* $\binom{n}{k}$*, a simple graph with* $|\mathsf{V}|$ *nodes (such that* $\mathsf{E} \subset [\mathsf{V}]_2$*) has at most* $\binom{|\mathsf{V}|}{2}$ *edges.*

Example 1.21. *Let* $\mathsf{V} := \{$*Teams of the Bundesliga*$\}$ *and* $\mathsf{E} := \{$*Matches of the latter half of the Bundesliga season*$\}$*, with* g *mapping each game into its two competitors. This defines a graph, which is in fact simple – as it does neither contain any loops (a team cannot play against itself) nor multiple edges (each team plays against any other at most once, in the latter half of the Bundesliga). Since each team plays against any other* exactly *once in the latter half of the Bundesliga, this graph is in fact complete, i.e., it consists of*

$$\binom{|\mathsf{V}|}{2}$$

matches.

Remark 1.22. *We have already emphasized that, by definition, in a graph there is no difference between the endpoints of an edge. If we wish to discern the role of an initial and a terminal endpoint, the notion of* directed graph *has to be introduced, see Chapter 4.*

There is a rich theory about directed graphs, too. Therefore, it often depends on concrete applications whether one prefers to discuss a directed or an undirected graph. For instance, there is no natural direction in a molecule's atomic bonds (Example 1.5), but it is natural to distinguish between home *and* away *games of the Bundesliga (Example 1.21).*

Example 1.23. *Let G be a finite group and S a set of generators of G such that $\{s^{-1} : s \in S\} = S$. Assume the identity of G* not *to belong to S. Then the* ***Cayley graph*** *associated with G, S is the directed graph whose node set is G and whose edges are given by (s, t), whenever $t^{-1}s \in S$.*

Exercise 1.24. *Determine the Cayley graph of the finite additive group $G = (\mathrm{Z}_3, +)$ with $S = \{1, 2\}$.*

Example 1.25. *Consider the pages in the World Wide Web as nodes and the links from page v to page w as edges directed from v to w. This defines a graph which is directed.*

Definition 1.26. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g), \mathsf{G}' = (\mathsf{V}', \mathsf{E}', g')$ be graphs.*

The graph G' is said to be ***contained*** *in G, or it is called* ***subgraph*** *of G, if $\mathsf{V}' \subset \mathsf{V}$, $\mathsf{E}' \subset \mathsf{E}$, and $g'(e) = g(e)$ for all $e \in \mathsf{E}'$.*

If $\tilde{\mathsf{V}} \subset \mathsf{V}$, then the subgraph ***induced by*** *$\tilde{\mathsf{V}}$, denoted by $\mathsf{G}[\tilde{\mathsf{V}}]$, is the graph that has node set $\tilde{\mathsf{V}}$, edge set $\tilde{\mathsf{E}} := \{e \in \mathsf{E} : g(e) \in [\tilde{\mathsf{V}}]_2\}$, and $\tilde{g}(e) = g(e)$ for all $e \in \tilde{\mathsf{E}}$.*

Definition 1.27. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g), \mathsf{G}' = (\mathsf{V}, \mathsf{E}, g)$ be graphs.*

Let G, G' satisfy $g(e) = g'(e)$ for all $e \in \mathsf{E} \cap \mathsf{E}'$. The ***union*** *of G, G' is defined as the graph $\mathsf{G} \cup \mathsf{G}' := (\mathsf{V} \cup \mathsf{V}', \mathsf{E} \cup \mathsf{E}', g_\cup)$, where $g_\cup$ is the natural extension of g and g' to $\mathsf{E} \cup \mathsf{E}'$. The* ***intersection*** *of G, G' is defined as the graph $\mathsf{G} \cap \mathsf{G}' := (\mathsf{V} \cap \mathsf{V}', \mathsf{E} \cap \mathsf{E}', g_\cap)$, where $g_\cap(e) := g(e) = g'(e)$ for all $e \in \mathsf{E} \cap \mathsf{E}'$.*

Let G, G' be disjoint. The ***product*** *$\mathsf{G} * \mathsf{G}'$ is defined as the graph with node set $\mathsf{V} \cup \mathsf{V}'$ and edge set $\mathsf{E} \cup \mathsf{E}'$ and such that any node of V is adjacent to any node of V'.*

Let $\mathsf{U} \subset \mathsf{V}$. We define the ***difference*** *$\mathsf{G} - \mathsf{U}$ as the graph with node set $\mathsf{V} \setminus \mathsf{U}$ and edge set which agrees with E up to those edges incident in any element of U.*

Let $\mathsf{F} \subset \mathsf{E}$. We define the ***difference*** *$\mathsf{G} - \mathsf{F}$ as the graph with node set V and edge set $\mathsf{E} \setminus \mathsf{F}$. By* ***complement*** *of G we mean the graph $\mathsf{G}^C := \mathsf{K}^{|\mathsf{V}|} - \mathsf{E}$.*

Let $\mathsf{G}' = (\mathsf{V}, \mathsf{E}', g')$ be another graph with same node set and such that $g'(e) = g(e)$ for all $e \in \mathsf{E} \cap \mathsf{E}'$. Consider the graph $\tilde{\mathsf{G}} := (\mathsf{V}, \mathsf{E} \cup \mathsf{E}', \tilde{g})$ for a mapping $\tilde{g} : \mathsf{E} \cup \mathsf{E}' \to \mathsf{V}$ such that $\tilde{g}(e) = g(e)$ if $e \in \mathsf{E}$ and $\tilde{g}(e) = g'(e)$ if $e \in \mathsf{E}'$. Then $\tilde{\mathsf{G}}$ is called a ***sum*** *and is denoted by $\mathsf{G} + \mathsf{G}'$. With an abuse of notation we sometimes write only $\mathsf{G} + \mathsf{E}'$.*

Exercise 1.28. *Are the product and the sum of graphs commutative? Are they associative?*

Exercise 1.29. *Prove that the automorphism group of a graph G coincides with the automorphism group of its complement G^C.*

Proposition 1.30. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph. If* $|\mathsf{E}| \geq 1$*, then there exists a subgraph* $\tilde{\mathsf{G}} = (\tilde{\mathsf{V}}, \tilde{\mathsf{E}}, \tilde{g})$ *such that*

$$2\delta(\tilde{\mathsf{G}}) > d_a(\tilde{\mathsf{G}}) \geq d_a(\mathsf{G}). \tag{1.1}$$

To fix the ideas, consider the simple case of $\mathsf{V} = \{v_1, v_2, v_3\}$ with v_1 adjacent to v_2 and $d(v_3, \mathsf{G}) = 0$. Then the graph $\tilde{\mathsf{G}}$ we are seeking for is obtained by deleting v_3, the node with lowest degree. More generally, the basic goal is to construct a subgraph containing as less nodes with as higher degree as possible.

Proof. Let $\mathsf{G}_0 := \mathsf{G}$. If G_0 does not contains any node v_0 such that

$$2d(v_0, \mathsf{G}) \leq d_a(\mathsf{G}_0),$$

i.e., each node v satisfies $2d(v, \mathsf{G}) > d_a(\mathsf{G}_0)$, then we are done, since already G satisfies the required condition.

Otherwise, set $\mathsf{G}_1 := \mathsf{G}_0 - \{v_0\}$, the subgraph obtained deleting v_0 and all the edges incident in v_0. Recursively, construct a sequence $\mathsf{G}_0, \mathsf{G}_1, \mathsf{G}_2, \ldots$ as in the first step: if no node v_i of G_i satisfies

$$2d(v_i, \mathsf{G}) \leq d_a(\mathsf{G}_i),$$

stop the process and set $\tilde{\mathsf{G}} := \mathsf{G}_i$. Otherwise, continue by setting $\mathsf{G}_{i+1} := \mathsf{G}_i - \{v_i\}$, so that the first inequality is satisfied at any step of the process. Observe that this recursive process will stop at some point, since the graph is assumed to be finite. Let us denote by i_0 the stopping step.

We still have to check the second inequality of (1.1). Observe that with v_i we are deleting at most $\frac{1}{2}d_a(\mathsf{G}_i) = \frac{d(\mathsf{G}_i)}{2|\mathsf{V}_i|}$ edges, since by assumption this is the highest possible degree of v_i, i.e.,

$$d(\mathsf{G}_i) - d(\mathsf{G}_{i+1}) = 2(|\mathsf{E}_i| - |\mathsf{E}_{i+1}|) \leq \frac{d(\mathsf{G}_i)}{2|\mathsf{V}_i|}.$$

Accordingly,

$$d(\mathsf{G}_i)|\mathsf{V}_{i+1}| = d(\mathsf{G}_i)(|\mathsf{V}_i| - 1) \leq d(\mathsf{G}_{i+1})|\mathsf{V}_i| \qquad \text{for all } i,$$

and in particular

$$d_a(\tilde{\mathsf{G}} = \frac{d(\mathsf{G}_{i_0})}{|\mathsf{V}_{i_0}|} \geq \frac{d(\mathsf{G}_0)}{|\mathsf{V}_0|} = d_a(\mathsf{G}).$$

This concludes the proof. □

Remark 1.31. *Observe that the final graph* $\tilde{\mathsf{G}}$ *cannot either be empty nor consist of a unique node, since this would imply that* $0 > 0$*,*

CHAPTER 2

Paths and cycles

The following intuitive definition could be made more precise by introducing suitable homomorphisms, see [**22**, § 1.2].

Definition 2.1. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph.*

(1) Two sequences $(v_0, v_1, \dots, v_n)$ *and* $(e_1, \dots, e_n)$ *are said to define an* ***edge sequence*** *if*
  (i) $v_i \in \mathsf{V}$ *for all* $i = 0, 1, \dots, n$,
  (ii) $e_i \in \mathsf{E}$ *for all* $i = 1, \dots, n$, *and*
  (iii) $g(e_i) = \{v_{i-1}, v_i\}$ *for all* $i = 1, \dots, n$.

(2) An edge sequence such that the edges $e_1, \dots, e_n$ *are pairwise different is called a* ***walk****.*

(3) A walk such that the nodes $v_0, v_1, \dots, v_n$ *are pairwise different is called a* ***path of length*** n ***with endpoints*** v_0, v_n *and is usually denoted by* P^n*.*

(4) A walk such that the nodes $v_0, v_1, \dots, v_{n-1}$ *are pairwise different but* $v_0 = v_n$ *is called a* ***cycle of length*** n *and is usually denoted by* C^n*.*

(5) An edge e *which is not contained in any cycle of* G *is called a* ***bridge****.*

By convention, a node v defines both a path and a cycle (without edges, hence of length 0) with same endpoints (v itself).

Remarks 2.2. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a simple graph.*

(1) Unlike in a general graph, a cycle in G *has necessarily length at least 3.*

(2) Since an edge of G *is uniquely determined by the adjacent nodes it connects, we will denote any edge sequence, and in particular any path and any cycle, simply by a/the sequence of nodes it contains, say* $(v_0, v_1, \dots, v_n)$ *or* $(v_0, v_1, \dots, v_n, v_0)$*.*

If a triangle is subgraph of G, then by definition G contains a cycle of length 3. This of course need not be the case – in fact, a graph need not contain any cycle, as one can see by considering any finite subgraph of the graph considered in Example 1.6. Still, the following holds.

Proposition 2.3. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph with* $\delta(\mathsf{G}) \geq 2$*. Then* G *contains a cycle.*

This motivates to introduce the notation $\zeta(\mathsf{G})$ for the number of cycles contained as subgraphs in G.

Proof. One can assume without loss of generality G to be a simple graph, since a loop or two multiple edges clearly constitute a cycle. Take a longest path – defined by $(v_0, v_1, \dots, v_n)$ and $(e_1, \dots, e_n)$ – in G. It is finite, since we are assuming G to be so. One has $d(v_n, \mathsf{G}) \geq \delta(\mathsf{G}) \geq 2$, hence v_n is necessarily adjacent to some of the nodes $v_0, \dots, v_{n-2}$, which we denote by v': if v_n were adjacent to a node $\tilde{v}$ not belonging to the path, then

one could construct a longer path defined in a natural way by $(v_0, v_1, \ldots, v_n, \tilde{v})$, in contrast with the assumptions. Thus, the cycle defined by $(v', \ldots, v_n, v')$ is contained in G. □

Under the assumption of simplicity, we can slightly improve Proposition 2.3.

Proposition 2.4 (G.A. Dirac, 1952)**.** *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a simple graph with $\delta(\mathsf{G}) \geq 2$. Then*

(1) G contains a path of length (at least) $\delta(\mathsf{G})$ and
(2) G contains a cycle of length (at least) $\delta(\mathsf{G}) + 1$.

Observe that this result does *not* hold under more general assumptions on V. For instance, in any graph with 2 nodes and 2 parallel edges the minimal degree is 2, but the only existing paths have length 1.

Proof. Consider a longest path $(v_0, v_1, \ldots, v_n)$ in G.

(1) As already remarked in the proof of Proposition 2.3, v_n can only be adjacent to other nodes belonging to the path. Since the path contains n further nodes, we see that the degree of v_n cannot be larger than n, the path's length. In other words, $n \geq d(v_n) \geq \delta(\mathsf{G})$.

(2) Consider now the least $i \in \{0, 1, \ldots, n_2\}$ such that v_i is adjacent to v_n. Then $(v_i, v_{i+1}, \ldots, v_n, v_i)$ defines a cycle. Regarding it as a subgraph C, it is clear that the minimal degree cannot be less than the minimal degree of G. Since $(v_i, v_{i+1}, \ldots, v_n, v_i)$ is a longest path in C, it has length $\geq \delta(\mathsf{G})$. Considering the edge connecting v_n and v_i, we obtain a cycle of length $\delta(\mathsf{G}) + 1$, as claimed. □

Definition 2.5. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a graph.*

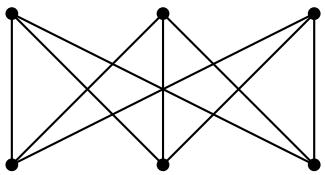
- *The length of a shortest cycle contained in G is called **girth** of G.*
- *The length of a longest cycle contained in G is called **circumference** of G.*

If G does not contain any cycle, both girth and circumference are defined to be ∞.

Admittedly, the notion of girth does not make much sense for a graph that is not simple.

Example 2.6. *1) Both girth and circumference of a cycle C^n amount to n.*

2) Consider a simple graph constructed in the following way: $\mathsf{V} = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $e \in \mathsf{E}$ if and only if it is of the form $e = v_i v_j$, with $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$. Such a graph is commonly denoted by $\mathsf{K}^{3,3}$.



The graph $\mathsf{K}^{3,3}$.

One sees that $\mathsf{K}^{3,3}$ has girth 4: on one hand for example $(v_1, v_4, v_2, v_5, v_1)$ defines a cycle, and on the other hand a cycle's minimal possible length is 3 (cf. Remark 2.2.(1)), but clearly a path of length 3 cannot connect either elements of $\{v_1, v_2, v_3\}$ or of $\{v_4, v_5, v_6\}$.

Definition 2.7. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph and fix two nodes* v, w*. The* ***distance*** *between* v *and* w*, denoted by* $\mathrm{dist}(v, w)$*, is the length* n *of a shortest path, defined by* $(v_0, v_1, \ldots, v_n)$ *and* $(e_1, \ldots, e_n)$*, such that* $v = v_0$ *and* $w = v_n$*.*

The greatest distance between any two nodes in G *is called* ***diameter*** *of* G *and denoted by* $\mathrm{diam}(\mathsf{G})$*.*

Example 2.8. *The diameter of a cycle* C^n *is* $\frac{n}{2}$*. The diameter of* K^n *is* 1*. The diameter of* $\mathsf{K}^{3,3}$ *is* 4*. The diameter of a path of length* n *is* n*.*

Proposition 2.9. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a simple graph containing a cycle. Then its girth satisfies* $g(\mathsf{G}) \leq 2\mathrm{diam}(\mathsf{G}) + 1$*.*

Proof. Let C be a shortest cycle in G, defined by $(v_0, \ldots, v_n, v_0)$. By definition, C has length $g(\mathsf{G})$. Assume that the assertion does not hold, i.e., that $g(\mathsf{G}) \geq 2\mathrm{diam}(\mathsf{G}) + 2$. In this case there exist along C two nodes v_i, v_j whose mutual distance (in C!) is at most $\mathrm{diam}(\mathsf{G}) + 1$. Let $\mathsf{C}_1, \mathsf{C}_2$ the paths (contained in C) defined by $(v_i, v_{i+1}, \ldots, v_j)$ and $(v_j, v_{j+1}, \ldots, v_n, v_0, \ldots, v_j)$. However, by definition of diameter of a graph, the distance between v_i and v_j (in G) is at most $\mathrm{diam}(\mathsf{G})$, and hence there exists a path P, contained in G but not in C, connecting these nodes and long at most $\mathrm{diam}(\mathsf{G})$. Then, the subgraphs $\mathsf{P} \cup \mathsf{C}_1$ and $\mathsf{P} \cup \mathsf{C}_2$ are both cycles. The shorter of them has length strictly less than $(\mathrm{diam}(\mathsf{G}) + 2) + \mathrm{diam}(\mathsf{G})$, a contradiction to the assumption that C is a shortest cycle. □

Larger girth forces the graph to have large number of nodes, too.

Proposition 2.10. *Let* G *a simple graph with minimal degree* $\delta \geq 3$ *and girth* $g \geq 3$*.*

(1) If g *is an odd number, say* $g = 2k + 1$*, then*

$$|\mathsf{V}| \geq 1 + \delta \sum_{j=0}^{k-1} (\delta - 1)^j.$$

(2) If g *is an even number, say* $g = 2k$*, then*

$$|\mathsf{V}| \geq 2 \sum_{j=0}^{k-1} (\delta - 1)^j.$$

Proof. (1) Pick $v \in \mathsf{V}$. Our aim is to count the nodes "radially" away from v. Then v has at least δ adjacent nodes $v_{(i)}$. If $k = 1$, we are done. If this is not the case, we can go on observing that each of the first (at least) δ adjacent nodes $v_{(i)}$ has at least $\delta - 1$ *further* adjacent nodes $v_{(ij)}$ (v is not considered, since we have already counted it). In fact, since the girth is $g = 2k + 1 \geq 5$, it is not possible that any $v_{(i)}$ agrees with any $v_{(ij)}$ – otherwise we would have found a circle of length 3 – nor that any $v_{(ij)}$ agrees with any $v_{(i'j')}$ – otherwise we would have found a circle of length 4. Summing up, we have found at least $1 + \delta + \delta(\delta - 1)$ pairwise different nodes. How far can we go with this procedure? Since there is no cycle of length $2k$, there is no node $w \in \mathsf{V}$ connected with v by at least *two* paths of length k (or less). Thus, we can go on with this procedure only up to the k^{th} step, since after it one may end up finding a node w along a circle crossing v, so that w might be counted on two different *radii*. The claim follows.

(2) This proof can be performed likewise, substituting v with a pair of disjoint nodes $\{v, w\}$ and counting the remaining nodes successively, along radii departing from v, w. $\square$

Definition 2.11. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph. A node* v *is said to be* ***central*** *if* $\max_{w \in \mathsf{V}} \operatorname{dist}(v, w)$ *is minimal in* V *– in this case, we call* $\operatorname{rad}(\mathsf{G}) := \max_{w \in \mathsf{V}} \operatorname{dist}(v, w)$ *the* ***radius*** *of* G.

Exercise 2.12. *Mimick the proof of Proposition 2.10 and show that a simple graph with radius* $\operatorname{rad}(\mathsf{G}) \leq k$ *and maximal degree* $\Delta(\mathsf{G}) \geq 3$ *has less than* $\frac{\Delta(\mathsf{G})}{\Delta(\mathsf{G})-2}(\Delta(\mathsf{G}) - 1)^k$ *edges.*

Exercise 2.13. *Show that for any simple graph* G *holds* $\operatorname{rad}(\mathsf{G}) \leq \operatorname{diam}(\mathsf{G}) \leq 2\operatorname{rad}(\mathsf{G})$.

Exercise 2.14. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a simple graph. The power graph* G^k *of* G *is the graph with same nodes of* G *and such that* v, w *are connected by an edge if they are connected by a path of length at most* k *in* G.

(1) Show that the $i - j$*-entry* $\alpha_{ij}^{(k)}$ *of the* k^{th}*-power of the adjacency matrix is* 1 *if and only if* v_i, v_j *are connected by a path of length exactly* k *in* G.

(2) Deduce that the adjacency matrix of G^k *is given by* $\sum_{i=1}^{k} \mathcal{A}^i$.

(3) Show that $\mathsf{G}^{\operatorname{diam}(\mathsf{G})} = \mathsf{K}^{|\mathsf{V}|}$.

CHAPTER 3

Connectedness

Definition 3.1. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph. A node* v *is said to be* ***connected*** *to a node* w *if there exists a path with endpoints* v, w*.*

Observe that while discussing connectedness of two nodes it is irrelevant whether the graph contains loops or multiple edges.

Exercise 3.2. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph. Define* $C := \{(v, w) \in \mathsf{V} \times \mathsf{V} : v = w$ *or* v *is connected to* $w\}$*. Prove that* C *is an equivalence relation on* V*.*

Definition 3.3. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph. The equivalence classes induced by the relation* C *are called* ***connected components*** *of* G*. The number of connected components of* G *is denoted by* $\kappa(\mathsf{G})$*. If* G *has one connected component, then it is called* ***connected****.*

The number $\mu(\mathsf{G}) := |\mathsf{E}| - |\mathsf{V}| + \kappa(\mathsf{G})$ *is called* ***cyclomatic number*** *of* G*.*

Example 3.4. *Let* $\mathsf{V} :=$ { *Teams competing in the FIFA World Championship* } *and* $\mathsf{E} :=$ { *Matches of the qualification round* }*, with* g *mapping each match into its two competitors. This defines a simple graph* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$*. Since in the qualification round the teams are divided into six subtournaments, each involving four teams, one sees that the graph is not connected. In fact, it consists of six connected components: each of those is in fact a complete graph.*

Exercise 3.5. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph. Show that* $e \in \mathsf{E}$ *is a bridge in* G *if and only if* $\mathsf{G} - e$ *is not connected.*

Definition 3.6. *Let* $k \in \mathrm{N}_0$ *and* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph. Assume that*

- $|\mathsf{V}| > k$ *and furthermore*
- $\mathsf{G} - \mathsf{V}'$ *is connected for any set* $\mathsf{V}' \subset \mathsf{V}$ *with* $|\mathsf{V}'| < k$*.*

Then G *is called* k***-connected****. The largest* $k \in \mathrm{N}_0$ *such that* G *is* k*-connected is called* ***connectivity*** *of* G *and is denoted by* $\lambda(\mathsf{G})$*.*

We sometimes call a node set V' **separating** if $\mathsf{G} - \mathsf{V}'$ is not connected. Then the above definition can be rephrased as follows: A graph is called k-connected if any separating set has cardinality at least k.

Remark 3.7. *If* G *is* k*-connected, then it is* k'*-connected for any* $k' < k$*. The second condition of the above definition is void if* $k = 0$*, hence each nonempty graph is* 0*-connected. Each connected graph with at least* 2 *nodes is* 1*-connected. Each path has connectivity* 1*. Each cycle has connectivity* 2*. Each complete graph* K^n *is* $(n-1)$*-connected (*$n - 1$ *is of course also its connectivity) – why? The connectivity of each disconnected graph is* 0 *– why?*

Definition 3.8. *Let $k \in \mathrm{N}_0$ and $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a graph. If $\mathsf{G} - \mathsf{E}'$ is connected for any set $\mathsf{E}' \subset \mathsf{E}$ with $|\mathsf{E}'| < k$, then G is called k-**edge-connected**. The largest $k \in \mathrm{N}_0$ such that G is k-edge-connected is called **edge-connectivity** of G and is denoted by $\sigma(\mathsf{G})$.*

In other words, a graph is k-edge-connected if it requires deletion of at least k edges in order to become disconnected.

Remark 3.9. *If G is k-edge-connected, then it is k'-connected for any $k' < k$. The condition in the above definition is void if $k = 0$, hence each nonempty graph is 0-edge-connected. Each connected graph is 1-edge-connected. The edge-connectivity of a connected graph G is 1 if and only if G contains a bridge – why? The edge-connectivity of each disconnected graph is 0 – why?*

Typical questions in graph theory go like "Does a graph's local property implies a global property? Or conversely, does a graph's global property implies a local property?". Intuitively, the global property of connectedness seems to be related to the local property of nodes' degree. Is it so?

Proposition 3.10 (H. Whitney, 1932)**.** *Connectivity, edge-connectivity and minimal degree of a graph without loops satisfy $\lambda(\mathsf{G}) \leq \sigma(\mathsf{G}) \leq \delta(\mathsf{G})$.*

Proof. The minimal degree is always ≥ 0. Thus, we can assume G to be non-trivial (i.e., $|\mathsf{E}| \geq 2$) and connected, since otherwise $\sigma(\mathsf{G}) = \lambda(\mathsf{G}) = 0$ if G is not connected or $\sigma(\mathsf{G}) = \lambda(\mathsf{G}) = 1$ if $|\mathsf{V}| = 2$ and $|\mathsf{E}| = 1$ – thus, the assertion follows.

Since $|\mathsf{E}| > 1$, $\delta(\mathsf{G}) \geq 1$. Pick a node v with $d(v, \mathsf{G}) = \delta(\mathsf{G})$ and consider the graph G' obtained by deleting all the $\delta(\mathsf{G})$ edges that are incident in v. Such a graph is not connected, hence G is not $(\delta(\mathsf{G}) + 1)$-edge-connected, i.e., its edge-connectivity is at most $\delta(\mathsf{G})$. This shows the latter inequality.

Let us prove that $\sigma(\mathsf{G}) \geq \lambda(\mathsf{G})$. First of all, observe that we may assume G to be simple without any loss of generality: in fact, one sees that the connectivity $\lambda(\mathsf{G})$ remains invariant upon adding loops and multiple edges, whereas the edge-connectivity may rise.

Let then G be a simple, connected graph. We consider the cases $\sigma(\mathsf{G}) = 1$ and $\sigma(\mathsf{G}) \geq 2$ separately. If $\sigma(\mathsf{G}) = 1$, then by Remark 3.9 G contains a bridge. But then one sees that $\lambda(\mathsf{G}) = 1$, since deleting either endpoint of the bridge makes the simple graph G disconnected.

Let finally consider the case of a simple, connected graph G with $\sigma(\mathsf{G}) \geq 2$. Thus, there exists a minimal edge set E' such that $\mathsf{G} - \mathsf{E}'$ is disconnected, i.e., an edge set $\mathsf{E}' = \{e_1, \ldots, e_{\sigma(\mathsf{G})}\}$ such that $\mathsf{G} - \mathsf{E}'$ is disconnected but $\mathsf{G} - \{e_2, \ldots, e_{\sigma(\mathsf{G})}\} = (\mathsf{G} - \mathsf{E}') + \{e_1\}$ is not. Then, e_1 is a bridge, say vw (we are using the notation introduced in Definition 1.1). Choose for any edge e_i an endpoint x_i, $i = 2, \ldots, \sigma(\mathsf{G})$, in such a way that $v \neq x_i \neq w$ (but the x_i's need not be pairwise different). Then these endpoints define a node set $\mathsf{V}'' := \{x_2, \ldots, x_{\sigma(\mathsf{G})}\}$ with $|\mathsf{V}''| < \sigma(\mathsf{G})$. Apparently, $\mathsf{G} - \mathsf{V}''$ might be connected but $\mathsf{G} - (\mathsf{V}'' \cup \{v\})$ or $\mathsf{G} - (\mathsf{V}'' \cup \{w\})$ are not, hence in particolar G is not $\sigma(\mathsf{G})$-connected. Thus,

$$\sigma(\mathsf{G}) \leq \min\{|\{v, x_2, \ldots, x_{\lambda(\mathsf{G})}\}|, |\{w, x_2, \ldots, x_{\lambda(\mathsf{G})}\}|\} \leq \lambda(\mathsf{G}).$$

This completes the proof. □

Remark 3.11. *It follows from Proposition 3.10 that each complete graph* K^{n-1} *has connectivity* $n-1$.

Definition 3.12. *Let* $\mathsf{G}=(\mathsf{V},\mathsf{E},g)$ *be a simple graph and* $A,B\subset\mathsf{V}$. *A path* $(v_0,v_1,\dots,v_n)$ *is called an* $A-B$***-path*** *if only the endpoints belong to* A *or* B, *i.e.,*

- $v_i\in A$ *if and only if* $i=0$,
- $v_i\in B$ *if and only if* $i=n$,
- $v_i\notin A\cup B$ *for* $i=1,\dots,n-1$.

Let now $\mathsf{V}'\subset\mathsf{V}$. *If each* $A-B$*-path in* G *contains a node from* V', *then* V' *is said to* ***separate*** A,B *in* G, *or to be a* $A-B$***-separating set***.

Observe that the definition does not exclude that $A\cap B\neq\emptyset$.

Exercise 3.13. *Let* $\mathsf{G}=(\mathsf{V},\mathsf{E},g)$ *be a simple graph and* $v,w\in\mathsf{V}$. *A node set* $\mathsf{V}'\subset\mathsf{V}$ *separates* $\{v\},\{w\}$ *if and only if (*$\mathsf{G}-\mathsf{V}'$ *is not connected and)* v,w *are contained into different connected components of* $\mathsf{G}-\mathsf{V}'$.

Theorem 3.14 (K. Menger, 1927)**.** *Let* $\mathsf{G}=(\mathsf{V},\mathsf{E},g)$ *be a simple graph and* $A,B\subset\mathsf{V}$. *Then the minimal cardinality of an* $A-B$*-separating set agrees with the maximal number of disjoint* $A-B$*-paths in* G.

In order to present the proof, we first need to present a basic construction.

Definition 3.15. *If* G *is a simple graph and* $e=(v,w)\in\mathsf{E}$, *then* G/e *is the graph with node set* $\mathsf{V}_e:=(\mathsf{V}\setminus\{v,w\})\cup\{v_e\}$ *that is obtained*

- *identifying both* v *and* w *with a new node* v_e,
- *replacing any edge* (v,z) *or* (w,z) *by an edge* (v_e,z), *and finally*
- *deleting the loop around* v_e *(corresponding to* e*) and the multiple edges that may arise.*

The new graph G/e *is said to be obtained* ***contracting*** *the edge* e.

Proof. If there are k disjoint $A-B$-paths in G, then it suffices to pick one node from each of them in order to construct a node set with k elements which is by definition an $A-B$ separating set. This shows that any separating set cannot have fewer elements than k.

Conversely, denote by h the smallest cardinality of an $A-B$-separating set in G: we are going to show that there are (at least) h disjoint $A-B$-paths in G.

If $|\mathsf{E}|=0$, then $|A\cap B|=\{v_1,\dots,v_h\}$ and there exist h different (and trivial) $A-B$ paths (namely, the 0-length-paths defined by $v_0,\dots,v_h$).

Let now the assertion hold for any graph with edge set of cardinality at most m. Consider a graph with edge set of cardinality $m+1$ and pick an edge $e=(v,w)$. We will use the induction hypothesis on G/e or $\mathsf{G}-e$.

We first consider the case that h disjoint $A-B$-paths exist in G/e. Since clearly these paths still exist after expanding G/e to G, the assertion holds.

If however no such h disjoint $A-B$-paths exist in G/e, by induction hypothesis the minimal cardinality of a node set separating A,B in G is less than h: let V'_e be an $A-B$ separating set of such a cardinality. Then necessarily $v_e\in\mathsf{V}'_e$, otherwise V'_e would be an

$A-B$-separating set in G, too, and thus necessarily of cardinality h, a contradiction. On the other hand, by setting $\mathsf{V}_e := (\mathsf{V}'_e \setminus \{v_e\}) \cup \{v, w\}$ we have constructed an $A-B$-separating set (in G) with cardinality exactly hk. (Observe that V_e is not a node set in G/e, but it is indeed a node set in $\mathsf{G}-e$.)

We are now in the position to apply the induction hypothesis to the smaller graph $\mathsf{G}-e$, whose edge set has cardinality m. Since $v, w \in \mathsf{V}_e$, each node set that separates $A - \mathsf{V}_e$ in $\mathsf{G}-e$ is an $A-B$-separating set in G, hence it has at least cardinality h, and so does (for a similar reason) each $B - \mathsf{V}_e$-separating set in $\mathsf{G}-e$. By induction there are h disjoint $A - \mathsf{V}_e$-paths in G as well as h disjoint $B - \mathsf{V}_e$-paths in G. Since V_e is an $A-B$-separating set, each of these $A - \mathsf{V}_e$-paths is disjoint from each of the $B - \mathsf{V}_e$-paths. Accordingly, we can "glue" them together in order to obtain h disjoint $A-B$-paths. □

A few related results follow promptly. Two paths connecting v, w are called **independent** if their only common nodes are v, w. In particular, observe that there are infinitely many independent trivial paths (i.e., paths of length 0) with a given endpoint v.

Corollary 3.16. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a simple graph and $v, w \in \mathsf{V}$. Assume $\mathrm{dist}(v, w) \geq 2$. Then the smallest cardinality of a node set $\mathsf{V}' \subset \mathsf{V} \setminus \{v, w\}$ separating v, w agrees with the largest cardinality of a set of pairwise independent paths connecting v, w in G.*

Proof. The claim follows from Menger's Theorem applied in the graph $\mathsf{G} - \{v, w\}$ to the sets A, B of all nodes adjacent to v, w, respectively. □

Remark 3.17. *Conversely, the above version of Menger's Theorem for individual nodes also implies the assertion in Theorem 3.14. In fact, let $A, B \subset \mathsf{V}$. Consider a new graph $\tilde{\mathsf{G}}$ obtained connecting a new node a to each node in A and a new node b to each node in B. Since by construction there is no edge (a, b), it is clear that the set of pairwise independent $\{a\} - \{b\}$ paths is bijective to the set of disjoint $A - B$ paths. On the other hand, each $A - B$-separating set is an $\{a\} - \{b\}$-separating set, and Menger's Theorem follows.*

Finally, we present a *global version* of Menger's Theorem.

Corollary 3.18 (H. Whitney, 1932)**.** *A simple graph G is k-connected if and only if there are (at least) k independent paths connecting any two nodes in G.*

Proof. Let us first assume that there are at least k independent paths connecting any two nodes in G. Then necessarily G has more than k nodes and by definition any node set separating G has at least cardinality k, i.e., G is k-connected.

Let conversely G be k-connected but such that a pair of nodes v, w is connected by at most $k-1$ independent paths. By Corollary 3.16 this is only possible if v, w are adjacent (as observed before, if $v = w$ all trivial paths are independent). Then, consider $\mathsf{G} - (v, w)$: this graph contains at most $k-2$ independent $\{v\} - \{w\}$-paths. Again by Corollary 3.16 v, w are therefore separated in $\mathsf{G} - (v, w)$ by a node set of cardinality $k-2$, say V' – i.e., v, w belong to two different connected components of $(\mathsf{G} - (v, w)) - \mathsf{V}'$. Now, by definition of k-connectedness we deduce that G has at least $k+1$ nodes, i.e., there exists a node z different from either v or w and also not belonging to the above defined separating set V'. Since z cannot belong to both connected components of $(\mathsf{G} - (v, w)) - \mathsf{V}'$ containing v and w, respectively, we deduce that V' separates z from v in $\mathsf{G} - (v, w)$, or V' separates z from w

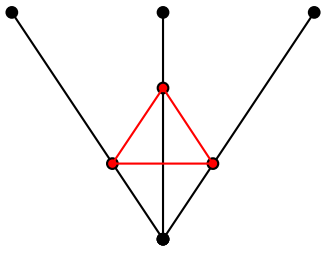
in $\mathsf{G}-(v,w)$. In either case, we would have found a set ($\mathsf{V}'\cup\{v\}$ or $\mathsf{V}'\cup\{z\}$) of cardinality less than k separating two nodes in G, a contradiction to the k-connectedness of G. □

In order to state another consequence of Menger's Theorem, we need to introduce a new notion.

Definition 3.19. *Let $\mathsf{G}=(\mathsf{V},\mathsf{E},g)$ be a simple graph. The **line graph** of G is the simple graph $\mathsf{G}_L=(\mathsf{V}_L,\mathsf{E}_L,g_L)$ with node set $\mathsf{V}_L:=\mathsf{E}$ and such that the edge (e,f) exists, i.e., such that two nodes $e,f\in\mathsf{E}$ are adjacent, if and only if $g(e)\cap g(f)\neq\emptyset$, i.e., e,f are adjacent edges in G.*

Example 3.20. *1) The line graph of a cycle C^m is a cycle C^m.*

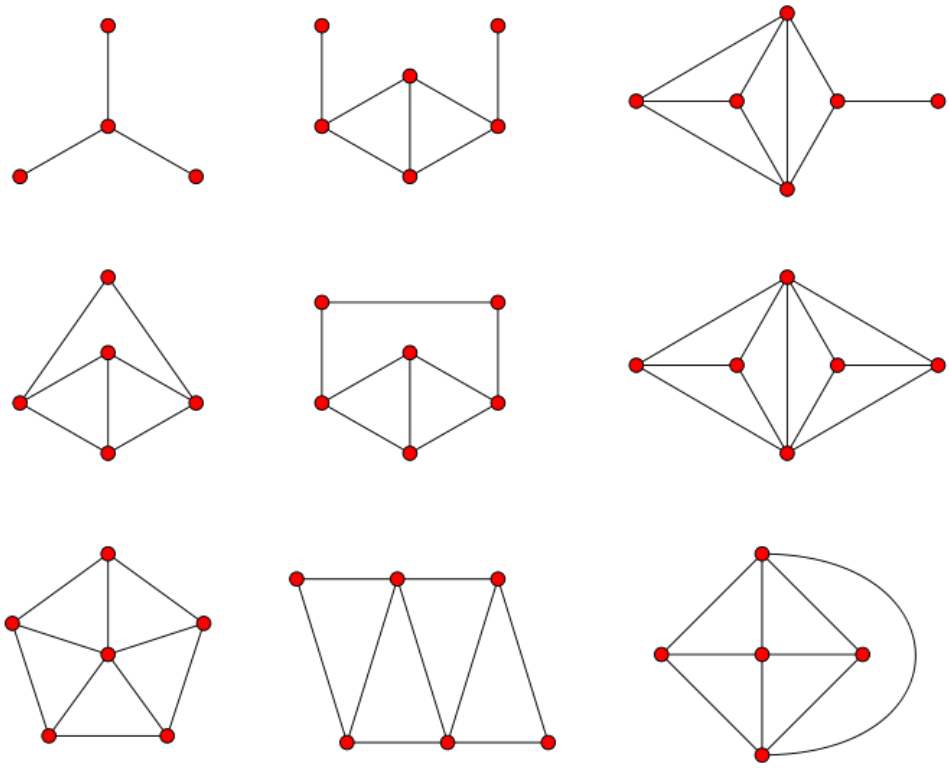
*2) Let G be a graph. In general, the line graph of G_L is different from G_L. To see this, consider the graph depicted below, sometimes called a **claw**.*



The claw (in black) and its line graph (in red).

The claw's graph line is clearly a triangle, but by 1) a triangle's graph line is again a triangle, and not a claw.

3) Not any graph is line graph of another graph. One can see that the claw is not the line graph of any graph. More generally, L.W. Beineke has proved in 1968 that a graph is line graph of another graph if and only if it does not *contain any of the following nine graphs as induced subgraphs.*



The nine forbidden line subgraphs (image taken from Wikipedia).

4) It has been proved by H. Whitney in 1932 that, with the only exception discussed in 2), two graphs are isomorphic if and only if their line graphs are isomorphic

Exercise 3.21. *Let* G *be a simple graph and* G_L *the associated line graph. Prove the following assertions.*

(1) If G *is connected, then also* G_L *is connected.*
(2) If $\delta(\mathsf{G}) \geq 1$*, i.e., if* G *contains no isolated nodes, then* G *is connected if and only if* G_L *is connected.*
(3) The adjacency matrix of G_L *is given by* $\mathcal{I}^T\mathcal{I} - 2I$*, where* I *denotes the identity* $|\mathsf{E}| \times |\mathsf{E}|$*-matrix.*

Exercise 3.22. *Let* G *be an* r*-regular graph,* $r \geq 1$*. Prove that the line graph of* G *is a* $2(r-1)$*-regular graph.*

Exercise 3.23. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a simple graph. Prove the following assertions.*

(1) Let $v, w \in \mathsf{V}$*,* $v \neq w$*. Then the smallest cardinality of an edge set* $\mathsf{E}' \subset \mathsf{E}$ *separating* v, w *agrees with the largest cardinality of a set of pairwise edge-disjoint paths connecting* v, w *in* G*.*
(2) G *is* k*-edge-connected if and only if it contains* k *pairwise edge-disjoint paths connecting any two vertices* v, w *in* G*.*

(Hint: Take into account G_L).

Exercise 3.24. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a simple graph and $v, w \in \mathsf{V}$. Formulate the notion of an* edge *set separating $\{v\}, \{w\}$ and prove the following versions of Menger's Theorem.*

(1) The smallest cardinality of an edge set separating v, w in G agrees with the largest cardinality of a set of pairwise edge-disjoint $\{v\} - \{w\}$ in G.

(2) The graph G is k-edge-connected if and only if there are k edge-disjoint paths connecting any two nodes in G.

Remark 3.25. *Let G be a connected graph. It has been observed by A. van Rooij and H. Wilf that when considering the sequence*

$$G, G_L, (G_L)_L, ((G_L)_L)_L, \dots$$

one of the following cases happens:

- *if G is 2-regular, then all the graphs in the sequence are pairwise isomorphic;*
- *if G is the claw, then G_L is a triangle and then by the above case all the further graphs are again triangles;*
- *if G is the path P^n, then G_L is the path P^{n-1}, $(G_L)_L$ is the path P^{n-2} and so on: the n-th entry of the sequence (and therefore also all the following ones) is the empty graph;*
- *if G is not any of the above graphs, then G_L is a graph with strictly more edges and nodes than G, and in fact the sequence of node and edge numbers of*

  $$G, G_L, (G_L)_L, ((G_L)_L)_L, \dots$$

  is strictly monotonically increasing, hence unbounded.

Exercise 3.26. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a simple graph, $\mathsf{V}' \subset \mathsf{V}$, and $v, w \in \mathsf{V} \setminus \mathsf{V}'$ such that X separates $\{v\}, \{w\}$ in G. Denote by C_v, C_w the connected components of $\mathsf{G} - \mathsf{V}$ containing v, w, respectively. Show that V' is a minimal $\{v\} - \{w\}$-separating set in G if and only if each node of V' is both adjacent to some node in C_v and to some node in C_w.*

Proposition 3.27 (G. Chartrand – F. Harary, 1968)**.** *Each simple but not complete graph $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ satisfies*

$$\lambda(\mathsf{G}) + |\mathsf{V}| \geq 2\delta(\mathsf{G}) + 2.$$

This suggests that *graphs with few nodes but high minimal degree have high connectivity.*

Proof. Let V' be a node set with cardinality $\lambda(\mathsf{G})$ (this is always possible, since by definition of k-connectedness G has more than $\lambda(\mathsf{G})$ nodes): by definition of k-connectivity, this implies that $\mathsf{G} - \mathsf{V}'$ is not-connected, i.e., it has at least two connected components. Let $\mathsf{G}_1, \mathsf{G}_2$ two connected components of $\mathsf{G} - \mathsf{V}'$ and v_1, v_2 nodes in $\mathsf{G}_1, \mathsf{G}_2$, respectively. Then, all nodes that are adjacent to v_1, v_2 in G, respectively, are contained in $\mathsf{G}_1, \mathsf{G}_2$, respectively, or belong to V'. Denote by $A_i = \{v \in \mathsf{V} : \mathrm{dist}(v, v_i) \leq 1\}$, $i = 1, 2$: on one hand we have $|A_1| + |A_2| \leq |\mathsf{V}| + |\mathsf{V}'|$. On the other hand each of these sets has cardinality $\geq 1 + \delta(\mathsf{G})$, and the claim follows. □

Proposition 3.28 (W. Mader, 1972; P. Sprüssel, 2005)**.** *Let $k \in \mathbb{N}$. Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a simple graph with $d_a(\mathsf{G}) \geq 4k$. Then there exists a $(k+1)$-connected subgraph G_0 of G such that*

$$d_a(\mathsf{G}_0) + 2k > d_a(\mathsf{G}). \tag{3.1}$$

PROOF. First of all, observe that the condition $d_a(\mathsf{G}) \geq 4k$ reads equivalently as

$$|\mathsf{E}| \geq 2k|\mathsf{V}|. \tag{3.2}$$

We consider the class of all subgraphs $\mathsf{G}' = (\mathsf{V}', \mathsf{E}', g')$ of G such that

$$|\mathsf{V}'| \geq 2k \qquad \text{and} \qquad \frac{|\mathsf{E}'|}{|\mathsf{E}|} > \frac{|\mathsf{V}'| - k}{|\mathsf{V}|}. \tag{3.3}$$

This class is nonempty, since G belongs to it (because $|\mathsf{V}| > \Delta(\mathsf{G}) \geq d_a(\mathsf{G}) \geq 4k$, so that certainly $|\mathsf{V}| > 2$), and we can therefore pick one of its elements – say $\mathsf{G}_0 := (\mathsf{V}_0, \mathsf{E}_0, g_0)$ – having smallest number of nodes. However, such a minimal graph cannot have exactly $2k$ nodes, otherwise

$$|\mathsf{E}_0| > \frac{|\mathsf{E}|}{|\mathsf{V}|}k \geq 2k^2 > 2k^2 - k = \frac{2k(2k-1)}{2} = \frac{|\mathsf{V}_0|(|\mathsf{V}_0|-1)}{2} = \binom{|\mathsf{V}_0|}{2}$$

(the first inequality follows from 3.2, the second from 3.3), a contradiction to the bound on $|\mathsf{E}_0|$ in Remark 1.20. Let us show that G_0 satisfies

$$\delta(\mathsf{G}_0) > \frac{|\mathsf{E}|}{|\mathsf{V}|} \tag{3.4}$$

(and therefore $|\mathsf{V}_0| \geq \frac{|\mathsf{E}|}{|\mathsf{V}|}$): if this is not the case, i.e. if $\delta(\mathsf{G}_0) \leq \frac{|\mathsf{E}|}{|\mathsf{V}|}$, then picking any node $w \in \mathsf{G}_0$ with $d(w, \mathsf{G}) \leq \frac{|\mathsf{E}|}{|\mathsf{V}|}$ we would have found a graph $\mathsf{G}_0 - \{w\}$ that still satisfies 3.3 – contradicting the hypothesis on minimality of G_0. Summing up, from (3.3) we obtain

$$\frac{|\mathsf{E}_0|}{|\mathsf{V}_0|} + k > \frac{|\mathsf{E}_0|}{|\mathsf{V}_0|} + \frac{k|\mathsf{E}|}{|\mathsf{V}|\,|\mathsf{V}_0|} > \frac{|\mathsf{E}|}{|\mathsf{V}|},$$

which in turn yields (3.1).

It remains to prove that G_0 is $(k+1)$-connected. Let this not be the case: then there exists a separating set $\tilde{\mathsf{V}}$ of cardinality at most k. Denote by $\tilde{\mathsf{V}}_1, \tilde{\mathsf{V}}_2$ two components separated by $\tilde{\mathsf{V}}$ and by $\mathsf{G}_1 = \mathsf{G}[\tilde{\mathsf{V}}_1 \cup \tilde{\mathsf{V}}]$, $\mathsf{G}_2 = \mathsf{G}[\tilde{\mathsf{V}}_2 \cup \tilde{\mathsf{V}}]$ the subgraphs induced in G by $\mathsf{V}_1 := \tilde{\mathsf{V}}_1 \cup \tilde{\mathsf{V}}$ and $\mathsf{V}_2 := \tilde{\mathsf{V}}_2 \cup \tilde{\mathsf{V}}$, respectively. Pick now a node $v_1 \in \tilde{\mathsf{V}}_1$ and observe that all the nodes that are adjacent to v_1 (there are $d(v_1, \mathsf{G}_1) \geq \delta(\mathsf{G}_0) > \frac{|\mathsf{E}|}{|\mathsf{V}|}$ of them, by (3.4)) are in V_1. Thus, G_1 has at least $\frac{|\mathsf{E}|}{|\mathsf{V}|}$ nodes. The same can be proved for G_2, of course. By assumption, both $\mathsf{G}_1, \mathsf{G}_2$ turn out to have at least $2k$ nodes. Recall that G_0 is by construction minimal in the class of those subgraphs of G satisfying (3.3). Accordingly, neither G_1 nor G_2 satisfies (3.3), i.e., $\frac{|\mathsf{E}_1|}{|\mathsf{E}|} \leq \frac{|\mathsf{V}_1|-k}{|\mathsf{V}|}$ and $\frac{|\mathsf{E}_2|}{|\mathsf{E}|} \leq \frac{|\mathsf{V}_2|-k}{|\mathsf{V}|}$. Summing up, one obtains

$$|\mathsf{E}_0| \leq |\mathsf{E}_1| + |\mathsf{E}_2| \leq \frac{|\mathsf{E}|}{|\mathsf{V}|}(|\mathsf{V}_1| + |\mathsf{V}_2| - 2k) \leq \frac{|\mathsf{E}|}{|\mathsf{V}|}(|\mathsf{V}_0| - k),$$

where we are using the fact that $\mathsf{V}_1 \cap \mathsf{V}_2 = \tilde{\mathsf{V}}$ has at most k elements. This is in contrast with (3.3) and concludes the proof. □

Exercise 3.29. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a simple graph.*

(1) Apply Remark 1.20 in order to prove that

$$|\mathsf{E}| \leq \binom{|\mathsf{V}| - \kappa(\mathsf{G}) + 1}{2},$$

where $\kappa(\mathsf{G})$ denotes as usual the number of connected components in G.
(Hint: We can assume without loss of generality G to consist of connected components that are complete graphs – why? Moreover, it is possible to modify the graph by replacing pairs of subgraphs (isomorphic to) $\mathsf{K}^n, \mathsf{K}^m$ by $\mathsf{K}^{n+1}, \mathsf{K}^{m-1}$, if $n \geq m$).
(2) Conclude that a sufficient condition for G to be connected is that $|\mathsf{E}| > \frac{1}{2}(|\mathsf{V}|-1)(|\mathsf{V}|-2)$.

To conclude this chapter, we turn to the question already adressed in the introduction and solve the Königsberg bridge problem 0.1.

Definition 3.30. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a graph.*
An edge sequence defined by $(v_0, \ldots, v_n)$ and $(e_1, \ldots, e_n)$ in such a way that $v_0 = v_n$ and the sequence contains each *element of E* at most once *is called a* ***tour****.*
If a tour contains each *element of E* exactly once*, then it is called a* ***Euler tour****.*
The graph G is called ***Eulerian*** *if there exists a Euler tour in G.*

Observe that while each edge has to be traversed exactly once, it is allowed to cross a node many times – i.e., we are not necessarily looking for a cycle.

Theorem 3.31 (L. Euler 1735, C. Hierholzer, 1873; O. Veblen, 1912)**.** *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a connected graph with $|\mathsf{E}| \geq 2$. Then the following assertions are equivalent.*
(a) G is Eulerian.
(b) Each node has even degree.
(c) There exist cycles $\mathsf{K}^1, \ldots, \mathsf{K}^h$ with $\mathsf{K}^i = (\mathsf{V}_i, \mathsf{E}_i, g_i)$ such that E is disjoint union of $\mathsf{E}_1, \ldots, \mathsf{E}_h$.

Proof. $(a) \Rightarrow (b)$ Consider a Euler tour defined by $(v_0, v_1, \ldots, v_n)$ and $(e_1, \ldots, e_n)$. If a node v appears k times in the node sequence, then the edge sequence necessarily includes $2k$ edges having v as an endpoint (where we are counting loops around v twice).

$(b) \Rightarrow (c)$ Since each node has even degree and the graph is connected, the minimal degree $\delta(\mathsf{G})$ is necessarily larger than 1. By Proposition 2.3 G contains a cycle with edge set E'. Also in $\mathsf{G}' - \mathsf{E}'$ all nodes have even degree, hence it contains a cycle... Repeating this procedure we finally obtain a decomposition of G in pairwise edge-disjoint cycles.

$(c) \Rightarrow (a)$ If G consists of K^1 only, then it is clearly Eulerian. If however this is not the case, there exists a second cycle K^2 that (due to connectedness) shares a node v_2 with K^1. If G consists of $\mathsf{K}^1, \mathsf{K}^2$ only, then consider the tour defined as follows: Pick an arbitrary node v_1 in K^1, follow the tour until one reaches v_2, then deviate, enter into K^2, follow K^2 until v_2 is reached again, and finally come back to K^1 and reach v_1 again. Since this is clearly an Eulerian tour, we are done. If however G is not saturated by $\mathsf{K}^1, \mathsf{K}^2$, then there is a further cycle K^3 that shares a node v_3 with K^1 or K^2... Extending this procedure until all cycles are considered we obtain an Euler tour – the claim follows. □

Remark 3.32. *The above proof suggests an algorithm for determining an Euler tour, see also* [**19**, pag. 323]*. In concrete applications it is often important to provide the algorithmic*

construction of a graph theoretical object as well as to determine the associated running costs on a computer. Answering these and similar questions is one of the goals of algorithmich graph theory*, which has turned into a research field in its own right, see e.g.* [**19**].

Example 3.33. *If* $\mathsf{V} = \{$*Ulm, Neu-Ulm, Neu-Ulmer Insel*$\}$ *and each bridge between any two nodes represents an edge, then the corresponding graph is* not *Eulerian: because the Neu-Ulmer Insel is a node of degree 3 (it is connected to Ulm by one bridge, whereas two bridges link it to Neu-Ulm).*

Exercise 3.34. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a connected graph with* $|\mathsf{E}| \geq 2$*. Show that* G *admits an edge sequence defined by* $(v_0, \ldots, v_n)$ *and* $(e_1, \ldots, e_n)$ *in such a way that* $v_0 = v_n$ *and for each* $e \in \mathsf{E}$ *there exist exactly two indices* $i, j = 1, \ldots, n$ *such that* $e = e_i = e_j$*.*

Exercise 3.35. *Let* e *be an edge of the complete graph* K^n*. For which* $n \geq 3$ *does* $\mathsf{K}^n - \{e\}$ *contain an edge sequence traversing each edge of* K^n *exactly once?*

Remark 3.36. *Consider a graph* G *that is not Eulerian. Let the graph is weighted by* ρ*. One can wonder how to find an edge sequence* $(e_0, e_1, \ldots, e_n)$ *containing each edge of* G *at least once and such that* $\sum_{i=1}^{n} \rho(e_i)$ *is minimal. In other words, does an optimal strategy for walking through a town traversing* each *street exist? This is the so-called* Chinese Postman Problem*, proposed by M.-K. Kuan in 1962.*

CHAPTER 4

Directed graphs and networks

Definition 4.1. *A **directed graph** or **digraph** is a 4-uple* $\overrightarrow{\mathsf{G}} := (\mathsf{V}, \mathsf{E}, \mathrm{init}, \mathrm{term})$*, where* V, E *are nonempty sets and* $\mathrm{init}, \mathrm{term} : \mathsf{E} \to \mathsf{V}$ *are two mappings. The elements of* E *are **directed edges** and are usually denoted by* $\overrightarrow{e}$, $\overrightarrow{f}$*, etc. The nodes* $v := \mathrm{init}(\overrightarrow{e})$ *and* $w := \mathrm{term}(\overrightarrow{e})$ *are called initial and terminal endpoint of* $\overrightarrow{e}$*, respectively, and one says that* $\overrightarrow{e}$ ***connects*** v **to** w *(instead of "connects* v *and* w*" as in the undirected case).*

Usually, directed graphs are represented as graphs whose edges e with initial endpoint v and terminal endpoint w are replaced by an arrow with head w and tail v.

Most notions introduced in the undirected case can be extended to the case of a directed graph. In particular, also directed graphs can be described by means of suitably modified adjacency and incidence matrices.

More precisely, we introduce the $|\mathsf{V}| \times |\mathsf{V}|$ **directed adjacency matrix** $\overrightarrow{\mathcal{A}} = (\overrightarrow{\alpha}_{ij})$, where each entry $\overrightarrow{\alpha}_{ij} \in \mathrm{N}_0$ gives the number of directed edges with initial endpoint v_i and terminal endpoint v_j (by convention, $a_{ii} = 1$ if there is a loop around v_i). Observe that $\overrightarrow{\mathcal{A}}$ is in general not symmetric; it consists of 0s and 1s only if there exists at most one directed edge (that is, at most one in either direction) connecting any two nodes.

We can also introduced the $|\mathsf{V}| \times |\mathsf{E}|$ **directed incidence matrix** $\overrightarrow{\mathcal{I}} = (\overrightarrow{\iota}_{ij})$, where

$$\overrightarrow{\iota}_{ij} := \begin{cases} -0 & \text{if } v_i \text{ is both initial and terminal endpoint of } e_j, \\ 1 & \text{if } v_i \text{ is (only) initial endpoint of } e_j, \\ -1 & \text{if } v_i \text{ is (only) terminal endpoint of } e_j, \\ 0 & \text{otherwise.} \end{cases}$$

Of course, -0 is just a symbol.

Definition 4.2. *Let* $\overrightarrow{\mathsf{G}} = (\mathsf{V}, \mathsf{E}, \mathrm{init}, \mathrm{term})$ *be a digraph. Then the number of edges with terminal endpoint* v *is called **indegree** of* v *and is denoted by* $d_i(v)$*, or sometimes by* $d_i(v, \overrightarrow{\mathsf{G}})$*. Likewise, the number of edges with terminal endpoint* v *is called **outdegree** of* v *and is denoted by* $d_t(v)$*, or sometimes by* $d_t(v, \overrightarrow{\mathsf{G}})$*. Of course, setting* $d_i(\overrightarrow{\mathsf{G}}) := \sum_{v \in \mathsf{V}} d_i(v)$ *and* $d_t(\overrightarrow{\mathsf{G}}) := \sum_{v \in \mathsf{V}} d_t(v)$*, the **total degree** of* G *is* $d(\overrightarrow{\mathsf{G}}) := d_i(\overrightarrow{\mathsf{G}}) + d_t(\overrightarrow{\mathsf{G}})$*.*

Example 4.3. *Consider a partially ordered set* $(P, \leq)$*. Then, one can define a directed graph setting* $\mathsf{V} := P$ *and considering an edge connecting* v *to* w *if and only if* $v \leq w$*.*

Remark 4.4. *Likewise, it is possible to introduce* oriented paths, oriented cycles, oriented bridges, oriented connectedness *(which is called* **strong connectedness***),... Some of the usual notions (like those of adjacent vertices/edges and automorphisms) can be defined in the context of oriented graphs just like in the non-oriented case.*

Also, a few of the results we have proved for undirected graphs have a pendant in the directed setting. For example, analogously to Theorem 3.31 one can prove that a directed graph is Eulerian if it is strongly connected and every node has equal indegree and outdegree. The main result in this chapter is Ford–Fulkerson's Max-flow-min-cut Theorem. Although it has been proved later, in the literature it is a favorite tool for proving Menger's Theorem – Ford–Fulkerson's theorem can be in fact interpreted as a directed version of the latter.

Observe that in spite of its usual attribution, this theorem has been proven in the same year and indepently by P. Elias, A. Feinstein, and C. E. Shannon, in the context of linear programming.

By definition of directed graph multiple edges between two given nodes are allowed in either direction. Still, for the sake of simplicity we focus on the case of *simple* digraphs.

Definition 4.5. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a simple graph. An **oriented edge** $\overrightarrow{e}$ is an edge (v, w) of G provided with an orientation: either from v to w or from w to v (thus, an oriented edge can be identified with an element of $\mathsf{V} \times \mathsf{V}$). We write $\overrightarrow{e} = \overrightarrow{(v, w)}$ and $\overleftarrow{e} = \overleftarrow{(v, w)} = \overrightarrow{(w, v)}$, respectively. An **oriented graph** $\overrightarrow{G}$ is a simple graph each of whose edges is provided with an orientation.*

For $\mathsf{V}_1, \mathsf{V}_2 \subset \mathsf{V}$ we introduce the notation $\overrightarrow{(\mathsf{V}_1, \mathsf{V}_2)} := \{\overrightarrow{e} = \overrightarrow{(v, w)} \in \overrightarrow{\mathsf{E}} : v \in \mathsf{V}_1,\ w \in \mathsf{V}_2\}$.

Remark 4.6. *In particular, in an oriented graph $\overrightarrow{e} \in \overrightarrow{(\mathsf{V}', \mathsf{V}')}$ if and only if $\overleftarrow{e} \in \overrightarrow{(\mathsf{V}', \mathsf{V}')}$, whenever $\mathsf{V}' \subset \mathsf{V}$.*

In an oriented graph there are by definition no loops, so that we can regard the above introduced directed adjacency and incidence matrices as not mere symbols, but also well-defined linear algebraic objects. Thus, a $|\mathsf{V}| \times |\mathsf{E}|$ **oriented incidence matrix** $\overrightarrow{\mathcal{I}} = (\overrightarrow{\imath}_{ij})$, where

$$\overrightarrow{\imath}_{ij} := \begin{cases} 1 & \text{if } v_i \text{ is (only) initial endpoint of } e_j, \\ -1 & \text{if } v_i \text{ is (only) terminal endpoint of } e_j, \\ 0 & \text{otherwise} \end{cases}$$

is naturally introduced. By definition, the entries of each column of $\overrightarrow{\mathcal{I}}$ sum up to 0.

Proposition 4.7. *Let G be a graph. Consider an arbitrary orientation of each edge $e \in \mathsf{E}$ and therefore an oriented graph $\overrightarrow{\mathsf{G}}$. Then the adjacency matrix $\mathcal{A}$ of G and the oriented incidence matrix $\overrightarrow{\mathcal{I}}$ are related by the formula*

$$\mathcal{D} - \mathcal{A} = \overrightarrow{\mathcal{I}}\,\overrightarrow{\mathcal{I}}^{T},$$

where $\mathcal{D}$ is the diagonal matrix whose diagonal entries δ_{ii} denote the degree of v_i in G.

This motivates to introduce the following.

Definition 4.8. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a simple graph. The **admittance matrix** $\mathcal{B}$ of G is defined by $\mathcal{B} := \mathcal{D} - \mathcal{A}$, where $\mathcal{D}$ is the diagonal matrix whose diagonal entries δ_{ii} denote the degree of v_i and $\mathcal{A}$ is the usual adjacency matrix.*

This matrix can clearly also be regarded as a mapping on the vector space $\mathrm{C}^{|\mathsf{V}|}$*, i.e., as a linear operator acting on the function space* $\mathsf{V} \to \mathrm{C}$*. In this context, it is also called* ***combinatorial*** *(or* ***graph****)* ***Laplacian*** *of* G*.*

Proof. A direct computation shows that the $i-j$-entry of $\overrightarrow{\mathcal{I}}\,\overrightarrow{\mathcal{I}}^T$ is given by

$$(\overrightarrow{\mathcal{I}}\,\overrightarrow{\mathcal{I}}^T)_{ij} = \sum_{k=1}^{|\mathsf{E}|} \overrightarrow{\iota}_{ik}\,\overrightarrow{\iota}_{jk} :$$

if $i = j$ this amounts to

$$(\overrightarrow{\mathcal{I}}\,\overrightarrow{\mathcal{I}}^T)_{ii} = \sum_{k=1}^{|\mathsf{E}|} |\overrightarrow{\iota_{ik}}|^2 = \sum_{k=1}^{|\mathsf{E}|} \iota_{ik} = d(v_i, \mathsf{G}),$$

by Remark 1.15. If on the other hand $i \neq j$, then $\overrightarrow{\iota}_{ik}\,\overrightarrow{\iota}_{jk} \neq 0$ if and only if the edge e_k is incident in v_i *and also* in v_j, meaning that v_i, v_j have to be both endpoints of e_k. This implies that $\overrightarrow{\iota}_{ik}\,\overrightarrow{\iota}_{jk} = -1$, as claimed. □

Remark 4.9. *The above introduced formalism allows for an interesting interpretation of directed graphs. Consider a graph* G *and an arbitrary orientation* $\overrightarrow{\mathsf{G}}$ *of it. Regard any* $w \in \mathrm{C}^{|\mathsf{E}|}$ *as the vector of currents flowing along the oriented edges of* $\overrightarrow{\mathsf{G}}$*:* $w_k = c > 0$ *(resp.,* < 0*) if a current of* c *ampere flows from the initial endpoint of* e_k *towards its terminal endpoint (resp., from the terminal endpoint of* e_k *towards its initial endpoint). Then, Kirchhoff's Current Law states that total current outflow from any node of an electric circuit is 0, that is,*

$$\overrightarrow{\mathcal{I}}\,w = 0.$$

Kirchhoff's Voltage Law can also be rephrased in a linear algebraic formalism: it states that the sum of the potential differences $p_{ij} := V_i - V_j$ *vanishes along any cycle of an electric circuit, that is,* $(z, p) = 0$ *for all cycles* $z \in \mathrm{C}^{|\mathsf{E}|}$ *(given a cycle* C *in* G*, this can be identified with a vector* z *with* $z_k = 1$ *if* $e_i \in \mathsf{C}$*,* $z_k = 0$ *otherwise).*

Exercise 4.10. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph. Denote by* $\zeta(\mathsf{G})$ *the number of cycles in* G*. We introduce the* $\zeta(\mathsf{G}) \times |\mathsf{E}|$***-cycle matrix*** $\mathcal{Z} = (z_{ij})$ *as follows:* $z_{ij} = 1$ *if* e_j *belongs to the* i^{th}*-cycle,* 0 *otherwise.*

Consider an arbitrary orientation $\overrightarrow{\mathsf{G}}$ *of* G *and the associated oriented incidence matrix* $\overrightarrow{\mathcal{I}}$*. Show that* $\mathcal{Z}\,\overrightarrow{\mathcal{I}}^T = 0$*. How can Kirchhoff's First Law be formulated in terms of the cycle matrix?*

Proposition 4.11. *Let* G *be a graph. Consider an arbitrary orientation of each edge* $e \in \mathsf{E}$ *and therefore an oriented graph* $\overrightarrow{\mathsf{G}}$*. If* G *has* $\kappa(\mathsf{G})$ *connected components, then* $\overrightarrow{\mathcal{I}}$ *has rank* $|\mathsf{V}| - \kappa(\mathsf{G})$*.*

Proof. Upon considering a block decomposition and restricting ourselves to a smaller matrix, it suffices to prove the assertion for $k = 1$, i.e., we assume G to be connected. It is clear that summing all the rows $\overrightarrow{\mathsf{G}}$ yields $\sum_{i=1}^{|\mathsf{E}|} \overrightarrow{\iota}_{ik} = 0$ for all $k = 1, \ldots, |\mathsf{V}|$, thus $\operatorname{rang} \overrightarrow{\mathcal{I}} \leq |\mathsf{V}| - 1$. It remains to prove the converse inequality.

Assume that $\operatorname{rang}\overrightarrow{\mathcal{I}} < |\mathsf{V}| - 1$, i.e., that all $(|\mathsf{V}|-1)\times(|\mathsf{V}|-1)$-submatrices of $\overrightarrow{\mathcal{I}}$ are singular, hence they contains linearly dependent rows. Take a node of the graph, say v_{i_0}. Since $\overrightarrow{\mathcal{I}}$ only contains $0, -1, 1$ as entries, this means that

$$\sum_{i\neq i_0} \overrightarrow{\iota}_{ij} = 0 \qquad \text{for all } j \in \mathsf{E}.$$

Each edge is incident in either 0 or 2 (one incoming and one outgoing) nodes *different from* v'. Accordingly, no edge is incident in v_{i_0}. This contradicts the assumption of connectedness of G and concludes the proof. □

Exercise 4.12. *Describe in detail how the general case of a graph with $\kappa(\mathsf{G})$ connected components can be reduced to the connected one.*

Proposition 4.13. *Let $\overrightarrow{\mathsf{G}}$ be a directed graph. Then its incidence matrix is* **totally unimodular**, *i.e., each of its square submatrices has determinant either* 0 *or* -1 *or* 1.

Proof. Let us prove the assertion by induction on the size of the square submatrix. The assertion is clearly true whenever we restrict ourselves to 1×1 submatrices.

Assume now the assertion to hold for $n\times n$-submatrices and consider an $(n+1)\times(n+1)$-submatrix E. If a column of E contains only 0, then we can develop the determinant along it and obtain $\det(E) = 0$. If moreover E contains for each column k two nonzero entries $\overrightarrow{\iota}_{ik}, \overrightarrow{\iota}_{jk}$, then necessarily $\overrightarrow{\iota}_{ik} = -\overrightarrow{\iota}_{jk}$ (i.e., e_k has endpoints v_i, v_j), and summing each row yields the null vector, i.e., $\det(E) = 0$ again.

Let us finally consider the case of E such that a column contains *exactly* one entry $\overrightarrow{\iota}_{ik} \neq 0$ and develop along this column. Since the matrix is totally unimodular, all of its entries must be either 0 or -1 or 1, hence $|\det(E)| = |\overrightarrow{\iota}_{ik}||\det(E_{ik})| = |\det(E_{ik})|$. Since E_{ik} is an $n\times n$-submatrix, the assertion follows. □

Exercise 4.14. *Let $\overrightarrow{\mathsf{G}}$ be a directed graph. Show that the incidence matrix $\mathcal{I}$ is totally unimodular if and only if the associated G is bipartite, cf. Definition 7.21 below.*

Exercise 4.15. *Show that the rank of the incidence matrix of a directed graph equals $|\mathsf{E}| - \mu(\mathsf{G})$.*

Definition 4.16. *A* ***network*** *is a 4-uple $\mathsf{N} := (\overrightarrow{\mathsf{G}}, v_{so}, v_{si}, \operatorname{cap})$, where $\overrightarrow{\mathsf{G}}$ is an oriented graph, $v_{so}, v_{si} \in \mathsf{V}$, $v_{so} \neq v_{si}$, are called* ***source*** *and* ***sink****, and $\operatorname{cap} : \mathsf{E} \to \mathrm{R}_+$ is a mapping, called* ***capacity****, such that $\operatorname{cap}(\overrightarrow{e}) = \operatorname{cap}(\overleftarrow{e}) =: \operatorname{cap}(e)$ for all $e \in \mathsf{E}$.*

For a function $f : \mathsf{E} \to \mathrm{R}$ and $\mathsf{V}_1, \mathsf{V}_2 \subset \mathsf{V}$ we write

$$f(\mathsf{V}_1, \mathsf{V}_2) := \sum_{\overrightarrow{e}\in\overrightarrow{(\mathsf{V}_1,\mathsf{V}_2)}} f(\overrightarrow{e}).$$

Exercise 4.17. *Let $\overrightarrow{\mathsf{G}}$ be an oriented graph and consider a function $f : \mathsf{E} \to \mathrm{R}$.*

(i) Let f satisfy

$$f(\overleftarrow{e}) = -f(\overrightarrow{e}) \qquad \text{for all edges } e \in \mathsf{E}. \tag{4.1}$$

Prove that $f(\mathsf{V}', \mathsf{V}') = 0$ for all $\mathsf{V}' \subset \mathsf{V}$.

(ii) Let f satisfy

$$f(\{v\}, \mathsf{V}) = 0 \qquad \text{for all } v \in \mathsf{V}. \tag{4.2}$$

Prove that $f(\mathsf{V}', \mathsf{V}) = 0$ for all $\mathsf{V}' \subset \mathsf{V}$.

(iii) Let f satisfy (4.1) and (4.2). Prove that $f(\mathsf{V}', \mathsf{V} \setminus \mathsf{V}') = 0$ for all $\mathsf{V}' \subset \mathsf{V}$.

A functions satisfying (4.1) and (4.2) is called a ***circulation*** *in N.*

Definition 4.18. *Let N be a network. A mapping $f : \mathsf{E} \to \mathrm{R}$ is called a* ***flow*** *in N if*

(i) $f(\overleftarrow{e}) = -f(\overrightarrow{e})$ for all edges $e \in \mathsf{E}$,

(ii) $f(\{v\}, \mathsf{V}) = 0$ for all $v \in \mathsf{V} \setminus \{v_{so}, v_{si}\}$, and

(iii) $|f(e)| \le c(e)$ for all $e \in \mathsf{E}$, independently on the orientation.

Exercise 4.19. *Let N be a network and f a flow on it. Show that*

$$f(\{v_{so}\}, \mathsf{V} \setminus \{v_{so}\}) = f(\mathsf{V} \setminus \{v_{si}\}, \{v_{si}\}).$$

Definition 4.20. *Let N be a network. A* ***cut*** *is a pair $(\mathsf{V}', \mathsf{V} \setminus \mathsf{V}')$, where the set $\mathsf{V}' \subset \mathsf{V}$ is such that $v_{so} \in \mathsf{V}'$ and $v_{si} \notin \mathsf{V}'$. The* ***capacity*** *of this cut is given by* $\mathrm{cap}(\mathsf{V}', \mathsf{V} \setminus \mathsf{V}')$.

A flow is not a circulation. In particular, one can see that in general a flow f does *not* satisfy $f(\mathsf{V}', \mathsf{V}) = 0$ for all $\mathsf{V}' \subset \mathsf{V}$. Instead, we have the following.

Lemma 4.21. *Let f be a flow in a network N. Then each cut $(\mathsf{V}', \mathsf{V} \setminus \mathsf{V}')$ in N satisfies $f(\mathsf{V}', \mathsf{V} \setminus \mathsf{V}') = f(\{v_{so}\}, \mathsf{V})$.*

Thus, $f(\mathsf{V}', \mathsf{V} \setminus \mathsf{V}')$ is independent of the chosen cut $(\mathsf{V}', \mathsf{V} \setminus \mathsf{V}')$: its common value $f(\{v_{so}\}, \mathsf{V})$ is denoted by $|f|$ and called the **total value** of the flow f.

Proof. First of all, observe that by definition of flow one has $f(\overleftarrow{e}) = -f(\overrightarrow{e})$ for all edges $e \in \mathsf{E}$, and by Remark 4.6 one has

$$f(\mathsf{V}', \mathsf{V}') = \sum_{\overrightarrow{e} \in \overrightarrow{(\mathsf{V}', \mathsf{V}')}} f(\overrightarrow{e}) = 0 \qquad \text{for any } \mathsf{V}' \subset \mathsf{V}.$$

Furthermore, one has

$$\begin{aligned} f(\mathsf{V}', \mathsf{V} \setminus \mathsf{V}') &= f(\mathsf{V}', \mathsf{V}) - f(\mathsf{V}', \mathsf{V}') \\ &= f(\{v_{so}\}, \mathsf{V}) + \sum_{v \in \mathsf{V}' \setminus \{v_{so}\}} f(\{v\}, \mathsf{V}) - 0 \\ &= f(\{v_{so}\}, \mathsf{V}), \end{aligned}$$

where the last equality follows from the second property defining a flow. □

Lemma 4.22. *Let f be a flow in a network N. Then each cut $(\mathsf{V}', \mathsf{V} \setminus \mathsf{V}')$ in N satisfies $\mathrm{cap}(\mathsf{V}', \mathsf{V} \setminus \mathsf{V}') \ge f(\{v_{so}\}, \mathsf{V})$.*

Proof. One has for each cut $(\mathsf{V}', \mathsf{V} \setminus \mathsf{V}')$

$$f(\mathsf{V}', \mathsf{V} \setminus \mathsf{V}') = \sum_{e \in \overrightarrow{(\mathsf{V}', \mathsf{V} \setminus \mathsf{V}')}} f(e) \le \sum_{e \in \overrightarrow{(\mathsf{V}', \mathsf{V} \setminus \mathsf{V}')}} |f(e)| \le \sum_{e \in \overrightarrow{(\mathsf{V}', \mathsf{V} \setminus \mathsf{V}')}} \mathrm{cap}(e) = \mathrm{cap}(\mathsf{V}', \mathsf{V} \setminus \mathsf{V}').$$

By Lemma 4.21, this concludes the proof. □

Remark 4.23. *It follows from the definition of flow that* $|f| = f(\mathsf{V}', \mathsf{V}\setminus\mathsf{V}') \leq \mathrm{cap}(\mathsf{V}', \mathsf{V}\setminus\mathsf{V}')$ *for any cut* $(\mathsf{V}', \mathsf{V}\setminus\mathsf{V}')$ *in a network* N*. In other words, for any flow* f *on* N *the total value of* f *is always less than or equal to the minimal capacity of a cut, and on the other hand no cut can have capacity less than* $|f|$*. In the following, which is usually called the* ***Max-flow-min-cut Theorem****, we see that there is always a cut such that this identity is satisfied.*

Theorem 4.24 (L.R. Ford – D.R. Fulkerson and P. Elias – A. Feinstein – C. E. Shannon, 1956)**.** *In a network* N *the maximal total value of a flow agrees with the minimal capacity of a cut.*

We prove the theorem in the special case of an N-valued capacity function. This case is particularly relevant, since then it is possible to provide a concrete algorithm for constructing a cut with minimal capacity, once a flow with maximal total value is given. Up to technical details, the general case of $\mathrm{cap} : \mathsf{E} \to \mathrm{R}_+$ can be proved in pretty much the same way, cf. [**2**, Thm. III.1].

Proof. The proof is based on the recursive construction of a sequence of Z-valued flows that is increasing with respect to the maximal value. Define $f_0(\overrightarrow{e}) = 0$ for all $e \in \mathsf{E}$. Assume now that flows $f_0, \ldots, f_n$ are constructed in such a way that $|f_i| < |f_j|$ for all $i < j$, $i, j \in \{0, 1, \ldots, n\}$. Since $|f_n(\overrightarrow{e})| \leq \mathrm{cap}(e)$ for all $e \in \mathsf{E}$ and all $n \in \mathrm{N}$, there exists a flow with maximal total value. Denote by $\mathsf{V}_n \subset \mathsf{V}$ the set containing all the nodes v that are connected with v_{so} by a walk W on each of whose edges e the strict inequality

$$f_n(\overrightarrow{e}) < \mathrm{cap}(e)$$

holds. Since v_{so} is connected with itself by the empty walk, $v_{so} \in \mathsf{V}_n$.

We are going to show that if f_n is a flow with maximal total value, then $v_{si} \notin \mathsf{V}_n$. Assume for a moment that this has already been proved. Then by definition the pair $(\mathsf{V}_n, \mathsf{V}\setminus\mathsf{V}_n)$ is a cut in N. On one hand, each flow f satisfies $f(\overrightarrow{e}) \leq \mathrm{cap}(e)$ for all $e \in \mathsf{E}$. On the other hand, if $e \in \mathsf{E}$ and $f_n(\overrightarrow{e}) < \mathrm{cap}(e)$, then by definition the endpoints of e cannot belong to V_n and $\mathsf{V}\setminus\mathsf{V}_n$, respectively. Accordingly, for each edge $\overrightarrow{e} \in \overrightarrow{(\mathsf{V}_n, \mathsf{V}\setminus\mathsf{V}_n)}$, respectively, one obtains $f_n(\overrightarrow{e}) = \mathrm{cap}(e)$, hence the total value $|f_n|$ agrees with $f_n\overrightarrow{(\mathsf{V}_n, \mathsf{V}\setminus\mathsf{V}_n)} = \mathrm{cap}(\mathsf{V}_n, \mathsf{V}\setminus\mathsf{V}_n)$, as desired.

Let us finally prove that if $v_{si} \in \mathsf{V}_n$, then f is not maximal, i.e., it is still possible to construct a further flow f_{n+1} with (strictly!) larger total value. To this aim, let $v_{si} \in \mathsf{V}_n$. Then by definition v_{so} is connected to v_{si}. Pick any walk $(v_{so} = v_0, v_1, \ldots, v_{t-1}, v_t = v_{si})$. Set

$$\epsilon := \min_{i\in\{0,\ldots,t-1\}} \left(c\overrightarrow{(v_i, v_{i+1})} - f\overrightarrow{(v_i, v_{i+1})} \right).$$

By construction ϵ is a strictly positive integer number. Then, we define a new flow f_{n+1} as follows:

- if $\overrightarrow{e}$ is along the walk W connecting v_{so} to v_{si}, then define $f_{n+1}(\overrightarrow{e}) := f_n(\overrightarrow{e}) + \epsilon$;
- in order to satisfy the definition of a flow, set $f_{n+1}(\overleftarrow{e}) := -f_n(\overrightarrow{e}) - \epsilon$;
- finally, leave the flow unchanged (i.e., $f_{n+1}(\overrightarrow{e}) = f_n(\overrightarrow{e})$) if e is not along W.

One sees that f_{n+1} is actually a flow: properties (i) and (iii) defining a flow are clearly satisfied. Morover, also property (ii) holds: if $v \in \mathsf{V}$ is not along W, then $f_{n+1}(v, \mathsf{V}) =$

$f_n(v, \mathsf{V})$; on the other hand, if v is indeed along such a walk, then the flow along the incoming walk's edge is decreased by ϵ and the flow along the outgoing walk's edge is augmented by ϵ, whereas f_{n+1}'s flows along further edges incident in v are the same of f_n's, so that again $f_{n+1}(v, \mathsf{V}) = f_n(v, \mathsf{V})$.

One also sees that f_{n+1} is Z-valued by construction. By definition of a walk, there is only one (oriented) edge e' with initial endpoint v_{so} and such that $f_n(e') \neq f_{n+1}(e')$ – in fact, $f_n(e') > f_{n+1}(e')$. Accordingly, the total value $f_{n+1}(v_{so}, \mathsf{V})$ of f_{n+1} is larger than the total value of f_n. In other words, we have constructed a flow f_{n+1} in N with total value larger than f_n.

Thus, as long as $t \in \mathsf{V}_n$ it is possible to construct a flow on N with larger total value, i.e., there cannot be a maximal flow. That is, a maximal flow is only possible if $t \notin \mathsf{V}_n$, corresponding to the existence of a cut. Since in that case the total value $|f_n|$ of a maximal flow agrees with $\mathrm{cap}(\mathsf{V}_n, \mathsf{V} \setminus \mathsf{V}_n)$, such a cut is minimal. □

Remark 4.25. *As already observed, Menger's Theorem can be derived from the Max-flow-min-cut Theorem. More precisely, Corollary 3.16 (which in turns implies Menger's Theorem) is a direct consequence of the Max-flow-min-cut Theorem, as we can see in the following way. Let v, w be non-adjacent nodes of a simple graph. Replace the graph V by a network by orienting all edges (allowing both orientations) and considering v, w as the network's source and sink, respectively. Assign capacity* 1 *to each edge, so that also the maximal flow (whose existence is claimed by the Max-flow-min-cut Theorem) is* Z*-valued, i.e., the maximal flow is* 0, 1, *or* -1 *in each edge, cf.* [**2**, Thm. III.2.5] *for details.*

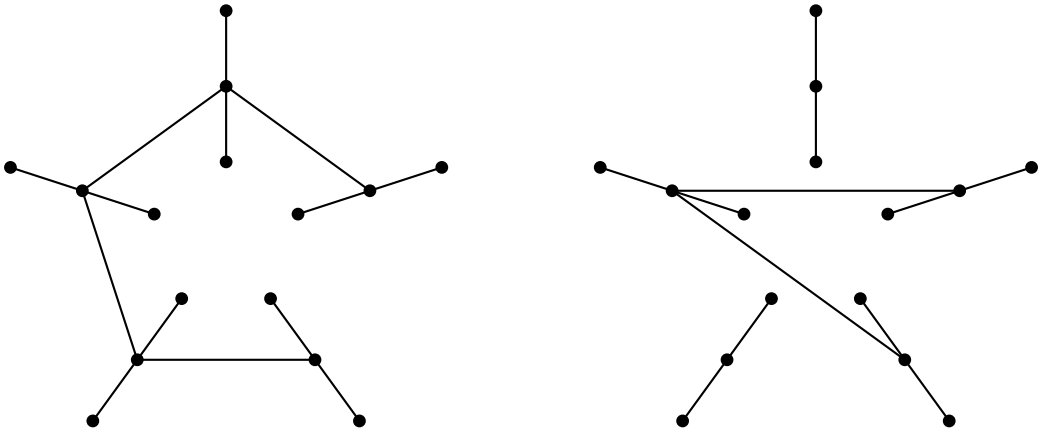
Remark 4.26. *Directed graphs, and in particular networks, play an important role in applied graph theory and computer sciences. For an introduction to these topics, see* [**4**].

CHAPTER 5

Trees and spanning trees

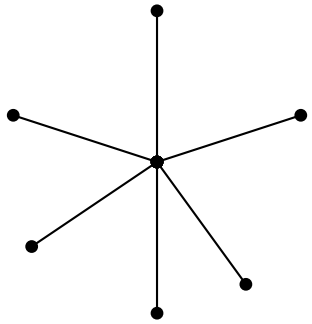
Definition 5.1. *A simple graph not containing any cycle as subgraph is called a* ***forest****. A connected forest is called a* ***tree****. Each node of degree* 1 *is called a* ***leaf****. If, up to a single node* v*, all nodes of a tree are leaves, then the tree is called a* ***star with center*** v*.*

Observe that the center of a star is the star's only central node.



A forest.

Remark 5.2. *Let* $m \geq 2$*. A star with* $m-1$ *edges is sometimes called an* m***-star*** *and denoted* S_m*. The* 4*-star is the claw introduced in Example 3.20.(2).*



A 7*-star.*

In particular, by definition any two nodes of a tree are connected by exactly one path.

Exercise 5.3. *Prove that the following conditions on a simple graph* G *with* n *nodes are equivalent.*

(a) G *is a tree.*

(b) G *is connected and has* $n-1$ *edges.*
(c) G *does not contain cycles and has* $n-1$ *edges.*

Corollary 5.4. *A tree with at least two nodes contains at least two leaves.*

PROOF. Count the n nodes of the tree T in such a way that their degree is increasing, i.e., $d(v_1,\mathsf{T}) \le d(v_2,\mathsf{T}) \le \ldots \le d(v_n,\mathsf{T})$. The minimal degree cannot be 0, since a tree is connected. In order to prove the assertion, assume T to have at most one node of degree 1, i.e., that $d(v_2,\mathsf{T}) \ge 2$. Thus, by Exercise 5.3 we deduce that

$$2(n-1) = 2|\mathsf{E}| = d(v_1,\mathsf{T}) + \ldots + d(v_n,\mathsf{T}) \ge 1 + d(v_2,\mathsf{T}) + \ldots + d(v_n,\mathsf{T}) \ge 1 + 2(n-1),$$

a contradiction. □

As a consequence of Exercise 1.14 we obtain the following characterization of a tree.

Exercise 5.5. *Let* $\mathsf{G} = (\mathsf{V},\mathsf{E},g)$ *be a nontrivial connected graph. Show that* G *is a tree if and only if*

$$|\mathsf{V}_{(1)}| = 2 + \sum_{i=3}^{\Delta(\mathsf{G})} (i-2)|\mathsf{V}_{(i)}|. \tag{5.1}$$

Example 5.6. *Molecules consisting of carbon and hydrogen of the form* C_nH_{2n+2}, $n \in \mathbb{N}$, *are called* ***n-alkanes*** *and* ***iso-alkanes*** *(depending on chemical properties). If we describe such molecules as a graph, associating to each atom a node, we can show that all alcanes are trees.*

In fact, associate to each carbon atom a node $v_1,\ldots,v_n$. *Since carbon has valence 4, each node* v_i *has degree 4. Moreover, associate to each hydrogen atom a node* $w_1,\ldots,w_{2n+2}$. *Since hydrogen has valence 1, each node* w_i *has degree 1.*

Accordingly, the graph G *describing this structure satisfies*

$$|\mathsf{V}_1| = 2n+2 = 2 + \sum_{i=3}^{\Delta(\mathsf{G})} (i-2)|\mathsf{V}_{(i)}|.$$

Since an atom clearly has a connected structure, we deduce by Exercise 5.5 that G *is a tree.*

Exercise 5.7. *Let* $\mathsf{G} = (\mathsf{V},\mathsf{E},g)$ *be a tree. Show that* G *has either one central node, or two adjacent central nodes.*
(Hint: What happens to central nodes of G if all leaves are removed, i.e., if we pass to $\mathsf{G} - \mathsf{V}_{(1)}$?)

Definition 5.8. *Let* $\mathsf{G} = (\mathsf{V},\mathsf{E},g)$ *be a graph.*

If G' *is a subgraph of* G, *then* V' *is said to* ***span*** G' *in* G *if* $\mathsf{V}' \subset \mathsf{V}$ *and furthermore for all* $e \in \mathsf{E}$ *such that* $g(e) = \{v,w\}$ *with* $v,w \in \mathsf{V}'$ *one also has* $e \in \mathsf{E}'$. *In other words,* V' *spans* G' *in* G *if and only if* G' *is the subgraph induced by* V' *in* G. *The subgraph* G' *is said to be* ***spanning*** *if it spans the whole* G, *i.e., if its node set agrees with the node set of* G.

If G' *is a subgraph of* G, *then* $\tilde{\mathsf{E}} \subset \mathsf{E}$ *is said to* ***generate*** G' *if* $\mathsf{E}' = \tilde{\mathsf{E}}$ *and* V' *is the set of all endpoints of the edges in* $\tilde{\mathsf{E}}$. *In this case we write* $\mathsf{G}[\tilde{\mathsf{E}}]$.

In the following, $\mu(\mathsf{G})$ and $\zeta(\mathsf{G})$ denote as usual the cyclomatic number of G and the number of cycles in G, respectively.

Theorem 5.9 (W. Ahrens, 1897)**.** *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a connected graph. Then*

$$\zeta(\mathsf{G}) \leq 2^{\mu(\mathsf{G})} - 1. \tag{5.2}$$

Proof. Consider a spanning tree T of G and define the set $\mathcal{A}$ of all nonempty subsets of $\mathsf{E}(\mathsf{G}) \setminus \mathsf{E}(\mathsf{T})$, whose cardinality is of course $2^{|\mathsf{E}(\mathsf{G})\setminus\mathsf{E}(\mathsf{T})|} - 1$, i.e., by Remark 5.12, $2^{\mu(\mathsf{G})} - 1$.

We denote by $\mathcal{C}$ the set of all cycles in G. We introduce a mapping $h : \mathcal{C} \ni \mathsf{C} \mapsto \mathsf{E}(\mathsf{C}) \setminus \mathsf{E}(\mathsf{T}) \in \mathcal{A}$, which is well-defined by construction of $\mathcal{A}$. By definition, $|\mathcal{C}| = \zeta(\mathsf{G})$, thus it remains to show that h is injective in order to complete the proof. Assume there exist two different cycles $\mathsf{C}_1, \mathsf{C}_2$ such that $h(\mathsf{C}_1) = h(\mathsf{C}_2)$, i.e., such that $\mathsf{E}(\mathsf{C}_1) \setminus \mathsf{E}(\mathsf{T}) = \mathsf{E}(\mathsf{C}_2) \setminus \mathsf{E}(\mathsf{T})$. Consider the graph $\mathsf{G}[\mathsf{E}(\mathsf{C}_1)\Delta\mathsf{E}(\mathsf{C}_2)]$ generated by the symmetric difference $\mathsf{E}(\mathsf{C}_1)\Delta\mathsf{E}(\mathsf{C}_2)$, i.e., the graph whose edge set is

$$\{e \in \mathsf{E}(\mathsf{C}_1) : e \notin \mathsf{E}(\mathsf{C}_2)\} \cup \{e \in \mathsf{E}(\mathsf{C}_2) : e \notin \mathsf{E}(\mathsf{C}_1)\}$$

and whose node sets agrees with the set of all endpoints of its edges. Obviously, $\mathsf{G}[\mathsf{E}(\mathsf{C}_1)\Delta\mathsf{E}(\mathsf{C}_2)]$ is a nonempty subgraph of T and moreover its minimal degree is at least 2, since nodes belonged to at least one cycle. By Proposition 2.3, $\mathsf{G}[\mathsf{E}(\mathsf{C}_1)\Delta\mathsf{E}(\mathsf{C}_2)]$ contains a cycle, a contradiction to the inclusion of this graph in a tree. □

Finding *trees* that span a given graph is an important task. In fact, a *spanning tree* permits to reach each node of the graph while avoiding unnecessary cycles – think of the problem of connecting all towns in a region to a railway network.

Proposition 5.10 (G. Kirchhoff, 1847)**.** *Each connected graph* G *is spanned by some tree.*

The idea is that it is always possible to consider a connected spanning subgraph of minimal length. A precise construction is given below.

Proof. We can assume G to contain cycles, otherwise it would already be a tree. Construct a sequence of subgraphs as follows:

- let e_1 be an edge of a cycle C_1 of G_0 and set $\mathsf{G}_1 := \mathsf{G}_0 - \{e_1\}$,

  ⋮
- let e_γ be an edge of a cycle C_γ of $\mathsf{G}_{\gamma-1}$ and set $\mathsf{G}_\gamma := \mathsf{G}_0 - \{e_\gamma\}$,

(Observe that all the graphs $\mathsf{G}_0, \ldots, \mathsf{G}_\gamma$ share the same node set).

This process has to stop at some step, say $\gamma \in \mathrm{N}$, whenever G_γ has no cycles. Thus, we have obtained a tree $\mathsf{T} = \mathsf{G} - \{e_1, \ldots, e_\gamma\}$ whose node set agrees with the node set of G. This completes the proof. □

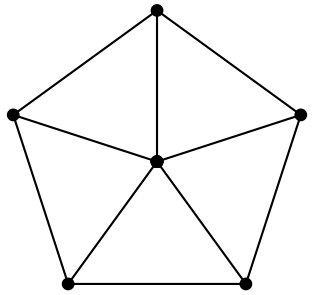
Exercise 5.11. *Does any 2-edge-connected simple graph also have two edge-disjoint spanning trees? Why?*

Remark 5.12. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a connected graph. Observe that if* T *with edge set* E' *(and of course with node set* $\mathsf{V}' = \mathsf{V}$*) is a spanning tree of* G*, then by Exercise 5.3* $|\mathsf{E}'| = |\mathsf{V}| - 1$ *and accordingly*

$$|\mathsf{E} \setminus \mathsf{E}'| = |\mathsf{E}| - |\mathsf{V}| + 1,$$

the cyclomatic number of G*.*

Exercise 5.13. *Consider the following construction of a graph* W^k_{n+1}, $k = 0, 1, \ldots$. *Take* $\mathsf{V} = \{v_0, v_1, \ldots, v_n\}$ *and consider a star whose center is* v_0 *and with node set* V. *The graph* W^k_{n+1} *is then defined as the graph that is constructed adding to this star* kn *edges – k (parallel) ones between each pair of nodes* $v_i, v_i + 1$, $i = 1, \ldots, n-1$ *as well as between* v_n, v_1. *(If* $k = 1$, *then such a graph is called* $(n+1)$-***wheel*** *and is often denoted by* W^{n+1}.*)*



The W^6.

Consider the cases

- $k = 0$,
- $k = 1$,
- $k \geq 2$.

In which of these cases does (5.2) *hold* as an equality*? Why?*

5.1. The Matrix–Tree Theorem

.

Aim of this section is to present an important result on spanning trees, which has likely alredy been known by G. Kirchhoff. We follow the proof proposed by H. Trent in 1954, as presented in [**12**, Kap. III], but several others are known.

Recall the following important result in linear algebra, which generalize the usual multiplicativity property of the determinant.

Theorem 5.14 (A.–L. Cauchy – J.P.M. Binet)**.** *Let* A *be an* $m \times n$*-matrix and* B *be an* $n \times m$*-matrix, with* $m \leq n$. *Let* $\mathcal{S}$ *be the set of the* $\binom{n}{m}$ m*-element subsets of* $\{1, \ldots, n\}$ *and let* $S \in \mathcal{S}$. *Denote by* A_S *(resp.,* B_S*) the* $m \times m$*-matrix whose columns (resp., rows) are those columns (resp., rows) of* A *(resp.,* B*) whose indices are those in* S. *Then the formula*

$$\det(AB) = \sum_{S \in \mathcal{S}} \det(A_S) \det(B_S)$$

holds.

Lemma 5.15. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph. If* G' *is a subgraph of* G *with* $|\mathsf{V}|$ *nodes and* $|\mathsf{V}| - 1$ *edges, then* G' *is a tree if and only if its incidence matrix has rank* $|\mathsf{V}| - 1$.

Proof. By Proposition 4.13 the incidence matrix of G' is totally unimodular. By Proposition 4.11, its determinant vanishes if and only if G' is not connected, i.e., if and only if G' is not a tree (by Exercise 5.3). □

In order to state the following **Matrix–Tree Theorem** we still have to introduce some notation. For a general $m \times n$-matrix A, we denote by $A_{[i]}$ the matrix obtained by removing from A the i^{th} row. If B is a square matrix, B_i denotes the matrix obtained by removing both the i^{th} row and the i^{th} column. It is easy to see that

$$B_{[i]}B_{[i]}^T = (BB^T)_i. \tag{5.3}$$

Although $B_{[i]}$ is not a square matrix (so that we cannot apply the determinant's multiplicativity), $\det((BB^T)_i)$ can be computed by the Cauchy–Binet theorem.

Theorem 5.16 (G. Kirchhoff, 1847)**.** *Let $\mathsf{G} = (\mathsf{E}, \mathsf{V}, g)$ be a graph with $|\mathsf{V}| \geq 2$. The number $z(\mathsf{G})$ of all spanning trees of G agrees with any minor $\det(\mathcal{B}_i)$ of order $|V| - 1$, $i = 1, \ldots, |\mathsf{V}|$.*

Remark 5.17. *Intuitively, the more spanning trees are contained in a graph, the more interconnected is the model described by the graph. Variants of the matrix tree theorem have therefore often been used in the topological investigation of (electrical, social, biological...) networks.*

Proof. Consider an arbitrary orientation of G and recall that by Proposition 4.7 the admittance matrix is given by $\mathcal{B} = \mathcal{D} - \mathcal{A} = \overrightarrow{\mathcal{I}}\,\overrightarrow{\mathcal{I}}^T$. Consider a minor $\det(\mathcal{B}_i)$ of order $|\mathsf{V}| - 1$, $i = 1, \ldots, |\mathsf{V}|$, or rather

$$\det\left(\overrightarrow{\mathcal{I}}_{[i]}\overrightarrow{\mathcal{I}}_{[i]}^T\right).$$

In order to compute this determinant by the Cauchy–Binet theorem, introduce the set $\mathcal{S}$ of $(|\mathsf{V}| - 1)$-element subsets of $\{1, \ldots, |\mathsf{E}|\}$ and consider all the $|\mathcal{S}|$ subgraphs G_S of G having edges $\{e_k : k \in S\}$. Observe that $\mathsf{G}_S = \mathsf{G}_{S'}$ if and only if $S = S'$. We denote by $\overrightarrow{\mathcal{I}}_S$ the associated $|\mathsf{V}| \times (|\mathsf{V}| - 1)$-incidence matrix. By Lemma 5.15, G_S is a tree if and only if $\operatorname{rank}(\overrightarrow{\mathcal{I}}_S) = |\mathsf{V}| - 1$. Accordingly, for some fixed $i \in \{1, \ldots, |\mathsf{V}|\}$, $\det((\overrightarrow{\mathcal{I}}_S)_{[i]}) \neq 0$ if and only if $\det((\overrightarrow{\mathcal{I}}_S)_{[i]})\det((\overrightarrow{\mathcal{I}}_S^T)_{[i]}) = |\det((\overrightarrow{\mathcal{I}}_S)_{[i]})|^2 = 1$ if and only if G_S is a tree.

Summing up,

$$\det(\mathcal{B}_i) = \sum_{S \in \mathcal{S}} \det((\overrightarrow{\mathcal{I}}_S)_{[i]})\det((\overrightarrow{\mathcal{I}}_S^T)_{[i]}) = \sum_{S \in \mathcal{S}} |\det((\overrightarrow{\mathcal{I}}_S)_{[i]})|^2,$$

and by the above observation such a sum counts exactly the (distinct) trees contained in G. In order to complete the proof it suffices to observe that for all $i \in \{1, \ldots, |\mathsf{V}|\}$ and for all $S \in \mathcal{S}$ one has $(\overrightarrow{\mathcal{I}}_S)_{[i]} = (\overrightarrow{\mathcal{I}}_{[i]})_S$. □

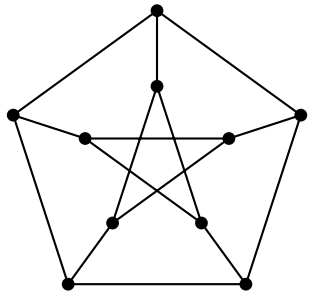
A direct consequence of the matrix-tree-theorem is the following.

Theorem 5.18 (A. Cayley, 1889)**.** *There exist exactly n^{n-2} spanning trees of the complete graph K^n, i.e., $\zeta(\mathsf{K}^n) = n^{n-2}$.*

Exercise 5.19. *1) Provide the details of the proof of Cayley's Theorem, proving the theorem by induction and/or using* `MAPLE`*.*

2) Find the number of spanning trees of the d-dimensional cube.

Exercise 5.20. *The Petersen Graph is a 3-regular graph with 10 nodes.*



The Petersen Graph.

Show (possibly with the help of MAPLE*) that the Petersen Graph has 2000 spanning trees.*

Remark 5.21. *Similarly to the Matrix–Tree Theorem, there exists a formula for counting the Euler tours of a directed graph. In fact, a directed graph* $\overrightarrow{\mathsf{G}}$ *contains exactly*

$$\det(\mathcal{B}_j)\prod_{v\in\mathsf{V}}(d_i(v)-1)!$$

Euler tours, for any minor $\det(\mathcal{B}_j)$ *of order* $|V|-1$*,* $j=1,\ldots,|\mathsf{V}|$*. This result is due to N.G. de Bruijn, T.P. Ehrenfest, C.A.B. Smith, and W.T. Tutte, and is therefore known as* BEST-Theorem*.*

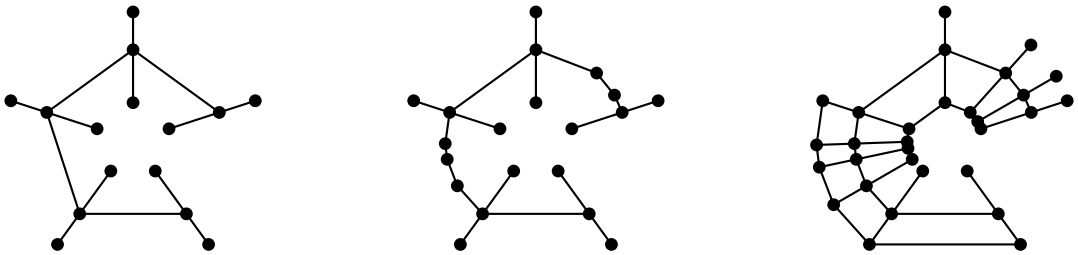
5.2. Kruskal's Tree Theorem

This section is based on [**8**, Chapt. 12].

Definition 5.22. *Let* $\mathsf{G}=(\mathsf{V},\mathsf{E},g)$ *be a graph.*

A further graph G' *is called a* ***subdivision*** *of* G *if* G *is isomorphic to* G' *up to replacing some edges of* G *by independent paths connecting the edges' endpoints.*

If a subdivision of G *is isomorphic to a subgraph of a further graph* $\tilde{\mathsf{G}}$*, then* G *is called a* ***topological minor*** *of* $\tilde{\mathsf{G}}$*.*



A graph G *(left), a subdivision of* G *(middle), and a graph* G *is a topological minor of (right).*

Definition 5.23. *A* ***quasi-ordering*** *on a set* M *is a reflexive and transitive relation* $\leq$ *on* M*, i.e. a subset of* $M\times M$ *such that for all* $x,y,z\in M$ *one has*

- $x\leq x$ *and*
- $x\leq y$ *and* $y\leq z$ *imply* $x\leq z$*.*

Example 5.24. *1) The usual $\leq$-relation defines a quasi-ordering on* R.

2) The relation $\leq_{\mathrm{R}^2} := \{(x_1, x_2), (y_1, y_2) : x_1 \leq y_1 \text{ and } x_2 \leq y_2\}$ defines a quasi-ordering on R^2. Observe that not all elements are comparable.

Given a sequence $(x_n)_{n\in\mathrm{N}} \subset M$, if there exist indices $i, j \in \mathrm{N}$, $i < j$, such that $x_i \leq x_j$, the pair (x_i, x_j) is called ***good*** *(with respect to the quasi-ordering $\leq$). If a sequence contains a good pair, it is called* ***good****, otherwise it is called* ***bad****. A quasi-ordering such that each sequence is good with respect to it is called a* ***well-quasi-ordering****.*

Example 5.25. *1) The sequence $(1, 2^{-1}, 3^{-1}, 4^{-1}, \ldots)$ is bad with respect to the usual $\leq$-order.*

2) The sequence $(2^{-1}, 1, 3^{-1}, 4^{-1}, \ldots)$ is good with respect to the usual $\leq$-order.

Our aim in the remaind of this chapter is to prove an interesting result concerning infinite sequences of (finite) trees. To this purpose, we first need a result on well-quasi-orderings: we only quote it from [**8**, § 12.1]. It is noteworthy that, while its statement only involves notions from the theory of ordered sets, its proof essentially relies upon graph theoretical methods. Some further interplay between these both fields can be found in [**8**, § 2.5].

Lemma 5.26. *A quasi-ordering $\leq$ on a set M is a well-quasi-ordering if and only if it contains neither an infinite* ***antichain****, i.e., an infinite set of pairwise non-comparable elements, nor an infinite sequence $(x_n)_{n\in\mathrm{N}}$ such that $x_0 > x_1 > \ldots$.*

Example 5.27. *1) The set $\{(n, n^{-1}) : n \in \mathrm{N}\}$ is an antichain with respect to $\leq_{\mathrm{R}^2}$.*

2) The set N of natural numbers is well-quasi-ordered with respect to the usual $\leq$-order. The set Z of integer numbers is not.

3) If for $p, q \in \mathrm{N}$ we write $p|q$ if p is a divisor of q, then $|$ does not define a well-quasi-ordering of N, since the prime numbers constitute an infinite antichain.

Example 5.28. *Fix (arbitrarily) a node v_R of a tree T, which we later on refer to as* ***root*** *– in this context T is said to be a* ***rooted tree****. We introduce a relation on the node set of T by setting $v \leq w$ if $\mathrm{dist}(v, v_R) \leq \mathrm{dist}(w, v_R)$. All nodes of T are comparable: by Lemma 5.26 this implies that $\leq$ is a well-quasi-ordering.*

Exercise 5.29 (J. Kőnig, 1916)**.** *Prove that a graph is bipartite if and only if it contains no cycle of odd length.*
(Hint: Consider a spanning tree with a root v_0 and the relation on V defined by vRw if the path connecting v, v_0 and w, v_0 have same length (mod2).)

Exercise 5.30. *Consider a relation $\preceq$ in R^2 defined as follows: $(x_1, y_1) \preceq (x_2, y_2)$ if $x_1 \leq x_2$* and *$y_1 \leq y_2$. Moreover, consider a set $M \subset \{(x, y) \in \mathrm{R}^2 : x > 0,\ y > 0\}$ and define an oriented graph $\overrightarrow{\mathsf{G}}$ setting $\mathsf{V} := M$ and connecting $w \in \mathsf{V}$ to $v \in \mathsf{V}$ if, among all nodes that are $\preceq w$, the euclidean distance between v, w is minimal. Answer the following questions and justify your answer.*

(1) Does $\preceq$ defines a quasi-ordering? Does it define a well-quasi-ordering?

(2) Is the graph $\overrightarrow{\mathsf{G}}$ a forest? If this is generally not the case, under which assumptions on M is this true?

It is possible to induce the quasi-order of a set M onto the set of its finite subsets by setting $M_1 \leq M_2$ if there is an injective mapping $f : M_1 \to M_2$ such that $x \leq f(x)$ for all $x \in M_1$, where the last inequality holds with respect to the quasi-order of M.

Lemma 5.31 (G. Highman, 1952)**.** *If a set is well-quasi-ordered, then so is the set of its finite subsets.*

Recall that a **partial order** on a set M is a quasi-ordering that is antisymmetric, i.e., such that for all $x, y \in M$ one has $x = y$ whenever $x \leq y$ *and* $y \leq x$ hold. (The ordering is said to be *partial* because not any pair $(x, y) \in M \times M$ needs to be element of the relation; if $x \leq y$ or $y \leq x$, then x, y are said to be *comparable*).

Example 5.32. *On a tree with root v_R it is possible to define a partial ordering (so-called **tree ordering**) by writing $v_1 \leq_{v_R} v_2$ if v_1 belongs to the unique path connecting the root v_R to v_2.*

*After fixing a node v_R, T is called **normal** if any two nodes of T are comparable (with respect to $\leq_{v_R}$ provided that they are adjacent in G. It is possible to obtain a slightly stronger result than Proposition 5.10: the spanning tree can be taken to be normal, cf.* [**8**, Thm. 0.5.6].

Example 5.33. *Consider again the well-quasi-ordering on a rooted tree introduced in Example 5.28. From $v \leq w$ and $w \leq v$ only follows that nodes v, w have the same distance from the root v_R, i.e., "they are at the same level of the tree", but in general this does* not *yield $v = w$. Therefore, $\leq$ is not a partial ordering.*

Definition 5.34. *Let T, T' be trees and fix roots v_R, v'_R, respectively. Write $\mathsf{T} \trianglelefteq \mathsf{T}'$ if T is a topological minor of T' with respect to an* order *isomorphism, i.e., if there exists an isomorphism ϕ of a subdivision of T into a subtree of T' such that $v \leq_{v_R} w$ implies $\phi(v) \leq_{v'_R} \phi(w)$ for all v, w nodes of T. We call $\trianglelefteq$ the **order-topological minor relation** and we say that T is an **order-topological minor** of T'.*

Lemma 5.35. *Let $\mathcal{V}$ be a set. The order-topological minor relation $\trianglelefteq$ is a quasi-ordering in the set of finite trees with node set $\mathsf{V} \subset \mathcal{V}$.*

Proof. Reflexivity is clear: it suffices to consider the identity of a tree T onto itself. Let now a subdivision of a tree T be order-isomorphic (via ϕ) to a subgraph of a further tree T', and a subdivision of T be order-isomorphic (via ψ) to a subgraph of a third tree T'', then it is clear that the same subdivision of T is order-isomorphic (via $\psi \circ \phi$) to a subgraph of T''. □

Theorem 5.36 (J.B. Kruskal, 1960)**.** *Let $\mathcal{V}$ be a set. The order-topological minor relation $\trianglelefteq$ is a well-quasi-ordering in the set of finite trees with node set $\mathsf{V} \subset \mathcal{V}$.*

Proof. Assume $\trianglelefteq$ not to be a well-quasi-ordering of the finite trees. To begin with, we construct recursively a bad sequence (in the sense of Definition 5.23) of rooted trees $(\mathsf{T}_n)_{n \in \mathbb{N}}$: we assume that a finite sequence of rooted trees $(\mathsf{T}_i)_{i \leq n}$ has already been found in such a way that a bad sequence of rooted trees begins with it. First of all, the considered set of finite trees is assumed to contain a bad sequence, which clearly begins with the empty sequence. Moreover, if we have determined $(\mathsf{T}_i)_{i \leq n}$, pick a rooted tree T_{n+1} which

is minimal (with respect to the cardinality of its node set) among those that appear on the $(n+1)^{\text{th}}$ coordinate of a bad sequence. This well-defines a bad sequence $(\mathsf{T}_n)_{n\in\mathrm{N}}$ of rooted trees, whose roots we denote by v_{R_n}. By definition of tree, each graph $\mathsf{T}_n - \{v_{R_n}\}$ is unconnected: denote by A_n the forest that arises as the set of connected components of $\mathsf{T}_n - \{v_{R_n}\}$. Observe that we may and do regard each A_n as a *finite* set of trees, each rooted in a node adjacent to v_{R_n} (there are $d(v_{R_n})$ of them). On each connected component in A_n there is a natural tree-order induced by the original tree-order $\leq_{v_{R_n}}$ of T_n as introduced in Example 5.32.

Assume for a moment that $\bigcup_{n\in\mathrm{N}} \mathsf{A}_n$ is well-quasi ordered. Once this has been actually proved, we consider the sequence $(\mathsf{A}_n)_{n\in\mathrm{N}}$ of finite subsets of $\bigcup_{n\in\mathrm{N}} \mathsf{A}_n$, we apply Lemma 5.31 and deduce that such a sequence is good, i.e., it contains a good pair, say $(\mathsf{A}_i, \mathsf{A}_j)$. By definition, this means that $\mathsf{A}_i \trianglelefteq \mathsf{A}_j$, i.e., there exists an injective mapping $f : \mathsf{A}_i \to \mathsf{A}_j$ such that $\mathsf{T} \trianglelefteq f(\mathsf{T})$ for all $\mathsf{T} \in \mathsf{A}_i$. Since each tree T inside the forest obtained deleting $\{v_{R_i}\}$ from T_i is an order-topological minor of the corresponding tree $f(\mathsf{T})$ in the forest $\{v_{R_j}\}$ from T_j, it is possible to extend f to an injective mapping $\tilde{f} : \mathsf{T}_i \to \mathsf{T}_j$ by setting $f(v_{R_i}) := v_{R_j}$. Since f preserves the order along each tree in A_i, also $\tilde{f}$ clearly preserves the order in the whole tree T_i, where (up to subdivision) the edge of T_i connecting v_{R_i} to the root of a tree T in A_i is mapped into a path in T_j univocally. This in turn yields that $(\mathsf{T}_n)_{n\in\mathrm{N}}$ is good, a contradiction to the fact that such a sequence is bad by construction.

In order to conclude the proof it thus remains to prove that $\bigcup_{n\in\mathrm{N}} \mathsf{A}_n$, is well-quasi-ordered. To this aim, take a sequence $(\mathsf{T}^k)_{k\in\mathrm{N}}$ in $\bigcup_{n\in\mathrm{N}} \mathsf{A}_n$ – i.e., a sequence of trees obtained deleting the root of rooted trees T_n, $n \in \mathrm{N}$ – and show that it is good. It is possible to pick for all $k \in \mathrm{N}$ an $n_k \in \mathrm{N}$ such that $\mathsf{T}^k \in \mathsf{A}_{n_k}$. Let $k^* \in \mathrm{N}$ satisfy $n_{k^*} \leq n_k$ for all $k \in \mathrm{N}$, i.e., let n_{k^*} be a minimum of $k \mapsto n_k$. Observe that by construction T^k is *strictly* contained in T_{n_k}, i.e.,

$$\mathsf{T}^k \trianglelefteq \mathsf{T}_{n_k} \qquad \text{and} \qquad \mathsf{T}^k \neq \mathsf{T}_{n_k}, \qquad k \in \mathrm{N}. \tag{5.4}$$

Consider the sequence

$$(\mathsf{T}_1, \ldots, \mathsf{T}_{n_{k^*}-1}, \mathsf{T}^{k^*}, \mathsf{T}^{k^*+1}, \ldots),$$

which is necessarily good, since $\mathsf{T}_{n_{k^*}}$ is by construction the minimal tree among those that appear as the $(n_{k^*})^{\text{th}}$ entry of a bad sequence. Thus, by definition this sequence contains a good pair, say $(\mathsf{T}, \mathsf{T}')$. Our goal is to prove that T, T' are entries of $(\mathsf{T}^k)_{k\in\mathrm{N}}$.

Observe that T, T' cannot possibly agree with some of the trees $\mathsf{T}_1, \ldots, \mathsf{T}_{n_{k^*}-1}$, otherwise this would contradict badness of $(\mathsf{T}_k)_{k\in\mathrm{N}}$. We can also exclude the case of $\mathsf{T} = \mathsf{T}_\ell$ and $\mathsf{T}' = \mathsf{T}^m$ for some $\ell \leq n_{k^*} - 1$ and $m \geq k^*$. This would namely yield $\mathsf{T} = \mathsf{T}_\ell \trianglelefteq \mathsf{T}_{n_m}$, i.e., $(\mathsf{T}_\ell, \mathsf{T}_{n_m})$ would be a good pair (this is justified by the fact that surely $n_{k^*} - 1 \leq n_m$). □

By Kruskal's theorem, the following holds.

Corollary 5.37. *In every infinite set of trees there are two such that one is an order-topological minor of the other.*

Remark 5.38. *Kruskal's Theorem plays a relevant role in computer science, cf.* [**10**] *for a thorough discussion of the consequences of Kruskal's result on the celebrated Knuth–Bendix Completion Algorithm.*

It is interesting to observe that general *graphs are* not *well-quasi-ordered by the topological minor relation. However, they are indeed well-quasi-ordered by a relation based on the notion of* minor *(see exercises to this lecture). This is the celebrated* ***graph minor theorem*** *proved by N. Robertson and P.D. Seymour between 1983 and 2004, one of the deepest results of discrete mathematics. A survey of their result and some of the involved techniques can be found in* [**14**].

5.3. Arborescences and Kirchhoff's laws

Before concluding this chapter we present an axiomatic approach to Kirchhoff's laws of electric circuits as developed in the 1940s a series of articles by R.L. Brooks, C.A.B. Smith, A.H. Stone, and W.T. Tutte, cf. [**20**, § VI.5] and [**21**, Chapt. 4]. Throughout this section we abstract from the usual notion of network by considering an oriented graph $\overrightarrow{\mathsf{G}}$ each of whose edges is provided with a capacity function cap. In usual (electrical) applications, cap(e) will represent the conductance of the edge e.

We make the notion of oriented spanning graph/tree precise.

Definition 5.39. *A* ***multiple arborescence*** *in* $\overrightarrow{\mathsf{G}}$ *is an (oriented) subgraph each of whose connected components is a rooted tree (with roots' indegree* $= 0$ *and all other nodes' indegree* $= 1$*) and such that its node set agrees with the node set of* G.

In particular, we can also regard an arborescence as an *oriented forest*, i.e., a set $(\overrightarrow{\mathsf{T}}_1, \dots, \overrightarrow{\mathsf{T}}_n)$.

A **single** (resp., **double**) **arborescence** is a multiple arborescence with one (resp., two) connected component(s).

Let N be a network. Moreover, let us denote by A^v (resp., A^v_w) the set of all single (resp., double) arborescences whose connected components are rooted in v (resp., in v and w). Let us introduce the notations[1]

$$\langle v \rangle := \sum_{\mathsf{T} \in \mathsf{A}^v} \prod_{e \in E(\mathsf{T})} \mathrm{cap}(e),$$

as well as

$$\langle v, w \rangle := \sum_{(\mathsf{T}_1, \mathsf{T}_2) \in \mathsf{A}^v_w} \prod_{e \in E(\mathsf{T}_1, \mathsf{T}_2)} \mathrm{cap}(e).$$

Observe that if $\mathrm{cap} \equiv 1$, then $\langle v \rangle$ and $\langle v, w \rangle$ simply count the single/double arborescences rooted in v and v, w, respectively.

More generally, we write $\mathsf{A}^{v_0 v_1 \dots v_m}$ (resp., $\mathsf{A}^{v_0 v_1 \dots v_m}_{w_0 w_1 \dots w_n}$) to denote the set of all single (resp., double) arborescences whose connected componentes are rooted in v (resp., in v and w) and containing $v_1, \dots, v_m$ (resp., $v_1, \dots, v_m$ *and* $w_1, \dots w_n$) as nodes – regardless of their

[1] Here and in the following $E(\mathsf{T})$ (resp., $E(\mathsf{T}_1, \mathsf{T}_2)$) denotes the edge set of an oriented tree $\overrightarrow{\mathsf{T}}$ (resp., of a graph consisting of two trees $\overrightarrow{\mathsf{T}_1}, \overrightarrow{\mathsf{T}_2}$).

order: that is, for example, $\mathsf{A}^{v_0v_1v_2\cdots} = \mathsf{A}^{v_0v_2v_1\cdots}$.[2] Then similarly we write

$$\langle v_0v_1\dots v_m\rangle := \sum_{\mathsf{T}\in\mathsf{A}^{v_0v_1\cdots v_m}} \prod_{e\in E(\mathsf{T})} \mathrm{cap}(e)$$

as well as

$$\langle v_0v_1\dots v_m, w_0w_1\dots w_n\rangle := \sum_{(\mathsf{T}_1,\mathsf{T}_2)\in\mathsf{A}^{v_0v_1\cdots v_m}_{w_0w_1\cdots w_n}} \prod_{e\in E(\mathsf{T}_1,\mathsf{T}_2)} \mathrm{cap}(e).$$

Lemma 5.40. *Let $v_0, v_1, \dots, v_m, z$ be a node of $\overrightarrow{\mathsf{G}}$. If there exist indices i, j such that $v_i = z = w_j$, then $\langle v_0v_1\dots v_m, w_0w_1\dots w_n\rangle = 0$.*

PROOF. A node cannot simultaneously belong to two different connected components of the same graph. □

Lemma 5.41. *Let $v_0, v_1, \dots, v_m, w_0, w_1, \dots, w_m, z$ be nodes of $\overrightarrow{\mathsf{G}}$. Then*
(1) $\langle v_0v_1\dots v_m\rangle = \langle v_0v_1\dots v_mz\rangle$ and
(2) $\langle v_0v_1\dots v_m, w_0w_1\dots w_n\rangle = \langle v_0v_1\dots v_mz, w_0w_1\dots w_n\rangle + \langle v_0v_1\dots v_m, w_0w_1\dots w_nz\rangle$.

PROOF. Each node belongs necessarily to a connected components of the same graph. □

Definition 5.42. *Let v_1, v_2, w_1, w_2 nodes of $\overrightarrow{\mathsf{G}}$. We then define the associated **transpedance** by*

$$[v_1w_1, v_2w_2] := \langle v_1v_2, w_1w_2\rangle - \langle v_1w_2, w_1v_2\rangle.$$

Lemma 5.43. *Let v_1, v_2, w_1, w_2 nodes of $\overrightarrow{\mathsf{G}}$. Then the following formulae hold:*
(1) $[v_1v_1, v_2w_2] = [v_1w_1, v_2v_2] = 0$.
(2) $[v_1w_1, v_2w_2] = \langle v_1v_2, w_1\rangle - \langle v_1w_2, w_1\rangle$.

PROOF. 1) The formula follows directly from the definition of transpedance.

2) Lemma 5.41 yields $\langle v_1v_2, w_1\rangle = \langle v_1v_2w_2, w_1\rangle + \langle v_1v_2, w_1w_2\rangle$ as well as $\langle v_1w_2, w_1\rangle = \langle v_1v_2w_2, w_1\rangle + \langle v_1w_2, w_1v_2\rangle$. Subtracting the second from the first equation we obtain the claimed identity. □

Corollary 5.44. *Let a, b, w_1, w_2, w_3 nodes of $\overrightarrow{\mathsf{G}}$. Then the transpedances satisfy the identity $[ab, w_1w_2] + [ab, w_2w_3] = [ab, w_1w_3]$.*

PROOF. The assertion is a direct consequence of Lemma 5.43.(2). □

Remark 5.45. *If in particular $a := v_{so}$ and $b := v_{si}$, then one can try to intepret the transpedances by means of more usual notions from the theory of electric circuits. The transitive relation expressed in Corollary 5.44 motivates to associate to each pair of nodes w_i, w_j an (abstract) difference of potential, i.e., $[v_{so}v_{si}, w_1w_2] := V(w_1) - V(w_2)$ in a circuit with source v_{so} and sink v_{si}. With this interpretation, Corollary 5.44 is in fact nothing but **Kirchhoff's Voltage Law**:* The sum of the electrical potential differences around any closed circuit must be zero. *In fact, it is also possible to derive an equivalent of*

[2] Conventionally, multiple occurrences of the same node in the node set corresponding to the same connected component are ignored, i.e., if $v_i = v_m$ for $i < m$, then $\langle v_0v_1\dots v_m\rangle = \langle v_0v_1\dots v_{m-1}\rangle$.

Kirchhoff's Current Law*, which states that* at any node of an electric circuit that does not represent a capacitor plate[3], the sum of currents flowing towards that node is equal to the sum of currents flowing away from it.

Proposition 5.46. *Let a, b, w nodes of $\overrightarrow{\mathsf{G}}$. Then the transpedances satisfy the identity*

$$\sum_{\overrightarrow{e}:\mathrm{term}(\overrightarrow{e})=v} [ab, \mathrm{init}(\overrightarrow{e})v]\mathrm{cap}(e) = \langle a\rangle\delta_{bv} - \langle b\rangle\delta_{av}, \tag{5.5}$$

where $\delta_{..}$ denotes the usual Kronecker delta.

Remark 5.47. *Observe that if $a = v_{so}$ and $b = v_{si}$, if $\mathrm{cap}(e)$ represents the conductance of e, and if we adopt the interpretation of the transpedances proposed in Remark 5.45, then by Ohm's law the current flowing through the edge $\overrightarrow{e} = \overrightarrow{(v, w)}$ is given by $[ab, vw]$, and the identity in Proposition 5.46 is just usual Kirchhoff's Current Law.*

In the proof we will make use of the following observations:

- if $\mathsf{T}_1, \mathsf{T}_2$ are trees and their intersection is a tree, then also their union is a tree (why?);
- if an oriented subgraph $\overrightarrow{\mathsf{T}} \subset \overrightarrow{\mathsf{G}}$ is such that the associated graph is a tree, and if each node is terminal endpoint of at most one oriented edge, then $\overrightarrow{\mathsf{T}}$ is a single arborescence (why?);
- if $\overrightarrow{\mathsf{T}}$ is a spanning (oriented) subgraph of $\overrightarrow{\mathsf{G}}$, then $\overrightarrow{\mathsf{T}}$ is a single arborescence rooted in a node w if and only if $\overrightarrow{\mathsf{T}}$ contains no cycle, $d_t(w, \overrightarrow{\mathsf{T}}) = 0$ and $d_t(z, \overrightarrow{\mathsf{T}}) = 1$ for any node $z \neq w$ – i.e., the root is terminal endpoint of no oriented edge and any other node is terminal endpoint of exactly one (why?);
- if $\overrightarrow{\mathsf{T}}$ is a single arborescence rooted in a node v, and if $\overrightarrow{e}$ is an edge in $\overrightarrow{\mathsf{T}}$, then $\overrightarrow{\mathsf{T}} - \{e\}$ is a double arborescence rooted in v and term$\overrightarrow{e}$ (why?) .

Proof. If $a = b$, (5.5) is trivially satisfied.

If instead $a \neq b$, then apply the definition of transpedance and rewrite (5.5) as

$$\sum_{\overrightarrow{e}:\mathrm{term}(\overrightarrow{e})=v} \langle a\ \mathrm{init}(\overrightarrow{e}), bv\rangle\mathrm{cap}(e) - \sum_{\overrightarrow{e}:\mathrm{term}(\overrightarrow{e})=v} \langle av, b\ \mathrm{init}(\overrightarrow{e})\rangle\mathrm{cap}(e) = \langle a\rangle\delta_{bv} - \langle b\rangle\delta_{av}, \tag{5.6}$$

which we are going to show.

The cases $v = a$ or $v = b$ are easily discussed: we only consider the latter. By convention and by Lemma 5.40, (5.6) reduces to

$$\sum_{\overrightarrow{e}:\mathrm{term}(\overrightarrow{e})=b} \langle a\ \mathrm{init}(\overrightarrow{e}), b\rangle\mathrm{cap}(e) = \langle a\rangle = \sum_{\overrightarrow{\mathsf{T}}\in\mathsf{A}^{v_0}} \prod_{e\in E(\mathsf{T})} \mathrm{cap}(e). \tag{5.7}$$

Thus, take an oriented edge $\overrightarrow{e}$ with terminal endpoint in b. We have to compute a sum (of capacity products) over a set of double arborescence $(\overrightarrow{\mathsf{T}}_1, \overrightarrow{\mathsf{T}}_2)$ with connected components rooted in a and $\mathrm{init}(\overrightarrow{e})$, respectively. Adding to the double arborescence the edge $\overrightarrow{e}$ we obtain a single arborescence rooted in in a. Conversely, if $\overrightarrow{\mathsf{T}}$ is a single arborescence rooted

[3] I.e., each node different from the source and the sink of an electric circuit.

in a, then $d_i(b, \overrightarrow{\mathsf{T}}) = 1$, i.e., there is only one oriented edge $\overrightarrow{e}$ whose terminal endpoint is b: deleting $\overrightarrow{e}$it we obtain $\overrightarrow{\mathsf{T}} - \{\overrightarrow{e}\}$, a collection of two unconnected components, with same node set of $\overrightarrow{\mathsf{T}}$ – i.e., a double arborescence rooted in a, b, and therefore associated to an addend in (5.7).

Finally, consider the case of $a \neq v \neq b$. All we want to show ist

$$\sum_{\overrightarrow{e}:\mathrm{term}(\overrightarrow{e})=v} \langle a\ \mathrm{init}(\overrightarrow{e}), bv\rangle \mathrm{cap}(e) = \sum_{\overrightarrow{e'}:\mathrm{term}(\overrightarrow{e'})=v} \langle av, b\ \mathrm{init}(\overrightarrow{e'})\rangle \mathrm{cap}(e'). \tag{5.8}$$

Take an addend in the left hand side, i.e., consider an oriented edge $\overrightarrow{e}$ such that $\mathrm{term}(\overrightarrow{e}) = v$. By definition of $\langle a\ \mathrm{init}(\overrightarrow{e}), bv\rangle$, we have to consider the set $\mathsf{A}_{bv}^{a\ \mathrm{init}(\overrightarrow{e})}$. Each element of this set, i.e., each double arborescence consisting of oriented trees $\overrightarrow{\mathsf{T}}_1$, $\overrightarrow{\mathsf{T}}_2$ rooted in a, b and containing $\mathrm{init}(\overrightarrow{e}), v$, respectively, contains exactly one directed edge $\overrightarrow{e'}$ with terminal endpoint v – in fact, such an edge is contained in $\overrightarrow{\mathsf{T}}_2$. Accordingly, $\overrightarrow{\mathsf{T}}_2 - \{\overrightarrow{e'}\}$ is itself a double arborescence $\overrightarrow{\mathsf{T}}'_2$, $\overrightarrow{\mathsf{T}}''_2$ rooted in $a, v = \mathrm{term}(\overrightarrow{e'})$. Let us now add the oriented edge $\overrightarrow{e}$ connecting $\mathrm{init}(\overrightarrow{e})$ to $\mathrm{term}(\overrightarrow{e}) = v$. Thus, $\overrightarrow{\mathsf{T}}^* := \overrightarrow{\mathsf{T}}''_2 + \{\overrightarrow{e'}\} + \overrightarrow{\mathsf{T}}_1$ is a connected subgraph, and in fact $\overrightarrow{\mathsf{T}}'_2$, $\overrightarrow{\mathsf{T}}^*$ is a double arborescence rooted in a, b. This process has turned a double arborescence corresponding to an addend in $\langle a\ \mathrm{init}(\overrightarrow{e}), bv\rangle$ into a double arborescence corresponding to an addend in $\langle av, b\ \mathrm{init}(\overrightarrow{e})\rangle$. The factors involved in $\langle a\ \mathrm{init}(\overrightarrow{e}), bv\rangle$ agree with the factors involved in $\langle av, b\ \mathrm{init}(\overrightarrow{e})\rangle$, with the sole exception of $\mathrm{cap}(e)$, which does not appear in the former product (but it does in the latter), and $\mathrm{cap}(e')$, which does not appear in the latter product (but it does in the former). In this way, the products have been shown to coincide.

By symmetry, this can be inverted, showing that the set of double arborescences corresponding to addends in $\langle a\ \mathrm{init}(\overrightarrow{e}), bv\rangle$ and the set of double arborescences corresponding to addends in $\langle av, b\ \mathrm{init}(\overrightarrow{e})\rangle$ have same cardinality. Moreover, since $\mathrm{cap}(e) = \mathrm{cap}(e')$, the corresponding products of capacities do not differ. This completes the proof of (5.8). □

CHAPTER 6

Blocks

Definition 6.1. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a connected graph,* $v \in \mathsf{V}$*. If* $\kappa(\mathsf{G} - \{v\}) > 1$*, i.e., if deleting* v *the graph becomes unconnected, then* v *is called a* ***cutvertex****.*

It is clear that a node is a cutvertex of a graph G if and only if it is a cutvertex of the graph obtained by G by deleting the loops. Nevertheless, some of the notions introduced below, most notably that of *block*, can become non-standard if loops are allowed. Therefore, we always avoid to consider loops throghout this section.

In some books the assumption of connectedness is dropped. Of course, all results in this chapter can be rephrased restricting ourselves to each connected component of an unconnected graph.

Example 6.2. *Any node of a path is a cutvertex. Complete graphs and cycles do not contain any cutvertex.*

Exercise 6.3. *Let* G *be a simple graph and* e *be a bridge of* G *whose both endpoints* v, w *satisfy* $d(v, \mathsf{G}) \geq 2$ *and* $d(w, \mathsf{G}) \geq 2$*. Show that* e *is a cutvertex of* G_L*, and that conversely each cutvertex of* G_L *is a bridge of* G *satisfying the above property.*

Proposition 6.4 (D. Kőnig, 1936)**.** *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a connected graph without loops. Then* $v \in \mathsf{V}$ *is a cutvertex if and only if there exist* $w, z \in \mathsf{V}$*, with* $w \neq v \neq z$*, such that* v *is contained in each path connecting* w, z*.*

Proof. If v is a cutvertex, then $\mathsf{G} - \{v\}$ is not connected. Pick two nodes w, z in two different connected components of $\mathsf{G} - \{v\}$.

Let conversely $w, z \in \mathsf{V}$ such that v is contained in each path connecting w, z, i.e., let $\{v\}$ separate $\{w\}, \{z\}$. Accordingly, $\mathsf{G} - \{v\}$ is not connected. This means that v is a cutvertex. □

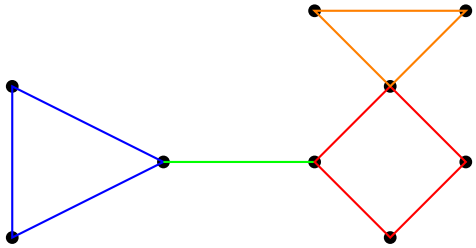
Exercise 6.5. *Show that a connected graph* G *without loops has connectivity* $\lambda(\mathsf{G}) = 1$ *if and only if* G *contains a cutvertex, or else if and only if* G *consists of two nodes and some connecting parallel edges.*

Definition 6.6. *A* ***block*** *of a connected graph without loops* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *is a maximal connected subgraph of* G *that does not contain any cutvertex of itself (but may indeed contain cutvertices of* G*).*

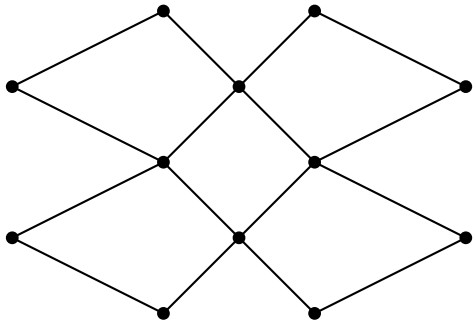
Lemma 6.7. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a connected graph without loops. A subgraph* G' *is a block of* G *if and only if either of the following holds:*

(1) G' *is a maximal* 2*-connected subgraph.*

(2) G' *is a bridge.*



A graph decomposed in four blocks.



A graph consisting of only one block.

Observe that a bridge is *not* 2-connected, since it only has two nodes and therefore does not satisfy the first condition in the definition of multiple connectedness.

PROOF. First of all, it is clear that both maximal 2-connected subgraphs and bridges define blocks: If G' is a 2-connected subgraph, then the deletion of a single node does not unconnect G' and a single edge is apparently a block, too.

Conversely, let G' be a block. It is maximal and connected by definition. If it has more than one edge, then it is necessarily 2-connected, since otherwise there would exist $v \in \mathsf{V}$ such that $\mathsf{G}' - \{v\}$ is not connected, i.e., we could find a cutvertex v. □

Remark 6.8. *If we drop the original connectedness assumption, then of course it could happen that* G *also contains of isolated nodes, i.e., nodes with degree* 0*. In the unconnected setting, isolated nodes are blocks, too.*

Observe that since blocks of G are maximal by definition, their pairwise intersection is either empty or it consists of a cutvertex of G only. Therefore, each edge of G belongs to exactly one block.

Theorem 6.9 (D. Kőnig, 1936)**.** *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a connected graph without loops and containing a cutvertex. Then the following assertions hold.*

(1) Different blocks have at most one common node.
(2) E *is disjoint union of its blocks' edge sets.*
(3) If a node $v \in \mathsf{V}$ *belongs to two blocks (or more), then it is a cutvertex.*
(4) A cutvertex $v \in \mathsf{V}$ *belongs to two blocks (or more) of* G*.*
(5) If $w, z \in \mathsf{V}$ *do not belong to the same block, then each path connecting* w, z *contains a cutvertex* v *such that* w, z *belong to different connected components of* $\mathsf{G} - \{v\}$*.*

Of course, if G is connected and does not contain any cutvertex, then it is already maximal and therefore the only block of itself.

PROOF. (1) Were two nodes simultaneously contained in two different blocks $\mathsf{G}_i, \mathsf{G}_j$, then the subgraph $\mathsf{G}_i \cup \mathsf{G}_j$ would be connected and it would contain no cutvertex – a contradiction to the definition of block since it is strictly larger than both G_i and G_j.

(2) First of all, observe that blocks have to be pairwise disjoint with respect to edges, otherwise two different blocks would also contain the endpoints of a common edge – a contradiction to (1). Moreover, either an edge belongs to no cycles – and in this case it is a bridge, i.e., in a block – or it indeed lays along a cycle – and therefore is contained in a 2-connected subgraph of G, and accordingly again in a block.

(3) By definition, if v, w are nodes of a block G', then each path connecting v, w includes at least one edge of G'. Thus, if v, w are nodes in a block G_1 and w, z are nodes in another block G_2, then each path connecting v, z contains w, which is therefore a cutvertex of G.

(4) By definition of cutvertex, the graph $\mathsf{G} - \{v\}$ is disconnected but G is not. Consider two components $\mathsf{G}_1, \mathsf{G}_2$ and take two edges, say e_1, e_2, connecting v with a node in each of these components, say v_1, v_2. By definition of cutvertex, these edges belong to two different blocks, hence v belongs to two different blocks, too.

(5) Consider a path defined by $(w = v_0, v_1, \ldots, v_n = z)$ and $(e_1, \ldots, e_n)$ and denote by G_1 the block containing e_1. Let $v_{i'}$ be the first node, i.e., the node with least index that does not belong to G_1 any more, but rather to (say) $\tilde{\mathsf{G}}$. Accordingly, $v_{i'-1}$ belongs to (at least) two blocks, namely $\mathsf{G}_1, \tilde{\mathsf{G}}$. By (3), $v_{i'-1}$ is a cutvertex, hence $\mathsf{G} - \{v_{i'-1}\}$ is unconnected. It remains to show that no further path connecting w, z is contained in $\mathsf{G} - \{v_{i'-1}\}$: assume that such a path exists, say defined by $(w = v'_0, v'_1, \ldots, v'_{n'} = z)$ and $(e'_1, \ldots, e'_{n'})$. Denote by l (resp., m) the largest (resp., the least) index $i \leq i'$ (resp., $i \geq i'$ such that v_l, v_m are contained in such a further path, i.e., $v_l = v'_{l'}$ and $v_m = v'_{m'}$ for suitable l', m'. Accordingly, gluing the paths defined by $(v_l, \ldots, v_m)$ and $(e_{1+1}, \ldots, e_m)$ and by $(v_l = v'_{l'}, \ldots, v'_{m'} = v_m)$ and $(e'_{1'+1}, \ldots, e'_{m'})$, respectively, we obtain a cycle. Since a cycle is 2-connected, this implies that all its nodes belong to a unique block, say G', and in particular that e_{r-1} belongs to both G_1 and $\tilde{\mathsf{G}}$. By (2), this yields that $\mathsf{G}_1 = \tilde{\mathsf{G}}$, i.e., $v_{i'}$ also belongs to G_1, a contradiction. □

Exercise 6.10. *Prove by induction that each 2-connected subgraph of a simple graph G can be obtained by considering a cycle G_0 and successively adding to the graph G_i a path connecting two nodes of G_i, thus defining a larger graph G_{i+1}.*

Exercise 6.11. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a graph with cutvertex set V' and block set $\mathcal{P}$. Define a new graph (so-called **block-graph**) whose node set is given by the (disjoint) union of V' and $\mathcal{P}$ and with an edge between a cutvertex $\tilde{v}$ and a block $\tilde{\mathsf{G}}$ if and only if $\tilde{v} \in \tilde{\mathsf{G}}$.*

(1) Show that the block graph of G is a forest – and a tree if and only if G is connected.

(2) Show that a block of G contains one cutvertex if and only if it is a leaf in the associated block-graph.

Lemma 6.12. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a connected graph without loops. Then each cycle in G is a cycle in some block of G.*

The converse is clearly true, too.

PROOF. A cycle in G is a connected subgraph. By Example 6.2 it does not contain any cutvertex of itself, hence it is contained in only one block. □

Definition 6.13. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a connected graph without loops.*

*(1) Let V be the disjoint union of $\mathsf{V}_1, \mathsf{V}_2$. If an edge $e \in \mathsf{E}$ such that $g(e) = \{v, w\}$ with $v \in \mathsf{V}_1$ and $w \in \mathsf{V}_2$, then e is said to **cross** $\mathsf{V}_1, \mathsf{V}_2$.*

(2) Let $\mathsf{E}' \subset \mathsf{E}$. If there exist $\mathsf{V}_1, \mathsf{V}_2$ such that
- *their disjoint union is V and*
- *E' is the set of all edges crossing $\mathsf{V}_1, \mathsf{V}_2$,*

*then E' is called a **cut** of G (associated with $\mathsf{V}_1, \mathsf{V}_2$).*

*(3) A **bond** of G is a nonempty cut of G that has minimal cardinality among all nonempty cuts of G.*

Exercise 6.14. *Prove that a spanning subgraph $\mathsf{G}' = (\mathsf{V}', \mathsf{E}', g')$ of a connected graph G is a spanning tree of G if and only if $\mathsf{G} - \mathsf{E}'$ contains no bond of G and adding any edge of G' to it creates a subgraph of G containing a bond of G.*

Lemma 6.15. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a connected graph without loops. Then each bond in some block of G is a bond in G.*

The converse is clearly true, too.

PROOF. Consider a cut E' in G associated to a partition $\mathsf{V} = \mathsf{V}_1 \dot\cup \mathsf{V}_2$, an edge $e \in \mathsf{E}'$, and the block G' containing e (which by construction has an endpoint in V_1 and an endpoint in V_2. Observe that there is no path in G such that only the endpoints of e belong to G' (if there were such a path, then one could "glue" it to G' and obtain a 2-connected subgraph that contains G', a contradiction to the maximality of a block). Accordingly, any path connecting the endpoints of e is also contained in G': thus, an edge crosses $\mathsf{V}_1, \mathsf{V}_2$ in G' if and only it crosses $\mathsf{V}_1, \mathsf{V}_2$ in G.

This shows that each cut in G' is a cut in G.

Taking the minimal elements in the class of cuts in G' and G, respectively, yields the assertion. □

Proposition 6.16. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a connected graph without loops and $e, f \in \mathsf{E}$, $e \neq f$. The following assertions are equivalent.*

(a) e, f belong to the same block of G.
(b) e, f belong to the same cycle of G.
(c) e, f belong to the same bond of G.

PROOF. $(a) \Rightarrow (b)$ By Lemma 6.7, the block e, f belong to is a (maximal) 2-connected subgraph of G. By Corollary 3.18, there exist two independent paths connecting any two nodes in the block. Since this holds in particular for the endpoints of e, f, these edges lay on a cycle.

$(b) \Rightarrow (c)$ Let e, f be edges in a cycle C and denote by K the connected component of G containing C. Consider $\mathsf{C} - \{e, f\}$, which is unconnected – in fact, it consists of two connected components $\mathsf{K}^1, \mathsf{K}^2$. Consider the associated node sets $\mathsf{V}_1, \mathsf{V}_2$, whose disjoint union gives the whole node set of C. Complete $\mathsf{V}_1, \mathsf{V}_2$ to a node partition of K, i.e., consider

two connected subgraphs $\tilde{\mathsf{K}}_1, \tilde{\mathsf{K}}_2$ of K such that the node set $\tilde{\mathsf{V}}_i$ of $\tilde{\mathsf{K}}_i$ contains V_i, $i = 1, 2$. Then the edges crossing $\tilde{\mathsf{V}}_1, \tilde{\mathsf{V}}_2$ form a cut of G that clearly contains e, f. Among all cuts constructed in this way we can consider the minimal one, the bond we are looking for.

$(c) \Rightarrow (a)$ If e, f belong to the same bond in G, then by Lemma 6.15 they also belong to the same bond in some block of G, and in particular are contained in the same block. □

We say that a graph G **is a block** if it consists of a block only, i.e., if it contains a cutvertex.

Proposition 6.17 (H. Whitney, 1932)**.** *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a connected graph without loops. Let $|\mathsf{V}| \geq 3$. Then the following assertions are equivalent.*

(a) For any two nodes of G there exists a cycle $\mathsf{C} \subset \mathsf{G}$ containing both of them.
(b) For any node and any edge of G there exists a cycle $\mathsf{C} \subset \mathsf{G}$ containing both of them.
(c) For any two edges of G there exists a cycle $\mathsf{C} \subset \mathsf{G}$ containing both of them.
(d) G is a block.

Proof. To begin with, we are going to prove that $(a), (b), (c)$ are all equivalent. The slightly more difficult part of the proof is to show that (a) is equivalent to (d).

$(a) \Rightarrow (c)$ Let $e, f \in \mathsf{E}$ with $g(e) = \{v_1, v_2\}$ and $g(f) = \{w_1, w_2\}$. Define a subdivision G' of G creating two new nodes v_n, w_n and replacing e by two new edges e_1, e_2 with endpoints v_1, v_n, v_n, v_2, and similarly f by two new edges f_1, f_2 with endpoints w_1, w_n, w_n, w_2. Observe that the set of cycles in G is bijective to the set of cycles in G' – and in particular $\zeta(\mathsf{G}) = \zeta(\mathsf{G}')$. By assumption, there exists a cycle C_n containing v_n, w_n. Since the only edges incident in v_n are e_1, e_2 and the only edges incident in w_n are f_1, f_2, we see that C_n contains e_1, e_2, f_1, f_2. The assertion follows removing v_n and w_n and "gluing" e_1, e_2 and f_1, f_2 together, i.e., passing from the subdivision back to G.

$(c) \Rightarrow (b)$ Take $e \in \mathsf{E}$ and $v \in \mathsf{V}$. Since G is connected, we can consider an edge f incident in v and (by assumption) a cycle containing e, f. Of course, such a cycle will also contain v.

$(b) \Rightarrow (a)$ Take $v, w \in \mathsf{V}$ and (again by connectedness of G) consider $e \in \mathsf{E}$ incident in v. Then again the cycle containing e, w, which exists by assumption, will necessarily also contain v.

$(d) \Rightarrow (a)$ Let $v \in \mathsf{V}$ and $Z(v)$ the set of all nodes that belong to cycles in G that also contain v. By assumption G does not contain any cutvertex. Moreover, since by assumption $|\mathsf{V}| \geq 3$, G cannot be a bridge, and by Lemma 6.7 it has to be 2-connected. In particular, all edges incident to v belong to a cycle that necessarily contains v, too – i.e., all nodes adjacent to v belong to $Z(v)$. We are going to show that, in fact, each node lies on a cycle *that also contains* v, i.e., $Z(v) = \mathsf{E}$. Assume that this is not true and pick $w \in \mathsf{E} \setminus Z(v)$. Consider a path defined by $(v = v_0, \ldots, v_m = w)$ and $(e_1, \ldots, e_m)$ (it exists due to connectedness of G) and let i be the least index such that $v_i \in \mathsf{E} \setminus Z(v)$, $i \in \{2, \ldots, m\}$ – recall that $v_1 \in Z(v)$, as observed above. Thus, $v_{i-1} \in Z(v)$. Choose a cycle C containing both v_i, v_{i-1} and a path W connecting v, v_i but *not* containing v_{i-1} (this is possible because v_{i-1} is not a cutvertex). Now, one sees that combining edges from C, W, and the path defined by $(v = v_0, \ldots, v_m = w)$ and $(e_1, \ldots, e_m)$, one can construct a cycle containg v, v_i. This is a contradiction to the construction of v_i.

$(a) \Rightarrow (d)$ Assume G to contain a cutvertex v. Then, by Proposition 6.4 there exist two nodes w, z such that v is contained in any path connecting w, z. This contradicts the assumption that w, z lie on a cycle, and hence that two independent paths connect them. □

Remark 6.18. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a connected graph without loops. Let $|\mathsf{V}| \geq 3$. Observe that if two nodes v, w do not lie on a same cycle, they do no belong to the same block. Since two different blocks can only be connected by a cutvertex, there exists a cutvertex $z \in \mathsf{V}$ such that each path connecting v, w contains z.*

CHAPTER 7

Hamiltonian cycles

Let us introduce a new notion. At a first glance it looks related to that of Eulerian graph, but in fact turns out to be much harder to characterize. Material of this section is mostly taken from [**22**, Kap. 4]. One can also check [**2**, § IV.3], [**8**, Chapt. 9], and [**23**, § I.2] for a large number of criteria necessary or sufficient for the Hamiltonian property.

Definition 7.1. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph. A subgraph* $\mathsf{C} \subset \mathsf{G}$ *is called a* ***Hamiltonian cycle*** *(in* G*) if* C *is a cycle and its node set agrees with* V*.*

The graph G *is called* ***Hamiltonian*** *if it contains a Hamiltonian cycle.*

We emphasize that neither a Hamiltonian graph needs to be Eulerian, nor a Eulerian graph needs to be Hamiltonian.

Many partial results are known and many, many open questions arise in connection with Hamiltonian graphs, cf. `http://www.densis.fee.unicamp.br/~moscato/Hamilton.html`.

Remark 7.2. *Observe that any graph* G *is Hamiltonian if and only if the graph* $\tilde{\mathsf{G}}$ *obtained by deleting loops and unifying multiple edges of* G *is Hamiltonian.*

Clearly, Hamiltonian graphs are necessarily connected.

Moreover, if G *is Hamiltonian and* C *is a Hamiltonian cycle contained in* G*, then* $d(v, \mathsf{C}) = 2$ *for all* $v \in \mathsf{G}$*, by definition. Therefore, a Hamiltonian graph cannot contain any cutvertex.*

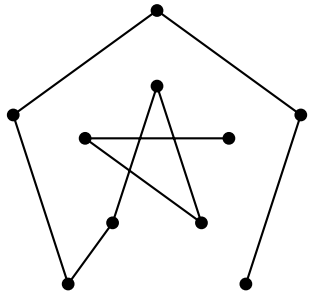
Example 7.3. *Each cycle is a Hamiltonian cycle in itself. Each complete graph* K^n *is Hamiltonian: it suffices to take the "perimeter" of* K^n*. A tree contains no cycle, hence in particular no tree (with more than one node) is Hamiltonian.*

Remark 7.4. *Similarly to the problem of finding an optimal edge sequence that "approximates" a Euler tour in a non-Eulerian weighted graph (Chinese Postman Problem), one can ask to find an optimal edge sequence that "approximates" a Hamiltonian cycle in a non-Hamiltonian weighted graph, i.e., an edge sequence containing each node of the graph with minimal weighted length. This is known as the* ***Travelling Salesman Problem****, first mentioned by K. Menger in 1930. Of course, each spanning tree is an edge sequence containing each node of the graph – but in general it is not the optimal one.*

Definition 7.5. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph. A set* $(\mathsf{P}_1, \ldots, \mathsf{P}_n)$ *of pairwise node-disjoint paths is called a* ***path cover*** *of* G *if each node of* G *belongs to one path* P_i*. The minimal number* $\pi(\mathsf{G})$ *of pairwise node-disjoint paths covering* G *– i.e., the minimal cardinality of a path cover – is called* ***path cover number****. If* $\pi(\mathsf{G}) = 1$*, each individual path that covers* G *is called* ***Hamiltonian path****.*

Example 7.6. *An m-star has a Hamiltonian path if and only if $m \leq 3$.*

The Petersen Graph has a Hamiltonian path. The path can be constructed transversing four edges along the perimeter, then reaching the inner star and transversing four of its edges.



A Hamiltonian path in the Petersen Graph.

Exercise 7.7. *Prove that the Petersen Graph is not Hamiltonian.*
(Hint: Observe that each node of the inner "star" as well as each node of the outer "pentagon" would have to be contained in a Hamiltonian cycle. Consider the number of edges that necessarily connect the star to the pentagon and rule out all possible combinations, using symmetry arguments to reduce the number of possible configurations).

Remark 7.8. *The Petersen Graph has further noteworthy properties. It is the smallest bridgeless 3-regular graph that is not Hamiltonian. Though, deleting any node makes it Hamiltonian. In fact, it is the smallest (with respect to the number of nodes) non-Hamiltonian graph which becomes Hamiltonian after deletion of any node – or equivalently: any two vertices can be connected by a Hamiltonian path.*

Proposition 7.9. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a Hamiltonian graph. If $\emptyset \neq \mathsf{V}' \subset \mathsf{V}$, then*

$$\kappa(\mathsf{G} - \mathsf{V}') \leq \pi(\mathsf{G} - \mathsf{V}') \leq |\mathsf{V}'|. \tag{7.1}$$

Proof. Let C be a Hamiltonian cycle and $\mathsf{V}' = \{v_1, \ldots, v_m\}$. Then, $\mathsf{C} - \{v_1\}$ is path, hence it is 1-connected. Thus, $\mathsf{C} - \{v_1, v_2\} = (\mathsf{C} - \{v_1\}) - \{v_2\}$ is either a path (if v_2 is an endpoint of $\mathsf{C} - \{v_1\}$), or it consists of two connected components, i.e., necessarily of two paths. Similarly, one sees that $\mathsf{C} - \{\mathsf{V}'\}$ consists of at most $|\mathsf{V}'|$ (disconnected) paths, showing that $\pi(\mathsf{C} - \mathsf{V}') \leq |\mathsf{V}'|$. On the other hand, it is clear that $\kappa(\mathsf{G} - \mathsf{V}') \leq \pi(\mathsf{G} - \mathsf{V}')$, since each path is a connected component. Thus, it remains to show that $\pi(\mathsf{G} - \mathsf{V}') \leq \pi(\mathsf{C} - \mathsf{V}')$. However, this is clear, since deleting a node v in C causes deletion of at most as many edges as it does in G. □

A first breakthrough in the theory of Hamiltonian cycles is the following, which is known as **Ore's Lemma**.

Proposition 7.10 (Ø. Ore, 1960)**.** *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a simple graph. Assume two non-adjacent nodes v, w of G to satisfy $d(v, \mathsf{G}) + d(w, \mathsf{G}) \geq |\mathsf{V}|$. Then G is Hamiltonian if and only if $\mathsf{G} + \{(v, w)\}$ is Hamiltonian.*

Proof. If G is Hamiltonian, then clearly also $\mathsf{G} + \{(v.w)\}$ is Hamiltonian, since the latter graph has the same node set of G.

Let conversely $\mathsf{G}+\{(v,w)\}$ be Hamiltonian and denote by C a Hamiltonian cycle in it. If $(v.w)$ is not contained in C, then clearly also $(\mathsf{G}+\{(v,w)\})-\{(v,w)\}=\mathsf{G}$ is Hamiltonian. If (v,w) is indeed contained in C, then G contains a Hamiltonian path with endpoints v,w, say $(v=v_1,\dots,v_{|\mathsf{V}|}=w)$. Since the path is Hamiltonian, by definition $\mathsf{V}=\{v_1,\dots,v_{|\mathsf{V}|})$. Define the index sets

$$A:=\{1\leq i\leq |\mathsf{V}|-1 \text{ s.t. } (v,v_{i+1})\in \mathsf{E}\}$$

and

$$B:=\{1\leq i\leq |\mathsf{V}|-1 \text{ s.t. } (v_i,w)\in \mathsf{E}\}.$$

Observe that by construction $|\mathsf{V}|\notin B$. Then $|\mathsf{A}|=d(v,\mathsf{G})$ and $|\mathsf{B}|=d(w,\mathsf{G})$, because $\mathsf{V}=\{v_1,\dots,v_{|\mathsf{V}|})$. Therefore

$$|\mathsf{V}|-1\geq |A\cup B|=|A|+|B|-|A\cap B|=d(v,\mathsf{G})+d(w,\mathsf{G})-|A\cap B|\geq |\mathsf{V}|-|A\cap B|,$$

hence $|A\cap B|\geq 1$, i.e., A,B are non-disjoint. Accordingly, there exists a "shortcut" in C, i.e., there exists $p\in A\cap B$. Accordingly, there is v_p such that $(w,v_p),(v,v_{p+1})\in\mathsf{E}$, and since v,w are not adjacent, p is necessarily different from $1,|\mathsf{V}|-1$. Summing up, the cycle

$$(v=v_1,v_2,\dots,v_p,w=v_{|\mathsf{V}|},v_{|\mathsf{V}|-1},\dots,v_{p+1},v)$$

is contained in G and it contains the nodes $v_1,\dots,v_{|\mathsf{V}|}$, i.e., all nodes. Accordingly, it is a Hamiltonian cycle. This completes the proof. □

Corollary 7.11 (Ø. Ore, 1960)**.** *Let $\mathsf{G}=(\mathsf{V},\mathsf{E},g)$ be a simple graph with $3\leq|\mathsf{V}|$. If*

$$d(v,\mathsf{G})+d(w,\mathsf{G})\geq|\mathsf{V}| \text{ for all non-adjacent } v,w\in\mathsf{V}, \tag{7.2}$$

then G is Hamiltonian.

Proof. Let v,w be two non-adjacent nodes. By Ore's Lemma, G is Hamiltonian if and only if $\mathsf{G}_1:=\mathsf{G}+\{(v,w)\}$ is Hamiltonian. Similarly, we can successively add an edge between any two previously non-adjacent nodes. Continuing this procedure eventually leads to construcion of $\mathsf{K}^{|\mathsf{V}|}$, the complete graph with $|\mathsf{V}|$ nodes. Since $\mathsf{K}^{|\mathsf{V}|}$ is Hamiltonian, so is G. □

Exercise 7.12. *Let $\mathsf{G}=(\mathsf{V},\mathsf{E},g)$ be a graph such that*

$$|\mathsf{E}|-2\geq\frac{(|\mathsf{V}|-1)(|\mathsf{V}|-2)}{2}.$$

Show that G is Hamiltonian.

This suggests that any graph with *enough edges* is Hamiltonian.

Definition 7.13. *Given a simple graph G, its **closure** $\overline{G}$ is the simple graph constructed from G by adding an edge (v,w) whenever v,w are non-adjacent nodes such that $d(v,\mathsf{G})+d(w,\mathsf{G})\geq|\mathsf{V}|$. The graph G is said to be **self-closed** if $\mathsf{G}=\overline{\mathsf{G}}$.*

Example 7.14. *A star is self-closed. A polygon with n nodes is self-closed if and only if $n\neq 4$ (if $n=4$, then its closure is K^4).*

Exercise 7.15. *Let G be a graph with 5 nodes. What is the maximal number of edge a graph G can have if it is self-closed? What is the minimal number of edges G can have if its closure sis K^5?*

Remark 7.16. *It can be proved that the closure of a graph is well-defined, i.e., that it does not depend on the order in which edges are added, see* [**15**, Thm. 5.11]

Ore's result can be extended as follows. It can be immediately proved by Ore's lemma and the definiton of closure.

Corollary 7.17 (A. Bondy – V. Chvátal, 1972)**.** *A graph is Hamiltonian if and only if its closure is Hamiltonian.*

An immediate consequence of Ore's Lemma is the following. Historically, it has been the first known sufficient condition implying that a graph is Hamiltonian.

Corollary 7.18 (G.A. Dirac, 1952)**.** *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a simple graph such that* $3 \leq |\mathsf{V}| \leq 2\delta(\mathsf{G})$*. Then* G *is Hamiltonian.*

Example 7.19. *An* n*-wheel satisfies the assumption of Corollary 7.18 if and only if* $n \leq 6$*, since the sum of degrees of any two non-adjacent nodes is always 6. Accordingly, each wheel* W^n *with* $n \leq 6$ *is Hamiltonian.*

Remark 7.20. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a simple graph. The* ***square*** G^2 *is the simple graph with the same node set of* G *and whose adjacency matrix is the square of* G*'s adjacency matrix. An interesting result due to H. Fleischner says that if* G *is* 2*-connected, than* G^2 *is Hamiltonian, cf.* [**8**, § 10.3]*. Observe that* 1*-connectedness (i.e., plain connectedness) of* G *is not sufficient to guarantee that either* G *or* G^2 *are Hamiltonian, as one can see considering as* G *a path with more than 4 edges. On the other hand, by Fleischner's theorem it is trivial that* k*-connectedness implies connectedness of* G^2*, and* a fortiori *of* G^k*.*

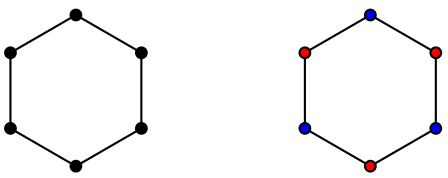
Definition 7.21. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a simple graph. Assume that* V *is the disjoint union of node subsets* $\mathsf{V}_1, \ldots, \mathsf{V}_N$*. If for each* $e \in \mathsf{E}$ $g(e) = \{v, w\}$ *implies that* $v \in \mathsf{V}_i$ *and* $w \in \mathsf{V}_j$ *for different* i, j *(i.e., vertices in the same set* V_i *may not be adjacent), then* G *is called* N***-partite****, and in particular* ***bipartite*** *if* $N = 2$*.*

If an N*-partite graph is simple and if for all* i *each node in* V_i *is adjacent to any other node in* V_j*,* $j \neq i$*, then* G *is called* ***complete*** N***-partite****. These graphs are commonly denoted by* $\mathsf{K}^{|\mathsf{V}_1|,\ldots,|\mathsf{V}_N|}$*.*

Remark 7.22. *By definition a graph is bipartite if and only if it has a cut containing all edges.*

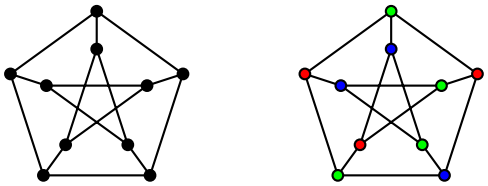
One can imagine a multipartite graph as a graph each of whose node is assigned a different colour (red, blue, green, etc.) and such that each edge has endpoints coloured differently.

Example 7.23. *A cycle* C_n *is bipartite if and only if* n *is even.*



Each (rooted) tree is bipartite, the nodes with odd (resp., even) distance from the root being red (resp., blue).

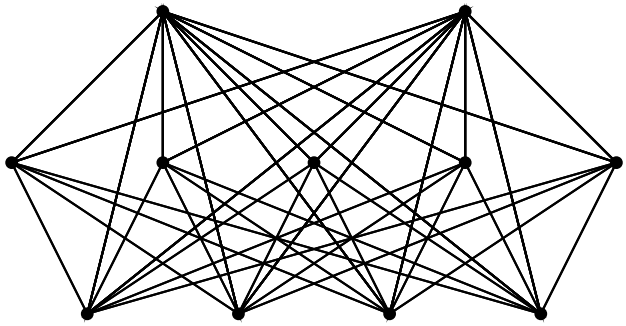
Exercise 7.24. *Prove that the Petersen graph is not bipartite.*



Exercise 7.25 (J. Kőnig, 1916)**.** *Prove that a graph is bipartite if and only if it contains no cycle of odd length.*
(Hint: Consider a spanning tree with a root v_0 and the relation on V defined by vRw if the path connecting v, v_0 and w, v_0 have same length (mod2).)

Exercise 7.26. *Show that the* $\mathsf{K}^{m_1,\dots,m_p}$ *has* $\sum_{1\le i<j\le p} m_i m_j$ *edges.*

The graph $\mathsf{K}^{3,3}$, which has already been introduced in Example 2.6, is an example of complete bipartite graph.



The graph $\mathsf{K}^{4,5,2}$.

Theorem 7.27. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph with* $|\mathsf{V}| \ge 3$ *that is complete multipartite with respect to the decomposition of* V *into node subsets* $\mathsf{V}_1, \dots, \mathsf{V}_N$*. Let* i^* *such that* V_{i^*} *is the node subset of largest cardinality, i.e.,* $|\mathsf{V}_i| \le |\mathsf{V}_{i^*}|$ *for all* $i \ne i^*$*. Then* G *is Hamiltonian if and only if* $\sum_{i\ne i^*} |\mathsf{V}_i| \ge |\mathsf{V}_{i^*}|$.

Proof. Consider the disjoint union $\mathsf{V}' := \dot{\bigcup}_{i\ne i^*} \mathsf{V}_i$. Observe that by assumption V' is nonempty and that $\mathsf{G} - \mathsf{V}'$ is the subgraph consisting of $|\mathsf{V}_{i^*}|$ isolated nodes. By Proposition 7.9

$$|\mathsf{V}_{i^*}| = \kappa(\mathsf{G} - \mathsf{V}') \le |\mathsf{V}'| = \sum_{i\ne i^*} |\mathsf{V}_i|.$$

Conversely, if $\sum_{i\ne i^*} |\mathsf{V}_i| \ge |\mathsf{V}_{i^*}|$, then clearly $|\mathsf{V}| \ge 2|\mathsf{V}_{i^*}|$ and therefore the degree of each node v in any of the partitions V_j is

$$d(v, \mathsf{G}) = |\mathsf{V}| - |\mathsf{V}_j| \ge |\mathsf{V}| - |\mathsf{V}_{i^*}| \ge \frac{|\mathsf{V}|}{2}.$$

Accordingly, the minimal degree satisfies $2\delta(\mathsf{G}) \le |\mathsf{V}|$ and by Dirac's result (Corollary 7.18) the graph is Hamiltonian. □

Example 7.28. *1) The graph* $\mathsf{K}^{4,5,2}$ *depicted above is Hamiltonian.*

2) In particular, the bipartite graph $\mathsf{K}^{m,n}$ is Hamiltonian if and only if $n = m$. This generalizes the (obvious) observation that an r-star is Hamiltonian if and only if $r = 2$, i.e., if and only if it is $\mathsf{K}^{1,1}$.

The following results are criteria showing that suitable properties of a simple graph imply that its line graph (introduced in Definition 3.19) is Hamiltonian.

Proposition 7.29. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a simple graph. If G is Eulerian, then its line graph G_L is both Eulerian and Hamiltonian. The converse implication does not hold.*

Proof. First of all, let $e = (v, w) \in \mathsf{E}$. Then, e is a node in G_L and its adjacent nodes in G_L are given by all edges of G that are incident in either v or w. Accordingly,

$$d(e, \mathsf{G}) = (d(v, \mathsf{G}) - 1) + (d(w, \mathsf{G}) - 1)\,.$$

If G is Eulerian, then by Theorem 3.31 $d(v, \mathsf{G})$, $d(w, \mathsf{G})$, and therefore also $d(e, \mathsf{G}_L)$ are even. This shows that G_L is Eulerian, too. Clearly, if G contains a closed edge sequence containing each edge exactly once (this is exactly the definition of a Euler tour), then this edge sequence also represent a Hamiltonian cycle, completing the proof.

To see that the converse implication does not hold, it suffices to consider $\mathsf{G} = \mathsf{K}^4$, which is not Eulerian, and its line graph on 6 nodes in which each edge is adjacent to 4 further nodes (each edge in K^4 is adjacent to all but one further edges). Such a line graph is clearly Eulerian and also Hamiltonian. □

Proposition 7.30. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a simple graph. If G is Hamiltonian, then also its line graph is Hamiltonian. The converse implication does not hold.*

Proof. Let $(v_1, \ldots, v_n, v_1)$ and $(e_1, \ldots, e_n)$ define a Hamiltonian cycle C in G. We construct a Hamiltonian graph in G_L algorithmically. Pick an edge that is incident in v_1 but is different from e_1, e_n). Construct a G_L-edge sequence by first considering all the G_L-nodes (there are $d(v_1, \mathsf{G}) - 2$ such nodes) given by the set of all edges incident in v_1 and different from either e_1, e_n (such nodes are adjacent in G_L by definition of line graph). Finally, reach e_1 and extend the edge sequence by considering all the G_L-nodes given by the set of all edges incident in v_2 such that

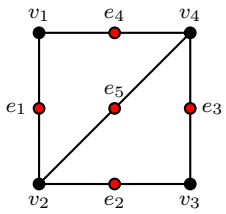
- they are not incident in v_1 (and have therefore not yet been considered) and
- are different from e_2.

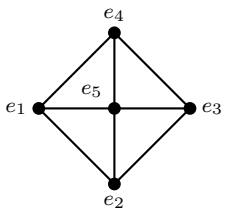
Continue in this way until v_n has been reached (this is possible exactly because G is Hamiltonian) and the edge sequence has been extended by considering all the G_L-nodes given by the set of all edges incident in v_n such that

- they are not incident in $v_1, \ldots, v_{n-1}$ (and have therefore not yet been considered) and
- are different from e_{n-1}.

To see that the converse implication does not hold, it suffices to consider $\mathsf{G} = \mathsf{K}^{1,3}$, the claw (which as a 3-star cannot be Hamiltonian), and its line graph $\mathsf{G}_L = \mathsf{K}^3$. □

Example 7.31. *Apply the algorithm introduced in the proof of Proposition 7.30 to the graph G below, which is Hamiltonian thanks to the cycle defined by (e_1, e_2, e_3, e_4).*





- *In v_1 there is nothing else to do but picking e_1.*
- *Reach v_2 and pick e_5 before reaching e_2.*
- *Reach v_3 and pick e_3, since there is no other possible choice.*
- *Reach v_4 and observe that the only possible choice is e_4, since e_5 is incident in v_2.*

Summing up, the constructed Hamiltonian cycle is defined by $(e_1, e_5, e_2, e_3, e_4)$. Of course, this Hamiltonian cycle is not unique and it depends on which node of G is labelled as v_1 (but this labelling is in fact arbitrary).

CHAPTER 8

Matching and factor theory

Definition 8.1. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph. A set* $\mathsf{E}' \subset \mathsf{E}$ *of pairwise non-adjacent edges that does not contain loops is called a* ***matching*** *of* G*. The maximal cardinality of a matching of* G *is called* ***matching number*** *of* G *and denoted by* $\alpha_0(\mathsf{G})$*.*

A ***matching*** E' ***of a node set*** $\mathsf{V}' \subset \mathsf{V}$ *is a matching such that each node* $v \in \mathsf{V}'$ *is endpoint of a unique edge* $e \in \mathsf{E}'$*.*

In the following we use a notation already introduced: for any graph G^* containing V', $N_{\mathsf{G}^*}(\mathsf{V}')$ is the set of all nodes from G^* adjacent to any node of V', i.e., $\{v \in \mathsf{V} : d(v, w) = 1 \text{ for some } w \in \mathsf{V}'\}$.

Problem 8.2. *The standard example of an application of the matching theory is to imagine two groups of n men and women. In a classical (and quite old-fashioned) formulation, a problem can be stated as follows:* Each woman would happily marry some subset of the men; and any man would be happy to marry a woman who wants to marry him. Consider whether it is possible to pair up the men and women so that every person is happy.

In other words: given a bipartite graph, $\mathsf{V} = \mathsf{V}_1 \dot\cup \mathsf{V}_2$, is it possible to find for each element v of V_1 an adjacent element $m(v) \in N_{\mathsf{G}^*}(\mathsf{V}') \subset \mathsf{V}_2$ in such a way that the mapping m is injective? In particular, observe that not any man, i.e., any element of V_2, has to be associated with a woman, i.e., an element of V_1.

The following result is usually referred to as **marriage theorem**.

Theorem 8.3 (P. Hall, 1935)**.** *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a bipartite simple graph such that* $\mathsf{V} = \mathsf{V}_1 \dot\cup \mathsf{V}_2$*. Then* G *contains a matching of* V_1 *if and only if the so-called* ***marriage condition***

$$|N_G(\tilde{\mathsf{V}})| \geq |\tilde{\mathsf{V}}| \qquad \text{for all } \tilde{\mathsf{V}} \subset \mathsf{V}_1$$

holds.

Proof. If the $|\mathsf{V}_1|$ elements of V_1 are altoghether adjacent to a subset of V_2 of cardinality less than $|\mathsf{V}_1| - 1$, then clearly no pairing is possible; that is, a matching can only exist if the marriage condition is satisfied.

In order to see that the marriage condition is also sufficient, let G not contain a matching. This means that the number of disjoint $\mathsf{V}_1 - \mathsf{V}_2$-paths contained in the graph is less than $|\mathsf{V}_1|$. By Menger's theorem, the minimal cardinality of a $\mathsf{V}_1 - \mathsf{V}_2$-separating set agrees with the maximal number of disjoint $\mathsf{V}_1 - \mathsf{V}_2$-paths. Hence, there is a $\mathsf{V}_1 - \mathsf{V}_2$-separating set V', say $\mathsf{V}' = \mathsf{V}'_1 \dot\cup \mathsf{V}'_2$ with $\mathsf{V}'_1 \subset \mathsf{V}_1$ and $\mathsf{V}'_2 \subset \mathsf{V}_2$, such that $|\mathsf{V}'| = |\mathsf{V}'_1| + |\mathsf{V}'_2| < |\mathsf{V}_1|$ and such that no node in $\mathsf{V}_1 \setminus \mathsf{V}'_1$ and no node in $\mathsf{V}_2 \setminus \mathsf{V}'_2$ are adjacent. Accordingly, $N_G(\mathsf{V}_1 \setminus \mathsf{V}'_1) \subset \mathsf{V}'_2$ and therefore

$$|N_G(\mathsf{V}_1 \setminus \mathsf{V}'_1)| \leq |\mathsf{V}'_2| < |\mathsf{V}_1| - |\mathsf{V}'_1| = |\mathsf{V}_1 - \mathsf{V}'_1|.$$

This completes the proof. □

Remark 8.4. *Hall's theorem hold in fact in the case of a non-simple graph, too, see* [**22**, § 6.2].

Lemma 8.5 (Adjacency inequality)**.** *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph without loops and* $\mathsf{V}' \subset \mathsf{V}$. *Then*

$$\sum_{v \in \mathsf{V}'} d(v, \mathsf{G}) \leq \sum_{w \in N_G(\mathsf{V}')} d(w, \mathsf{G}).$$

In particular the inequality

$$\delta(\mathsf{G})|\mathsf{V}'| \leq \Delta(\mathsf{G})|N_G(\mathsf{V}')|$$

holds.

Definition 8.6. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph. For* $\mathsf{V}_1, \mathsf{V}_2 \subset \mathsf{V}$ *we adopt in the following the notation*

$$(\mathsf{V}_1, \mathsf{V}_2) := \{e \in \mathsf{E} : g(e) = \{v, w\} \text{ with } v \in \mathsf{V}_1,\ w \in \mathsf{V}_2\},$$

similarly to the oriented case discussed in Definition 4.5. In the particular case of $\mathsf{V}_2 = \mathsf{V} \setminus \mathsf{V}_1$ *we call* $(\mathsf{V}_1, \mathsf{V} \setminus \mathsf{V}_1)$ *the* ***edge-boundary*** *of* V_1, *sometimes denoted by* $\partial \mathsf{V}_1$.

Proof. If $v \in \mathsf{V}'$, then

$$\sum_{v \in \mathsf{V}'} d(v, \mathsf{G}) = \sum_{v \in \mathsf{V}'} |(\{v\}, N_G(\mathsf{V}'))| = \sum_{w \in N_G(\mathsf{V}')} |(\{w\}, \mathsf{V}')| \leq \sum_{w \in N_G(\mathsf{V}')} d(w, \mathsf{G}).$$

This concludes the proof. Observe that the second equality depends on the fact that G does not contain loops. □

Corollary 8.7. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a bipartite simple graph such that* $\mathsf{V} = \mathsf{V}_1 \dot\cup \mathsf{V}_2$. *Assume that no node in* V_1 *is isolated. If* $\min_{v \in \mathsf{V}_1} d(v, \mathsf{G}) \geq \max_{w \in \mathsf{V}_2} d(w, \mathsf{G})$, *then* G *contains a matching of* V_1.

Proof. Let $\mathsf{V}_1' \subset \mathsf{V}_1$. Since $N_G(\mathsf{V}_1') \subset \mathsf{V}_2$, it follows from the adjacency inequality that

$$|\mathsf{V}_1'| \min_{v \in \mathsf{V}_1} d(v, \mathsf{G}) \leq \sum_{v \in \mathsf{V}_1'} d(v, \mathsf{G}) \leq \sum_{w \in N_G(\mathsf{V}_1')} d(w, \mathsf{G}) \leq |N_G(\mathsf{V}_1')| \max_{w \in \mathsf{V}_2} d(w, \mathsf{G}) \leq |N_G(\mathsf{V}_1')| \min_{w \in \mathsf{V}_1} d(v, \mathsf{G}).$$

By assumption, $1 \leq \min_{v \in \mathsf{V}_1} d(v, \mathsf{G})$ and therefore Hall's marriage condition is satisfied. Accordingly, by Theorem 8.3 a matching of V_1 exists. □

The following results consider the case of a general matching.

Proposition 8.8 (G. Kőnig, 1916)**.** *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a bipartite, k-regular simple graph. Then* G *contains a matching* E' *such that the set of endpoints of* E' *agrees with* V.

Proof. Let $\mathsf{V} = \mathsf{V}_1 \dot\cup \mathsf{V}_2$. Due to k-regularity, Corollary 8.7 applies and we deduce existence of a matching E_1 of V_1 as well as of a matching E_2 of V_2. Since each node of V_1 is endpoint of a unique edge of the former matching, $|\mathsf{E}_1| = |\mathsf{V}_1| \leq |\mathsf{V}_2|$, and since each node of V_2 is endpoint of a unique edge of the latter matching, $|\mathsf{V}_1| \geq |\mathsf{V}_2| = |\mathsf{E}_2|$. Summing up, we have found two matchings, each of whose is a matching of both $\mathsf{V}_1, \mathsf{V}_2$. □

A matching E' of both partitions of a bipartite graphs is sometimes called **perfect**.

Exercise 8.9. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a bipartite simple graph such that* $\mathsf{V} = \mathsf{V}_1 \dot\cup \mathsf{V}_2$. *Let* $d \in \mathrm{N}$. *Show that if*

$$|N_G(\tilde{\mathsf{V}})| \geq |\tilde{\mathsf{V}}| - d \qquad \text{for all } \tilde{\mathsf{V}} \subset \mathsf{V}_1$$

holds, then G *contains* $|\mathsf{V}| - d$ *disjoint edges.*

Proposition 8.10 (Ø. Ore, 1955)**.** *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a bipartite graph with* $\mathsf{V} = \mathsf{V}_1 \dot\cup \mathsf{V}_2$. *If* $\hat{\mathsf{E}}$ *is a matching in* G *of maximal cardinality, then*

$$|\mathsf{V}_1| = |\hat{\mathsf{E}}| + \max_{\mathsf{V}_1' \subset \mathsf{V}_1} \left(|\mathsf{V}_1'| - |N_G(\mathsf{V}_1')| \right). \tag{8.1}$$

Formula (8.1) is known as the Kőnig–Ore Formula.

Proof. Consider a new graph $\tilde{\mathsf{G}}$ obtained by adding $m := \max_{\mathsf{V}_1' \subset \mathsf{V}_1} (|\mathsf{V}_1'| - |N_G(\mathsf{V}_1')|)$ new nodes to the partition V_2, each of them connected to each node in V_1. Call $\tilde{\mathsf{V}}_2$ such an enlargement of V_2. In this manner $\tilde{\mathsf{G}}$ admits in a natural way a bipartition, as its node set is $\tilde{V} = \mathsf{V}_1 \dot\cup \tilde{\mathsf{V}}_2$. Observe that for all $\mathsf{V}_1' \subset \mathsf{V}_1$ one has $|N_{\tilde{\mathsf{G}}}(\mathsf{V}_1')| = |N_G(\mathsf{V}_1')| + m$. Since by definition of m one has $|\mathsf{V}_1'| \leq m + |N_G(\mathsf{V}_1')|$, one concludes that $|\mathsf{V}_1'| \leq |N_{\tilde{\mathsf{G}}}(\mathsf{V}_1')|$ for all $\mathsf{V}_1' \subset \mathsf{V}_1$. By Hall's marriage theorem there exists a matching $\tilde{\mathsf{E}}$ of V_1 in $\tilde{\mathsf{G}}$. Clearly, $\tilde{\mathsf{E}} \setminus \mathsf{E}$ contains at most m edges, and $\tilde{\mathsf{E}}$ induces in G a matching E^* with $|\mathsf{E}^*| + m \geq |\tilde{\mathsf{E}}| = |\mathsf{V}_1|$. Therefore, a matching $\hat{\mathsf{E}}$ with maximal cardinality satisfies the estimate $|\hat{\mathsf{E}}| \geq |\mathsf{E}^*| \geq |\mathsf{V}_1| - m$.

It remains to prove that $|\mathsf{V}_1| \geq |\hat{\mathsf{E}}| + \max_{\mathsf{V}_1' \subset \mathsf{V}_1} (|\mathsf{V}_1'| - |N_G(\mathsf{V}_1')|)$. Again, add $|\mathsf{V}_1| - |\hat{\mathsf{E}}|$ new nodes to V_2 and connect each of them to each node in V_1 – this yields a new graph G°. Complete the matching $\hat{\mathsf{E}}$ to a new matching E° in G°: of course $|\mathsf{E}^\circ| = |\mathsf{E}^\circ| + |\mathsf{V}_1| - |\mathsf{E}^\circ| = |\mathsf{V}_1|$. Again by Hall's Theorem,

$$|\mathsf{V}'| \leq |N_{\mathsf{G}^\circ}(\mathsf{V}')| \leq |N_G(\mathsf{V}')| + |\mathsf{V}_1| - |\mathsf{E}^\circ|$$

for all $\mathsf{V}' \subset \mathsf{V}_1$, and this yields the claim. □

Definition 8.11. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph. A set* $\mathsf{V}' \subset \mathsf{V}$ *is called a* ***covering*** *if each edge in* E *is incident in (at least) a node belonging to* V'. *The minimal cardinality of a covering of* G *is called* ***covering number of*** G *and is denoted by* $\beta(\mathsf{G})$.

Exercise 8.12. *Let* $n \in \mathrm{N}$, $n \geq 3$. *Compute the covering number of* $\mathsf{K}^n - \{e\}$, *where* e *is any edge of* K^n.

Lemma 8.13. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph without loops. Then the matching number of* G *is not larger than the covering number of* G, *i.e.,* $\alpha_0(\mathsf{G}) \leq \beta(\mathsf{G})$.

If moreover a matching and a covering have same cardinality n, *then* $\alpha_0(\mathsf{G}) = n = \beta(\mathsf{G})$.

Proof. Let $\tilde{\mathsf{E}}$ be a matching with maximal cardinality and $\tilde{\mathsf{V}}$ be a covering with minimal cardinality. Then by definition each edge in $\tilde{\mathsf{E}}$ has at least one endpoint in V', hence $\alpha_0(\mathsf{G}) = |\tilde{\mathsf{E}}| \leq |\tilde{\mathsf{V}}| = \beta(\mathsf{G})$.

The latter assertion promptly follows, if for any matching E' one has $|\mathsf{E}'| \geq \beta(\mathsf{G})$. □

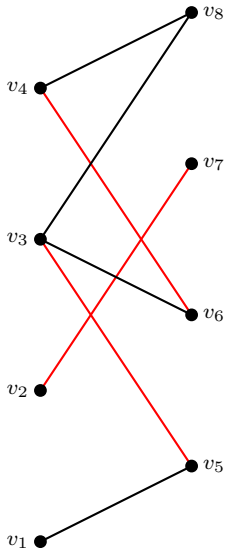
Definition 8.14. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a bipartite graph with* $\mathsf{V} = \mathsf{V}_1 \dot\cup \mathsf{V}_2$. *Let* E' *be a matching in* G. *An* E'***-alternating path*** *is a nontrivial path defined by* $(v_0, \ldots, v_n)$ *and* $(e_1, \ldots, e_n)$ *such that*

- $v_0 \in \mathsf{V}_1$,
- *no edge of* E' *is incident in* v_0,
- $e_i \notin \mathsf{E}'$ *for all odd indices* i,
- $e_i \in \mathsf{E}'$ *for all even indices* i.

An E'*-alternating path is called* E'***-augmenting*** *if* $v_n \in \mathsf{V}_2$ *and* $e_n \notin \mathsf{E}'$*, i.e., if no edge of* E' *is incident in* v_n*, either (in particular, an augmenting path has odd length).*

Remark 8.15. *Finding an* E'*-augmenting path permits to* augment *a matching* E'*: in fact, it suffices to replace the edges contained in the augmenting path that belong to* E' *(i.e., the edges with even index) by those that do not belong to* E' *(i.e., thos with odd index) to obtain a matching* E'' *with larger cardinality. In particular, if* E' *is a matching with maximal cardinality, then no* E'*-augmenting path exists.*

Example 8.16. *Consider the bipartite graph depicted below.*



A matching E' of non-maximal cardinality in a graph.

The red edges form a matching E' *of the graph. The path defined by the node sequence* $(v_1, v_5, v_3, v_6, v_4)$ *is an* E'*-alternating path. Furthermore, the node sequence* $(v_1, v_5, v_3, v_6, v_4, v_8)$ *defines an* E' *augmenting path, giving rise to a new matching of cardinality 4.*

Theorem 8.17 (D. Kőnig, 1931)**.** *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a bipartite graph without loops. Then the matching number of* G *agrees with the covering number of* G*, i.e.,* $\alpha_0(\mathsf{G}) = \beta(\mathsf{G})$.

Proof. Let $\mathsf{V} = \mathsf{V}_1 \dot\cup \mathsf{V}_2$. Let E' be a matching with maximal cardinality. Define a node set V' as follows: for any edge $e \in \mathsf{E}'$ with $g(e) = \{v_1, v_2\}$ pick the endpoint

- $v_2 \in \mathsf{V}_2$, if v_2 is endpoint of an E'-alternating path, or
- $v_1 \in \mathsf{V}_1$ otherwise.

Assume for a moment that we have already shown that V' is a covering: since each edge of E' has to be incident in (at least) one node of any covering, no covering can have less nodes than V', i.e., V' has minimal cardinality. By construction, $|\mathsf{V}'| = |\mathsf{E}'|$ and the assertion follows.

In order to show that V' is a covering, take $e \in \mathsf{E}$ and show that one of its endpoints v_1, v_2 lies in V'. This is clear by construction if $e \in \mathsf{E}'$; if this is not the case, by maximality E' contains an edge e' (say, with $g(e') = \{v_1', v_2'\}$) sharing an endpoint with e.

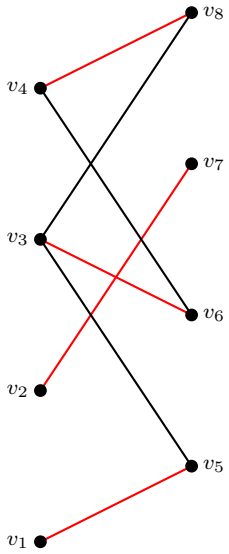
If no edge of E' is incident in v_1, but rather in v_2, then e would be an E'-alternating path and v_2 would be in V' by construction (as it is endpoint of $e' \in \mathsf{E}'$, too).

If on the other hand the node shared by e, e' is $v_1 \in \mathsf{V}_1$, but $v_1 \notin \mathsf{V}'$, then $v_2' \in \mathsf{V}'$ (again by construction of V', since e is an alternating path). Denote by P the alternating path ending in v_2' (it exists by definition of V'). Accordingly, either the path P cut at v_2 (if v_2 is contained in P) or another path P' obtained glueing P with e, e' (if v_2 is not contained in P) is an alternating path. In either case, such an alternating path ends in v_2 but cannot be augmenting (cf. Remark 8.15), hence v_2 has to be endpoint of some edge $e'' \in \mathsf{E}'$ and by construction $v_2 \in \mathsf{V}'$.

This completes the proof. □

Remark 8.18. *Observe that Kőnig's Theorem does not hold in the general case of possibly non-bipartite graphs. E.g., if* $\mathsf{G} = \mathsf{C}^3$, $\alpha_0(\mathsf{C}^3) = 1$ *while* $\beta(\mathsf{C}^3) = 2$.

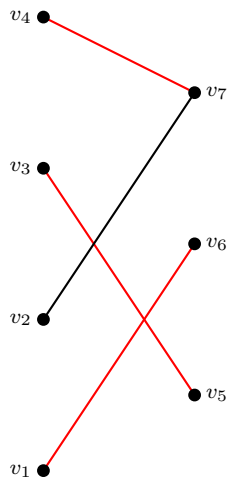
Example 8.19. *Consider again the bipartite graph introduced in Example 8.16. Since we have found an* E'*-augmenting path, we are able to provide a new matching* E' *of larger cardinality (in fact, of maximal cardinality).*



A matching E' of maximal cardinality in a graph.

In order to find a covering of G *of cardinality 4, apply the construction presented in the proof of Kőnig's Theorem to the edges from* E'. *Observe that the graph does not contain any* E'*-alternating path, since each node in* V_1 *is endpoint of an edge from* E', *hence the covering* V' *consists of the nodes* v_1, v_2, v_3, v_4.

Example 8.20. *Consider a new graph as in the picture below.*



A matching E' of maximal cardinality in a graph.

Observe that an E'-alternating path can only begin in v_2. In fact, the graph only contains the E'-alternating paths define dy node sequences (v_2, v_7, v_4) and its subpath (v_2, v_7). Accordingly, the construction presented in the proof of Kőnig's Theorem leads to a covering $V' = \{v_1, v_3, v_7\}$.

Definition 8.21. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a graph. A subgraph G' is called a k-**factor** of G if it is spanning and also k-regular, i.e., $d(v, \mathsf{G}') = k$ for all $v \in \mathsf{V}$.*

Remark 8.22. *1) By definition, a Hamiltonian cycle in a graph is a 2-factor. A Hamiltonian cycle of even length is a 1-factor.*

2) A path $(e_1, e_2, \ldots, e_m)$ of length m has a 1-factor if and only if m is odd, and the 1-factor is then given by the disjoint union of all edges with odd index, $\{e_1, e_3, \ldots, e_m\}$ together with their endpoints.

3) It is clear that G' is a 1-factor of G if and only if its edge set E' is a perfect matching of V.

4) It is also clear that if a graph contains a 1-factor whose edge set is E', then its number of nodes is twice $|\mathsf{E}'|$ – and in particular is an even number.

5) Observe that a graph G has a 1-factor if and only if the graph $\tilde{\mathsf{G}}$ obtained by deleting loops and unifying multiple edges of G has a 1-factor.

Corollary 8.23. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a bipartite simple graph such that $\mathsf{V} = \mathsf{V}_1 \dot\cup \mathsf{V}_2$. If G is k-regular, then G has a 1-factor.*

Proof. If a bipartite graph G is k-regular, then one sees (intuitively, by symmetry; or formally, by induction on k) that $|\mathsf{V}_1| = |\mathsf{V}_2|$. By Remark 8.22.3), it now suffices to show that G contains a matching of V_1: to this aim, we check the marriage condition. Let $\mathsf{V}' \subset \mathsf{V}_1$: by assumption, $k|\mathsf{V}'|$ edges connect V' to $|N_{\mathsf{G}}(\mathsf{V}')|$, a subset of the set of all edges incident in the nodes in $N_{\mathsf{G}}(\mathsf{V}')$ – whose cardinality is clearly $k|N_{\mathsf{G}}(\mathsf{V}')|$. Accordingly, $k|\mathsf{V}'| \leq k|N_{\mathsf{G}}(\mathsf{V}')|$, hence $|\mathsf{V}'| \leq |N_{\mathsf{G}}(\mathsf{V}')|$ and the marriage condition is satisfied. □

Corollary 8.24 (J. Petersen, 1891)**.** *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a $2k$-regular graph for some $k \in \mathbb{N}$. Then G has a 2-factor.*

PROOF. Without loss of generality we can assume G to be connected. Since each node has even degree, G contains a Euler tour defined by $(v_0, \ldots, v_n)$ and $(e_1, \ldots, e_n)$ with $v_0 = v_n$. Let us now consider a new graph $\tilde{\mathsf{G}}$ obtained by "splitting the nodes", i.e., by replacing each node v_i, $i = 0, \ldots, n-1$, by two ("incoming" and "outgoing") nodes v_i^+, v_i^-, and each edge e_i (with endpoints v_i, v_{i+1}) by a new edge with endpoints v_i^+, v_{i+1}^-. In particular, since no edge has been inserted between v_i^+, v_i^-, the new edges with endpoints v_{i-1}^+, v_i^- and v_i^+, v_{i+1}^- are not adjacent any more. The Eulerian tour comes into each node k times and goes out of each node k times, as it contains by definition all edges. Accordingly, $\tilde{\mathsf{G}}$ is simple and by construction bipartite (with respect to the sets of incoming and outgoing nodes). Accordingly, by Corollary 8.23 we obtain a 1-factor of $\tilde{\mathsf{G}}$. If incoming and outgoing nodes are glued together again, in order to build up the original graph, then we have constructed a 2-factor of G. □

The following is known as **Tutte's 1-factor theorem**.

Theorem 8.25 (W.T. Tutte, 1947)**.** *A graph $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ has a 1-factor if and only if the number $o(\mathsf{G} - \mathsf{V}')$ of connected components of $\mathsf{G} - \mathsf{V}'$ with odd node set's cardinality is at most equal with $|\mathsf{V}'|$, for all $\mathsf{V}' \subset \mathsf{V}$.*

Sometimes

$$o(\mathsf{G} - \mathsf{V}') \leq |\mathsf{V}'| \qquad \text{for all } \mathsf{V}' \subset \mathsf{V}$$

is dubbed **Tutte's condition**.

The following proof has been provided by I. Anderson in 1971.

PROOF. Let us first prove by induction on $|\mathsf{V}|$ that Tutte's condition is also sufficient. This is clearly true if $|\mathsf{V}| = 0$. Assume now that G is a graph with $|\mathsf{V}| \geq 1$ that satisfied Tutte's condition. Let also the assertion (i.e., existence of a 1-factor provided that Tutte's condition is satisfied) hold for all graphs whose node set's cardinality is less than $|\mathsf{V}|$. The proof will be based on properties of a set $\mathsf{V}_0 \subset \mathsf{V}$ of maximal cardinality among those satisfying

$$o(\mathsf{G} - \mathsf{V}') = |\mathsf{V}'| \tag{8.2}$$

To begin with, we prove that a node set that satisfies (8.2) exists and contains at least one node. By Tutte's condition applied to $\mathsf{V}' = \emptyset$ we deduce that $o(\mathsf{G}) = 0$, i.e., G has an even number of nodes. Let now v be any node: then $\mathsf{G} - \{v\}$ contains an odd number of nodes, therefore it has to feature at least one connected component containing an odd number of nodes. On the other hand, by Tutte's condition $o(\mathsf{G} - \{v\}) \leq 1$ and we conclude that $\mathsf{G} - \{v\}$ consists of *exactly* one connected component (which contains an odd number of nodes), i.e.,

$$o(\mathsf{G} - \{v\}) = |\{v\}| \qquad \text{for all } v \in \mathsf{V}, \tag{8.3}$$

as we wanted to prove.

Now, consider the connected components of $\mathsf{G} - \mathsf{V}'$, say $\Omega_1, \ldots, \Omega_{|\mathsf{V}_0|}$ (with a node set of odd cardinality) and $\mathsf{H}_1, \ldots, \mathsf{H}_m$ (with a node set of even cardinality).

To begin with, we prove that each H_i has a 1-factor, $i = 1 \ldots, m$. This is due to the fact that if V' is a set of nodes belonging to H_i, then

$$o(\mathsf{G} - (\mathsf{V}_0 \cup \mathsf{V}')) \leq |\mathsf{V}_0 \cup \mathsf{V}'| = |\mathsf{V}_0| + |\mathsf{V}'|$$

by Tutte's condition and because V' is a set of nodes of H_i, with $\mathsf{H}_i \subset \mathsf{G} - \mathsf{V}_0$, i.e., $\mathsf{V}' \cap \mathsf{V}_0 = \emptyset$. For the same reason

$$o(\mathsf{G} - (\mathsf{V}_0 \cup \mathsf{V}')) = o(\mathsf{G} - \mathsf{V}_0) + o(\mathsf{H}_i - \mathsf{V}'),$$

and since V_0 satisfies (8.2) we conclude that $o(\mathsf{H}_i - \mathsf{V}') \leq |\mathsf{V}'|$, i.e., Tutte's condition is satisfied by H_i. Since H_i has less nodes than G (due to the fact that $\mathsf{H}_i \subset \mathsf{G} - \mathsf{V}_0$ with $|\mathsf{V}_0| \geq 1$), we can apply the induction assumption and conclude that actually H_i has a 1-factor.

Take now $i = 1, \ldots, |\mathsf{V}_0|$ and $v \in \Omega_i$. Let us now prove that also $\Omega_i - \{v\}$ has a 1-factor. In fact, assume this to be false and observe that $\Omega_i - \{v\}$ has less nodes than G (and in fact, additionally $|\Omega_i - \{v\}|$ is even). Thus, the induction assumption applies and we can consider a node set V' contained in $\Omega_i - \{v\}$ such that Tutte's condition is not fulfilled, i.e.,

$$o(\Omega_i - (\{v\} \cup \mathsf{V}')) > |\mathsf{V}'| \tag{8.4}$$

for some $\mathsf{V}' \subset \Omega_i - \{v\}$. We now distinguish the cases of $|\mathsf{V}'|$ odd and $|\mathsf{V}'|$ even. In the former case, also $(\Omega_i - \{v\}) - \mathsf{V}' = \Omega_i - (\mathsf{V}' \cup \{v\})$ has odd cardinality, therefore also $o(\Omega_i - (\mathsf{V}' \cup \{v\}))$ is odd (otherwise one would have both an even number of connected components with even and with odd cardinality, i.e., the whole set $\Omega_i - (\mathsf{V}' \cup \{v\})$ would have even cardinaliy, a contradiction). Similarly, if instead $|\mathsf{V}'|$ is even, also $(\Omega_i - \{v\}) - \mathsf{V}' = \Omega_i - (\mathsf{V}' \cup \{v\})$ has even cardinality, therefore also $o(\Omega_i - (\mathsf{V}' \cup \{v\}))$ is even. Summing up, $|\mathsf{V}' \cup \{v\}|$ is odd (resp., even) if and only if $o(\Omega_i - (\mathsf{V}' \cup \{v\}))$ is odd (resp., even), i.e., their difference $o(\Omega_i - (\mathsf{V}' \cup \{v\})) - |\mathsf{V}' \cup \{v\}|$ is always even. Accordingly, it follows from (8.4) that

$$o(\Omega_i - (\{v\} \cup \mathsf{V}')) \geq |\mathsf{V}'| + 2. \tag{8.5}$$

We finally observe that Tutte's condition is satisfied by $\mathsf{G} - \{v\}$, hence

$$|\mathsf{V}_0| + |\mathsf{V}'| + 1 = |\mathsf{V}_0 \cup \{v\} \cup \mathsf{V}'| \geq o(\mathsf{G} - (\mathsf{V}_0 \cup \{v\} \cup \mathsf{V}')). \tag{8.6}$$

After removing V_0 from G we end up with $o(\mathsf{G} - \mathsf{V}_0)$ connected components whose node set has odd cardinality, and on the other hand in one of these connected components, Ω_i, we obtain $o(\Omega_i - \mathsf{V}' - \{x\})$ connected components whose node set has odd cardinality, after removing $\mathsf{V}' \cup \{x\}$). Summing up,

$$o(\mathsf{G} - (\mathsf{V}_0 \cup \{v\} \cup \mathsf{V}')) + 1 = o(\mathsf{G} - \mathsf{V}_0) + o(\Omega_i - \mathsf{V}' - \{v\}). \tag{8.7}$$

Since Tutte's condition is satisfied by $\Omega_i - \{v\}$

$$o(\mathsf{G} - \mathsf{V}_0) + o(\Omega_i - \mathsf{V}' - \{v\}) \geq |\mathsf{V}_0| + |\mathsf{V}'| + 2. \tag{8.8}$$

Combining (8.6)–(8.7)–(8.8), we conclude that

$$o(\mathsf{G} - (\mathsf{V}' \cup \mathsf{V}_0 \cup \{v\})) = |\mathsf{V}_0 \cup \{v\} \cup \mathsf{V}'|,$$

i.e., (8.2) is satisfied by $\mathsf{V}_0 \cup \{v\} \cup \mathsf{V}'$, a contradiction to the fact that V_0 has by construction maximal cardinality among those satsifying (8.2). We have thus proved that $\Omega_i - \{v\}$ has a 1-factor.

Furthermore, we can prove that that for each Ω_i there is an edge e_i connecting it with V_0 and that these edges $e_1, \ldots, e_{|\mathsf{V}_0|}$ are pairwise disjoint. In fact, consider a bipartite graph $\tilde{\mathsf{G}} = (\tilde{\mathsf{V}}, \tilde{\mathsf{E}}, \tilde{g})$ with respect to $\tilde{\mathsf{V}} = \tilde{\mathsf{V}}_1 \dot\cup \mathsf{V}_0$, where $\tilde{\mathsf{V}}_1 = \{\Omega_1, \ldots, \Omega_{\mathsf{V}_0}\}$, and such that any Ω_i is adjacent (in $\tilde{\mathsf{G}}$) to some $v \in \mathsf{V}_0$ if and only if there is an edge from E connecting v to some node in Ω_i. Clearly, in order to prove the claim it suffices to show that $\tilde{\mathsf{G}}$ contains a matching of V_0. This can be checked by Hall's marriage theorem: if $\mathsf{V}^* \subset \tilde{\mathsf{V}}_1$, then $|\mathsf{V}^*| \leq o(\mathsf{G} - N_{\tilde{\mathsf{G}}}(\mathsf{V}^*))$ and by Tutte's condition $o(\mathsf{G} - N_{\tilde{\mathsf{G}}}(\mathsf{V}^*)) \leq |N_{\tilde{\mathsf{G}}}(\mathsf{V}^*)|$. In other words, Hall's theorem applies to $\tilde{\mathsf{V}}_1$ in $\tilde{\mathsf{G}}$ and we conclude that a matching of $\tilde{\mathsf{G}}_1$ exists.

We are in the position to combine the above partial results. Consider first the graph $\mathsf{G}_1 := \mathsf{G}[\{e_1, \ldots, e_{|\mathsf{V}_0|}\}]$ generated by the set of pairwise disjoint edges e_i connecting each Ω_i with V_0, $i = 1, \ldots, |\mathsf{V}_0|$. Moreover, each subgraph $\Omega_i - \{v_i\}$ contains a 1-factor: denote their union by G_2. Finally, no edge connecting V_0 to any node of $\mathsf{H}_1, \ldots, \mathsf{H}_m$ is contained in either G_1 or G_2. Accordingly, we have no constraint in considering an arbitrary 1-factor of each subgraph H_i: denote their union by G_3. In this way we have constructed a 1-factor $\mathsf{G}^\circ = \mathsf{G}_1 \dot\cup \mathsf{G}_2 \dot\cup ; \mathsf{G}_3$ of G.

We finally prove that Tutte's condition is necessary. Let G have a 1-factor G'. Consider an arbitrary set $\mathsf{V}' \subset \mathsf{V}$. Then in each connected component $\tilde{\mathsf{G}}$ of $\mathsf{G} - \mathsf{V}'$ there is an even number of nodes contained in G'; while those nodes that are contained in $\tilde{\mathsf{G}}$ but not in G' are endpoints of an edge from G' whose other endpoint belongs to V'. In particular, for every connected component $\tilde{\mathsf{G}}$ of $\mathsf{G} - \mathsf{V}'$ with odd cardinality there is an edge of G' connecting a node in $\tilde{\mathsf{G}}$ and a node from V'. Since however G' consists by definition of disjoint edges, $\mathsf{G} - \mathsf{V}'$ contains at most $|\mathsf{V}'|$ connected components with odd cardinality. $\square$

CHAPTER 9

Symmetry in graphs

The notion of symmetric graphs has been introduced by R.M. Foster in 1932.

Definition 9.1. *Two nodes v, w of a graph $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ are called **similar** if there exists an automorphism π on G – i.e., a permutation $\pi : \mathsf{V} \to \mathsf{V}$ respecting adjacence – such that $\pi(v) = w$. Two edges e, f with $g(e) = \{v_1, v_2\}$ and $g(f) = \{w_1, w_2\}$ are called **similar** if there exists an automorphism π on G such that $\pi(\{v_1, w_1\}) = \{v_2, w_2\}$.*

In order to avoid trivial assertions, we assume graphs to be simple and exclude isolated nodes throughout this chapter.

Definition 9.2. *A graph $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ such that $\delta(\mathsf{G}) \geq 1$ is called*

- ***node-symmetric**, or sometimes **node-transitive**, if any two nodes are similar;*
- ***edge-symmetric**, or sometimes **edge-transitive**, if any two edges are similar;*
- ***symmetric** if it is both node-symmetric and edge-symmetric.*

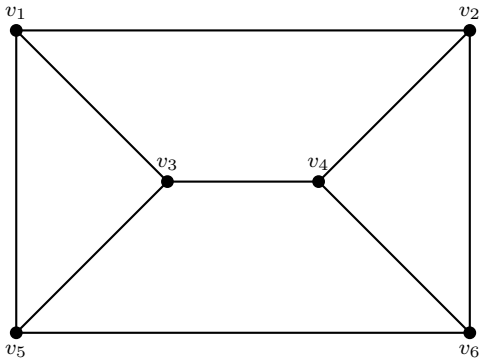
Exercise 9.3. *Prove that a node-symmetric graph is necessarily regular.*

Exercise 9.4. *Prove that the line graph of an edge-symmetric graph is node-symmetric.*

Exercise 9.5. *Clearly, each graph automorphism π can be described by means of a $|\mathsf{V}| \times |\mathsf{V}|$-matrix, which we also denote by π. Prove that a $|\mathsf{V}| \times |\mathsf{V}|$-matrix π is a graph automorphism if and only if π commutes with the adjacency matrix A, i.e., if and only if*

$$\pi A = A\pi.$$

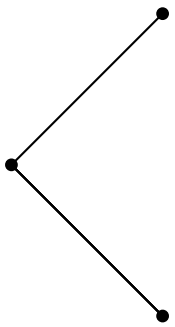
Example 9.6. *1) Consider the following simple graph G:*



We prove that G is not edge-symmetric, and in particular that (v_3, v_4) is not similar to (v_4, v_6). To this aim assume that an automorphism π on G exists such that (v_3, v_4) is mapped to (v_4, v_6). We only consider the case of $\pi(v_3) = v_4$ and $\pi(v_4) = v_6$ – the case

$\pi(v_3) = v_6$ and $\pi(v_4) = v_4$ being analogous. By definition of automorphism, v_6 has to be adjacent to $\pi(v_2), \pi(v_6)$. This is only possible if either $\pi(v_2) = v_2$ and $\pi(v_6) = v_5$, or if $\pi(v_2) = v_5$ and $\pi(v_6) = v_2$. But neither solution is allowed, since this would imply that $\pi(v_2), \pi(v_6)$ are not adjacent. This contradiction concludes the proof.

2) Consider the graph P^2.



As already remarked, a node-symmetric graph is necessarily regular. It follows that the graph P^2 (and, more generally, any star) is not node-symmetric.

Exercise 9.7. *An* edge automorphism *π is defined as a permutation of the set E such that if edges e, e' are adjacent, then also $\pi(e), \pi(e')$ are adjacent.*

(1) Prove that the set of all edge automorphisms defines a group, the so-called ***edge automorphism group*** $\mathrm{Aut}^*(\mathsf{G})$.

(2) Show that the edge automorphism group $\mathrm{Aut}^*(\mathsf{G})$ of a graph G is isomorphic to the automorphism group $\mathrm{Aut}(\mathsf{G}_L)$ of its line graph G_L.

Neither node-symmetric graphs need to be edge-symmetric, nor edge-symmetric graphs need to be node-symmetric.

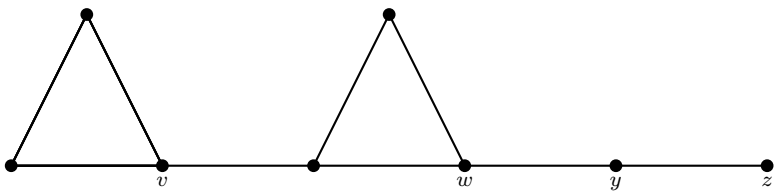
Exercise 9.8. *Prove that the graphs introduced in Example 9.6 are node-symmetric and edge-symmetric, respectively.*

Exercise 9.9. *Prove that similarity is an equivalence relation.*

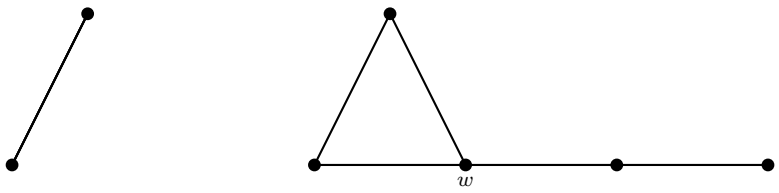
Theorem 9.10. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a simple graph such that $\delta(\mathsf{G}) \geq 1$. If two nodes $v, w \in \mathsf{V}$ are similar, then $\mathsf{G} - \{v\}$ is isomorphic to $\mathsf{G} - \{w\}$. The converse is not true.*

Proof. Observe that by definition for each automorphism π onto G and each node $v \in \mathsf{V}$, the graphs $\mathsf{G} - \{v\}$ and $\mathsf{G} - \{\pi(v)\}$ are isomorphic: this proves the former assertion.

To see that the converse is not true, consider the following example.

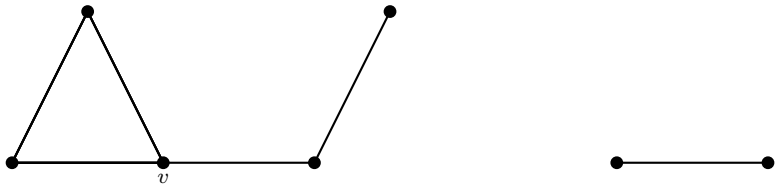


Deleting v and w we obtain the graphs



$\mathsf{G} - \{v\}$.

and



$\mathsf{G} - \{w\}$.

respectively. Thus, clearly $\mathsf{G} - \{v\}$ and $\mathsf{G} - \{w\}$ are isomorphic, but v, w are not similar. In fact, the node z is the only one with degree 1, hence any automorphism maps z into itself. Since adjacency is invariant under automorphisms, also y and consequently w are only similar to themselves. □

Proposition 9.11. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be an edge-symmetric simple graph such that $\delta(\mathsf{G}) \geq 1$. Then G is node-symmetric or bipartite.*

Graphs that are edge-symmetric but not node-symmetric are usually called **semi-symmetric**.

Proof. Let $e \in \mathsf{E}$ with $g(e) = \{v_1, v_2\}$. Due to edge-symmetry, there are at least $|\mathsf{E}|$ automorphisms onto G, say π_i mapping e to e_i, $i = 1, \ldots, |\mathsf{E}|$. Let $\mathsf{V}_1 := \{\pi_1(v_1), \ldots, \pi_{|\mathsf{E}|}(v_1)\}$ and $\mathsf{V}_2 := \{\pi_1(v_2), \ldots, \pi_{|\mathsf{E}|}(v_2)\}$. Now observe that since G does not contain isolated nodes, for any node there exists an edge incident in it, i.e., any of the edges e_i. Summing up, any node is endpoint of $\pi_j(e)$ for some $j = 1, \ldots, |\mathsf{E}|$, hence $\mathsf{V} = \mathsf{V}_1 \cup \mathsf{V}_2$.

Let first $\mathsf{V}_1 \cap \mathsf{V}_2 = \emptyset$. Consider an edge $f \in \mathsf{E}$ with endpoints v, w. We are going to show that $v \in \mathsf{V}_1$ and $w \in \mathsf{V}_2$. Let $j \in \{1, \ldots, |\mathsf{E}|\}$ such that $\pi_j(e) = f$. Accordingly, the endpoints of e are similar (with respect to the same automorphism!) to the endpoints of f: say (without loss of generality) $\pi_j(v_1) = v$ and $\pi_j(v_2) = w$. Then, $v \in V_1$ and $w \in V_2$.

Let now $\mathsf{V}_1, \mathsf{V}_2$ be non-disjoint: say, assume that $\pi_i(v_1) = \pi_j(v_2)$ for some $i, j \in \{1, \ldots, |\mathsf{E}|\}$. We are going to prove that G is node-symmetric, i.e., that any two nodes $v, w \in \mathsf{V}$ are similar. This is already clear if $v, w \in \mathsf{V}_1$ or $v, w \in \mathsf{V}_2$, since in that case $\pi_\ell(v_1) = v$ and $\pi_k(v_1) = w$ for some $\ell, k \in \{1, \ldots, |\mathsf{E}|\}$, i.e., $v = \pi_\ell(\pi_k^{-1}(v))$. Thus, let us consider the case of $v \in \mathsf{V}_1$ and $w \in \mathsf{V}_2$. Then $\pi_\ell(v_1) = v$ and $\pi_k(v_2) = w$ for some $\ell, k \in \{1, \ldots, |\mathsf{E}|\}$, and we conclude that $v = \pi_\ell(v_1) = \pi_\ell(\pi_i^{-1}(\pi_j(v_2))) = \pi_\ell(\pi_i^{-1}(\pi_j(\pi_k^{-1}(w))))$, i.e., v, w are similar. □

Definition 9.12. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a graph such that $\delta(\mathsf{G}) \geq 1$. Let $e \in \mathsf{E}$ with $g(e) = \{v, w\}$. Then the pair $\{d(v, \mathsf{G}), d(w, \mathsf{G})\}$ is called **degree of** e, and G is called **edge-regular** if all edges have same degree.*

Example 9.13. *Show that each bipartite graph of the form* $\mathsf{K}^{m,n}$ *is edge-regular with degree* $\{m, n\}$.

Exercise 9.14. *How does the notion of edge degree in a graph* G *relate to that of (node) degree in the associated line graph* G_L*?*

Corollary 9.15. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be an edge-symmetric simple graph such that* $\delta(\mathsf{G}) \geq 1$. *Then the following assertions hold.*

- *If the degree of any edge is* $\{m, n\}$ *with* $m \neq n$, *then* G *is bipartite.*
- *If* G *has an odd number of nodes and the degree of any edge is* $\{m, n\}$ *with* $m = n$, *then* G *is node-symmetric.*
- *If* G *has an even number of nodes and it is* r*-regular with* $2r \geq |\mathsf{V}|$, *then* G *is node-symmetric.*

Remark 9.16. *Of course, the above theorem does not exclude that* G *is both bipartite and node-symmetric: this is e.g. the case of* C^6.

Exercise 9.17. *Is the Petersen graph node-symmetric? Is it edge-symmetric? Explain your answer.*

The essential feature of symmetric graphs is the possibility of translate graph theoretical problems into group theoretical language, and then solve then by group methods. A simple example of application of symmetry methods is given in the following. Observe that the notions of adjacence and hence of graph automorphism are also defined for oriented graphs. Therefore, we can extend the definition of node/edge-symmetric graph to the oriented case.

Proposition 9.18. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a node-symmetric simple oriented graph. If* G *is connected, then it is strongly connected.*

Proof. For each $v \in \mathsf{V}$ denote by $B(v) \subset \mathsf{V}$ the set of vertices that can be reached by directed paths starting in v. Let now $w \in B(v)$: then clearly $B(w) \subset B(v)$. Since the graph is vertex-symmetric, there is an automorphism mapping v into w, and therefore $B(v)$ into $B(w)$. Accordingly, the subsets $B(v), B(w) \subset \mathsf{V}$ have the same cardinality, and therefore one also has $v \in B(w)$. This completes the proof. □

Remark 9.19. *What if a graph is not symmetric, but is not far away from being so? Erdős and Rényi introduced in 1963 the notion of* ***degree of asymmetry*** $A(G)$ *of a graph* G *as the minimal number of edges that have to be added to and/or removed from* G *in order to make it symmetric. Among other things, they proved that* $A(\mathsf{G}) \leq \frac{1}{2}(|\mathsf{V}| - 1)$, *cf.* [**3**, Thm. 9.12].

CHAPTER 10

Planar graphs and colourability

One of the most important features of graphs is the possibility they allow to be used to model concrete situations in a 2-dimensional "world", like the surface of a 3-dimensional domain, like the sphere S^2. Therefore, it is particularly relevant to study the interactions between graph theory and euclidean geometry. We recall the following elementary notion.

Definition 10.1. *An n-dimensional **simple arc** is a continuous function $s : [0,1] \to \mathrm{R}^n$ whose restriction to $[0,1)$ is injective, and $s(0), s(1)$ are its endpoints. The **interior** is the set $s(0,1)$.*

Definition 10.2. *Let $n \in \mathrm{N}$. We say that a graph G can be **embedded** in R^n if it is possible*

(1) to associate to each node v_i a point $x_i \in \mathrm{R}^n$ in a bijective way, and
(2) to connect two points x_i, x_j by an n-dimensional simple arc s_{ij} if and only if v_i, v_j are adjacent, in such a way that different arcs do not share any internal points.

Example 10.3. *Consider a finite number of points on the n-dimensional sphere. Connect them by arcs along the sphere. This defines a graph embedded in R^n.*

Accordingly, we identify edges with simple arcs. With an abuse of terminology we therefore speak of "interior e° of an edge e". Similarly, we also consider objects like $\mathrm{R}^3\mathsf{G}$ – by which we mean $\mathrm{R}^3 \setminus \bigcup_{j=1}^{|\mathsf{E}|} s_j[0,1]$, where s_j is the simple arc associated with the edge e_j.

Proposition 10.4. *Each graph can be embedded in R^3.*

Proof. The embedding can be performed as follows. First of all, associate a point of the x-axis with each node. Then, embed each edge e_i into a different plane P_i containing the x-axis: this can be done connecting its endpoints v, w by any curve in P_i that touches the x-axis *only* in v, w. □

Definition 10.5. *A graph is called **planar** if it can be embedded in R^2.*

Observe that if G' is a cycle in a planar graph G, then after embedding in R^2 its edges constitute a Jordan curve. Recall the following fundamental result – so-called **Jordan curve theorem** – which we shall not prove – cf. `http://en.wikipedia.org/wiki/Jordan_curve_theorem`.

Proposition 10.6 (O. Weblen, 1905)**.** *Let s be a Jordan curve (i.e., a simple closed curve in R^2). Then the complement of the image of s consists of two distinct connected components. One of these components is bounded (the* interior*) and the other is unbounded (the* exterior*). The image of s is the boundary of each component.*

Example 10.7. *The triangle and more generally all polygons are planar. Wheels are planar, too.*

Each graph containing a nonplanar subgraph is itself nonplanar. It easily follows that no graph containing as a subgraph *a subdivision* of a nonplanar graph can be planar. The proof of this fact is less trivial than one might guess: we refer to [**20**, Thm. XI.19].

Proposition 10.8. *Each subgraph of a planar graph* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *is planar.*

Corollary 10.9. *Let* s_1, s_2, s_3 *be three* 2*-dimensional simple arcs with same endpoints* v, w *but otherwise disjoint. Then the following assertions hold.*

(1) $\mathrm{R}^2 \setminus (s_1 \cup s_2 \cup s_3)$ *consists of three disjoint connected components. Their boundaries are* $s_1 \cup s_2, s_2 \cup s_3, s_1 \cup s_3$*.*

(2) Let s *be a further simple arc connecting two points of* $s_1 \setminus \{v, w\}, s_3 \setminus \{v, w\}$*. By the Jordan curve theorem there is only one connected component of* $\mathrm{R}^2 \setminus (s_1 \cup s_3)$ *that contains* s_2*, say* F*: if the interior of* s *is contained in* F*, then the interiors of* s *and* s_2 *are nondisjoint.*

Definition 10.10. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a planar graph. By the Jordan curve theorem, each cycle* C_i *in* G *defines two closed connected components* $F_i, F_i^C \subset \mathrm{R}^2$ *– one bounded and one unbounded. All of the components* F_i *as well as the unbounded component* $\cap_i F_i^C$ *are called* ***faces****. The face set of a planar graph is often denoted by* F*. If the boundaries of two faces* F, F' *are non-disjoint and their intersection consists of infinitely many points, then* F, F' *are called* ***adjacent****.*

If G *is a forest, it is clearly planar. Then* G *does not contain any cycle and the Jordan curve theorem does not apply. However,* $\mathrm{R}^2 \setminus \mathsf{G}$ *is connected and we conventionally call it a face, too.*

Exercise 10.11. *How many faces are defined by a cycle* C^n*? And by a wheel* W^n*?*

In the following we denote by ∂F the boundary of a face F..

A famous formula relating the number of nodes, edges and faces of a planar graph (or more generally, in fact, of a 3-d structure) is due to L. Euler. Before stating and proving it, we will recall the following result. We do not provide its proof and refer to [**8**, Lemma 4.2.2].

Lemma 10.12. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a planar graph and* e *be an edge of* G*. Then the following assertions hold.*

(a) Let Γ *be the boundary of a face of* G*. Then either* e *is contained in* Γ *or else the set* e° *of all points in the interior of* e *is disjoint from* Γ*.*

(b) If e *lies on a cycle of* G*, then* e *is contained in the boundaries of exactly two faces of* G*.*

(c) If e *lies on* no cycle *of* G*, then* e *lies on the boundary of exactly one face of* G*.*

Remark 10.13. *Thus, by Lemma 10.12.(c) a forest has one face.*

Proposition 10.14 (Euler's formula)**.** *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a connected planar graph with face set* F*. Then the formula*

$$|\mathsf{V}| + |\mathsf{F}| - |\mathsf{E}| = 2$$

holds.

Remark 10.15. *Euler's formula does not hold for unconnected graphs: simply think of a forest with more than one connected components. However, it can be proved that any planar graph satisfies* $|\mathsf{V}| + |\mathsf{F}| - |\mathsf{E}| = 1 + \kappa(\mathsf{G})$.

Euler's formula has been found by L. Euler and formally proved by A.L. Cauchy for surfaces of convex polyhedra. The formula reads essentially in the same wayin the presented case of planar graphs. Observe that it does not hold for nonconvex polyhedra.

Several other proofs are known: cf. `http://www.ics.uci.edu/~eppstein/junkyard/euler`

Proof. If $|\mathsf{F}| = 1$, then by the Jordan curve theorem the graph's embedding in R^2 cannot contain any closed curve, i.e., it is (connected and) acyclic and therefore a tree. By Remark 10.13, it has one face. By Exercise 5.3, the formula follows.

Let now consider the case of a graph with $|\mathsf{F}| \geq 2$, i.e., assume the graph to contain a cycle. Take an edge e along a cycle. Then by Lemma 10.12.(b) $\mathsf{G} - \{e\}$ has *one edge less*, the same number of nodes, and *one face less* then G or it. Thus, Euler's formula holds for $\mathsf{G} - \{e\}$ if and only if it holds for G. It therefore suffices to find out whehter Euler's formula actually applies to $\mathsf{G} - \{e\}$, since its validity for the latter graph is equivalent to its validity for G. Observe that since e lies on a cycle, $\mathsf{G} - \{e\}$ is connected. In this way, we can recursively reduce G to a tree (it is the same procedure as in the algorithm for finding a spanning tree of a graph, cf. Proposition 5.10). We can now apply the reasoning presented in the case of a tree. This concludes the proof. □

Remark 10.16. *Consider the case of a connected planar graph and denote by* $\mathsf{F}_{(i)}$ *the subset of* F *containing those faces having exactly* i *edges in their boundary. Of course,* $\mathsf{F} = \bigcup_{i \in \mathrm{N}} \mathsf{F}_{(i)}$ *(the union is in fact disjoint) and accordingly* $|\mathsf{F}| = \sum_{i \in \mathrm{N}} |\mathsf{F}_{(i)}|$. *If in particular* G *has no bridge, then each edge is in the boundary of two faces and accordingly* $\sum_{i \in \mathrm{N}} i|\mathsf{F}_{(i)}| = 2|\mathsf{E}|$.

Exercise 10.17. *Show that the complete graph* K^n *is planar if and only if* $n \leq 4$.

Deduce that each simple graph with $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *with* $|\mathsf{V}| \leq 4$ *is planar.*

Exercise 10.18. *(1) Each polyhedron (a 3-dimensional geometric object with 2-dimensional flat faces and 1-dimensional straight edges) can be considered as a graph. Show that in fact each polyhedron defines (i.e., it can be drawn as) a planar graph..*

(2) Conclude that Euler's formula holds for polyhedra, too.

(3) Deduce that if each face of a polyhedron is a pentagon, then the polyhedron necessarily contains exactly 12 pentagons.

(4) Prove: If more generally the face set of a polyhedron only consists of p *pentagons and* h *hexagons, then* $p = 12$.

Proposition 10.19. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a simple and connected planar graph. Let the girth* $g(\mathsf{G}) \geq 3$. *Then,* G *has at most*

$$\max \left\{ \frac{g(\mathsf{G})}{g(\mathsf{G}) - 2}(|\mathsf{V}| - 2), |\mathsf{V}| - 1 \right\} \tag{10.1}$$

edges.

Remark 10.20. *The above formula is well-defined if* $g(\mathsf{G}) < \infty$, *i.e., if* G *is not acyclic – that is, if* G *is a forest. However, even if* G *is a forest, then we already know that it*

contains at most $|\mathsf{V}| - \kappa(\mathsf{G}) \leq |\mathsf{V}| - 1$ *edges. In other words, the above formula also holds in the limit* $g(\mathsf{G}) \to \infty$*, and in the following we will quote* (10.1) *without distinguishing the cases of* G *acyclic and non-acyclic.*

Proof. We can assume without loss of generality G to be connected. We will prove the assertion by induction on $|\mathsf{V}|$.

If $|\mathsf{V}| \leq g(\mathsf{G}) - 1$, then G does not contain any cycle, i.e., it is a forest and accordingly $|\mathsf{E}| \leq |\mathsf{V}| - 1$.

Let $|\mathsf{V}| \geq g(\mathsf{G})$ and (10.1) hold for smaller $|\mathsf{V}|$. We consider two cases:

a) G has a bridge e, so that $\mathsf{G} - e$ consists of two disjoint subgraphs $\mathsf{G}_1 = (\mathsf{V}_1, \mathsf{E}_1, g_1)$ and $\mathsf{G}_2 := (\mathsf{V}_2, \mathsf{E}_2, g_2)$ (with either of the graphs possibly reduced to an isolated node).

By the induction hypothesis one sees that

$$\begin{aligned} |\mathsf{E}| = |\mathsf{E}_1| + |\mathsf{E}_2| + 1 \quad &\leq \quad \max\left\{\frac{g(\mathsf{G}_1)}{g(\mathsf{G}_1)-2}(|\mathsf{V}_1|-2), |\mathsf{V}_1|-1\right\} \\ &\qquad + \max\left\{\frac{g(\mathsf{G}_2)}{g(\mathsf{G}_2)-2}(|\mathsf{V}_2|-2), |\mathsf{V}_2|-1\right\} + 1 \\ &\leq \quad \max\left\{\frac{g(\mathsf{G})}{g(\mathsf{G})-2}(|\mathsf{V}|-2), |\mathsf{V}|-1\right\} \end{aligned}$$

(using the induction hypothesis is consistent by Remark 10.20).

b) G has no bridge. In this case, each face contains on its boundary at least $g(\mathsf{G})$ edges since its boundary is necessarily a cycle. Hence

$$2|\mathsf{E}| = \sum_{i \in \mathrm{N}} i|\mathsf{F}_{(i)}| = \sum_{i \geq g(\mathsf{G})} i|\mathsf{F}_{(i)}| \geq \sum_{i \in \mathrm{N}} g(\mathsf{G})|\mathsf{F}_{(i)}| = g(\mathsf{G})|\mathsf{F}|.$$

Accordingly, by Euler's formula $|\mathsf{E}| = |\mathsf{V}| + |\mathsf{F}| - 2 \leq |\mathsf{V}| + 2\frac{|\mathsf{E}|}{g(\mathsf{G})} - 2$, thus

$$|\mathsf{E}| \leq \frac{g(\mathsf{G})}{g(\mathsf{G})-2}(|\mathsf{V}|-2).$$

This concludes the proof. □

Example 10.21. *The complete graph* K^5 *has girth* $g(\mathsf{G}) = 3$ *and* $\binom{5}{2} = 10$ *edges, thus it cannot be planar by Remark 10.38.*

Also $\mathsf{K}^{3,3}$ *cannot be planar, since it has girth 4 and hence by Proposition 10.19 it could have at most 8 edges, if it were planar.*

Exercise 10.22. *Prove that the Petersen graph is not planar.*

Example 10.23. *A tessellation is, in general, a way of covering a domain* $\Omega \subset \mathrm{R}^2$ *by juxtaposition of (finitely many) 2-dimensional polygons with neither overlaps nor gaps. This clearly defines a planar graph. There are only three possible tessellation obtained by juxtaposition of only one figure: the triangular, square and hexagonal tessellations. It is possible to prove that there are exactly seventeen ways to tessellate a planar domain periodically, cf.* `http://en.wikipedia.org/wiki/Wallpaper_group`*. They are called* ***wallpaper groups****.*

Exercise 10.24. *1) Crosswords schemes can be seen as good examples of planar graphs. Propose the construction of a graph* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *associated with a crossword scheme.*

2) One of the most famous and authoritative crosswords in the world is that published daily in the New York Times. Currently it is edited by Will Shortz. He has formulated the following set of rules for authors:

(i) The pattern of black-and-white squares must be symmetrical. Generally this rule means that if you turn the grid upside-down, the pattern will look the same as it does right-side-up.

(ii) Do not use too many black squares. In the old days of puzzles, black squares were not allowed to occupy more than 16% of a grid. Nowadays there is no strict limit, in order to allow maximum flexibility for the placement of theme entries. Still, "cheater" black squares (ones that do not affect the number of words in the puzzle, but are added to make constructing easier) should be kept to a minimum, and large clumps of black squares anywhere in a grid are strongly discouraged.

(iii) Do not use unkeyed letters (letters that appear in only one word across or down). In fairness to solvers, every letter has to be appear in both an Across and a Down word.

(iv) Do not use two-letter words. The minimum word length is three letters.

(v) The grid must have all-over interlock. In other words, the black squares may not cut the grid up into separate pieces. A solver, theoretically, should be able to able to proceed from any section of the grid to any other without having to stop and start over.

(vi) Long theme entries must be symmetrically placed. If there is a major theme entry three rows down from the top of the grid, for instance, then there must be another theme entry in the same position three rows up from the bottom. Also, as a general rule, no nontheme entry should be longer than any theme entry.

(vii) Do not repeat words in the grid.

(viii) Do not make up words and phrases. Every answer must have a reference or else be in common use in everyday speech or writing.

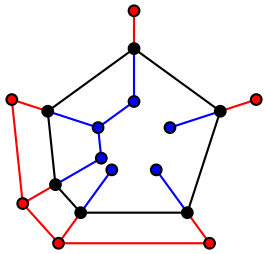
Which of the above rules can be formulated in terms of graph theoretical notions?

10.1. Kuratowski's Theorem

By Example 10.21, neither K^5 nor $\mathsf{K}^{3,3}$ are planar. By Example 10.7, no subdivision of these both graphs can be subgraph of a planar subgraph. The main result of this chapter, Kuratovski's theorem, shows in fact that this characterizes planar graphs already. The relevance of this criterion is that planarity seems to be a *global* property, but can be proved/disproved by *locally* looking at the connectivity properties of individual nodes.

Prior to proving Kuratowski's Theorem, we introduce the following notions.

Definition 10.25. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a planar graph containing a cycle* C *as a subset. By the Jordan curve theorem, two connected components are defined: a bounded one* (F) *and an unbounded one* (F^C)*. Due to planarity, the interior of each edge is either contained in* F*, or in* F^C*. The connected components of the subgraph of* G *generated by all the edges whose interior is contained in* F *(resp.,* F^C*) are called the* ***internal pieces*** *(resp.,* ***external pieces****) of* G*.*



A cycle in a graph (in black) and the associated internal and external pieces (in blue and red, respectively).

Lemma 10.26. *(1) Each simple, 2-connected planar graph* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *can be embedded in* R^2 *in such a way that any given face is the unbounded one.*

(2) In particular, it can be embedded in R^2 *in such a way that any given edge (and hence any given node, too) is in the boundary of the unbounded face.*

The proof is based on the notion of **stereographic projection** of a sphere $\mathsf{S}^n \subset \mathrm{R}^{n+1}$ on R^n, i.e., the *picture* one obtains by

- tracing a segment between each point x of S^n and the north pole N of S^n,
- extending it to a line l of R^{n+1},
- considering a hyperplane R^n outside S^n, and
- associating with x the point Px, where $\{Px\} := l \cap \mathrm{R}^n$.

The mapping $\mathsf{S}^n \setminus \{N\} \ni x \mapsto Px \in \mathrm{R}^n$ is a bijection.

Remark 10.27. *Observe that in particular each planar graph can be naturally embedded on a sphere* S^2 *in such a way that the edges correspond to simple arcs on the surface that do not share internal points.*

In particular, if a planar graph contains a cycle, then it is possible to embed it in S^2 *in such a way that the cycle corresponds to the equator of* S^2*.*

Proof. (1) Now, pick a face F. Embed G in a sphere S^2 (this is always possible, due to planarity) and choose as north pole N a point inside F. Clearly, $P(F \setminus \{N\})$ turns out to be the unbounded outer face of the graph's drawing.

(2) Pick an edge e. Consider any face of whose boundary e belongs to and apply (1). □

Lemma 10.28 (H. Whitney, 1932)**.** *A simple graph* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *is planar if and only if each of its blocks is planar.*

Proof. One implication is clear.

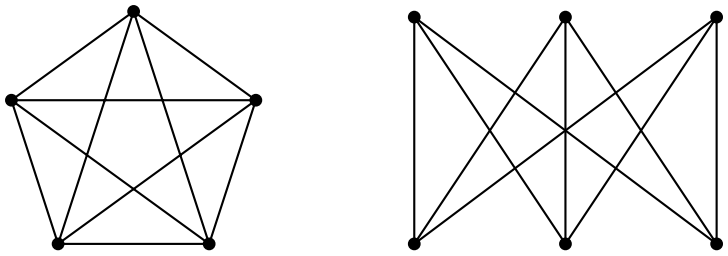
Let now assume that each of the blocks of G is planar. Without loss of generality we can assume G to be connected. We perform the proof by induction on $\beta(\mathsf{G})$, the number of blocks in G. If $\beta(\mathsf{G}) = 1$, i.e., if G is a block, then it is planar by assumption. Let us now prove the induction step: let G consist of $\beta(\mathsf{G})$ blocks, each of which planar. Consider a block $\mathsf{G}' = (\mathsf{V}', \mathsf{E}', g')$: by Theorem 6.9 it contains a cutvertex, say v. By definition, $\mathsf{G} - \{v\}$ is unconnected and consists of two connected components with node sets V_1 and V_2, respectively. Consider the subgraphs $\mathsf{G}_1 := \mathsf{G} - (\mathsf{V}_1 \setminus \{v\})$ and $\mathsf{G}_2 := \mathsf{G} - (\mathsf{V}_2 \setminus \{v\})$: both of them contains less than $\beta(\mathsf{G})$ blocks, all of which are by assumption planar. By induction hypotheses, both G_1 and G_2 are therefore planar. Since by Lemma 10.26 we can

assume without loss of generality that v lies on the boundary of the unbounded face of G_1, one can "glue" G_1 to G_2 in v in such a way that G is planar. □

Basing on the above result, G.A. Dirac and S. Schuster were able in 1954 to deliver a new proof of an already known result due to K. Kuratowski. Several other proofs of this result are known, including a constructive one due to W.T. Tutte, cf. [**20**, Chapter 11]. A slight emprovement of this result, yielding a *convex* drawing, can be found in [**8**, § 3.4]. Here we follow the proof presented in [**13**, Chapter 11].

Proposition 10.29 (K. Kuratowski, 1930)**.** *A simple graph $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ is planar if and only if it does not contain a subgraph that is a subdivision of K^5 or $\mathsf{K}^{3,3}$.*

In other words, a simple graph is planar if and only if neither K^5 nor $\mathsf{K}^{3,3}$ are topological minors of it.



The graphs K^5 and $\mathsf{K}^{3,3}$.

Proof. One implication is clear by Example 10.21.

The converse implication is proved by contradiction and is based on the following points:

Consider a graph $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ that has minimal number of edges among those non-planar graphs *not* containing a subdivision of either K^5 or $\mathsf{K}^{3,3}$. Therefore, all graphs with less than $|\mathsf{E}|$ edges are either non-planar *and* containing a subdivision of $\mathsf{K}^{3,3}$ or of K^5, or they are planar. By Exercise 10.17, G has at least 5 nodes and by Lemma 10.28 we can assume without loss of generality G to be a block.

(1) Show that $\delta(\mathsf{G}) \geq 3$.

In particular, G cannot be a bridge (of itself), hence it is a 2-connected graph – and by Proposition 3.10 also 2-edge-connected. Take some $e = (v, w) \in \mathsf{E}$ and observe that $\mathsf{G}_p := \mathsf{G} - \{e\}$ is planar. Moreover, G_p is surely 1-edge-connected, hence connected.

(2) Show that there is a cycle $\mathsf{C} \subset \mathsf{G}_p$ that contains v, w.

Among such cycles, consider one that divides R^2 in two connected components (by the Jordan curve theorem) such that the bounded one surrounds a maximal number of faces of G: we call it $\mathsf{C}_{v.w}$ and denote by $(v = v_1, \ldots, w = v_{i_0}, \ldots, v_n)$ and $(e_1, \ldots, e_n)$ the nodes and edges that define it. Take v_i, v_j, $i \neq j \pm 1$, and denote by $\overline{\mathsf{C}}_{ij}$

- either the path defined by $(v_i, \ldots, v_j)$ if $i < j$,
- or the path defined by $(v_i, \ldots, v_1, \ldots, v_j)$ if $j < i$.

Furthermore, denote by C°_{ij}

- either the path defined by $(v_{i+1}, \ldots, v_{j-1})$ if $i < j - 1$,
- or the path defined by $(v_{i+1}, \ldots, v_1, \ldots, v_{j-1})$ if $j < i - 1$.

(3) Each external piece is incident in exactly two nodes of C, one belonging to $\mathsf{C}^\circ_{1i_0}$ and one to $\mathsf{C}^\circ_{i_01}$.

(4) There exists an external piece $\tilde{\mathsf{G}}$, say incident in the nodes v_ℓ and v_k belonging to $\mathsf{C}^\circ_{1i_0}$ and to $\mathsf{C}^\circ_{i_01}$, respectively, such that an internal piece I that is incident in $\mathsf{C}^\circ_{1i_0}$, $\mathsf{C}^\circ_{i_01}$, $\mathsf{C}^\circ_{\ell k}$, and $\mathsf{C}^\circ_{k\ell}$ exists – say in the nodes $w_1 \in I \cap \mathsf{C}^\circ_{1i_0}$, $w_2 \in I \cap \mathsf{C}^\circ_{i_01}$, $w_3 \in I \cap \mathsf{C}^\circ_{\ell k}$, $w_4 \in I \cap \mathsf{C}^\circ_{k\ell}$.

(Of course, w_1, w_2, w_3, w_4 need not be the unique nodes with these properties).

(5) Finally, we consider four cases:
   (i) w_3 and w_4 belong to $\mathsf{C}^\circ_{1i_0}$ and $\mathsf{C}^\circ_{i_01}$, respectively;
   (ii) w_3 and w_4 both belong to either $\mathsf{C}^\circ_{1i_0}$ or $\mathsf{C}^\circ_{i_01}$
   (iii) $w_3 = w$ and $w_4 \neq v$;
   (iv) $w_3 = w$ and $w_4 = v$ – and we can also assume that $w_1 = v_\ell$ and $w_2 = v_k$, since otherwise one of the previous cases would hold;

   and show that each of them leads to a contradiction. Thus, the assertion is finally proved.

To conclude, let us prove the above mentioned points.

(1) If a node v of degree 1 exists, the edge $e = (v, w)$ incident in v can be removed from G obtaining a graph $\mathsf{G}_p := \mathsf{G} - \{e\}$ with less edges and not containing a subdivision of K^5 or $\mathsf{K}^{3,3}$, either – hence, a planar graph. Take a face F on whose boundary w lies. Then, it is possible to draw in the interior of F another point and a simple arc connecting it to w and not crossing any other simple arc. Clearly, this is a planar embedding of $\mathsf{G} = \mathsf{G}_p \cup \{e\}$, a contradiction.

Assume now that a node v of degree 2 exists and denote by w, z its only adjacent nodes. If the edge (w, z) exists, then consider $\mathsf{G} - \{(w, z)\}$, a subgraph of the original graph. Since it has less edges of G, and in particular it cannot contain subdivisions of K^5 or $\mathsf{K}^{3,3}$ as G does not, we conclude by construction of G that $\mathsf{G} - \{(w, z)\}$ is planar. Since $(w, v), (v, z) \in \mathsf{E}$ and $d(v, \mathsf{G}) = 2$, it is certainly possible to draw a simple arc between (w, z) in such a way that also G remains planar, a contradiction.

If instead in the graph G the nodes v, w are not adjacent, consider the new graph $(\mathsf{G} - \{(v, w), (v, z)\}) \cup \{(w, z)\}$. Again, it has less edges than G, and in fact G is a subdivision of it. For this reason, $(\mathsf{G} - \{(v, w), (v, z)\}) \cup \{(w, z)\}$ cannot contain a subdivision of K^5 or $\mathsf{K}^{3,3}$, either, since G does not. It follows as above that $(\mathsf{G} - \{(v, w), (v, z)\}) \cup \{(w, z)\}$ is planar. Replacing (w, z) by the two original edges $(v, w), (v, z)$ does not affect planarity: i.e., G is planar, again a contradiction.

(2) Assume that there exists no cycle $\mathsf{C} \subset \mathsf{G}_p$ that contains v, w. Then, it follows from Proposition 6.17 that v, w belong to different blocks of G_p (observe that G_p need not be a block, despite G is). We can assume without loss of generality that G_p consists of 2 blocks – in not, repeat the following argumentation for each block.

By Remark 6.18 there exists a cutvertex z of G_p that belongs to each path connecting v, w. Consider the graph $\mathsf{G}'_p := \mathsf{G}_p \cup \{(v, z), (w, z)\}$. Of course, z is a cutvertex of G'_p, too – i.e., v, w belong to two different blocks of G'_p: by Theorem 6.9 these blocks necessarily contain z. Since each of these blocks (say, $\mathsf{G}_1, \mathsf{G}_2$) contains at least one edge, each of them also has less edges than G. However, if G_1 contains a subdivision of K^5 or $\mathsf{K}^{3,3}$, then also G contains a subdivision of the same forbidden graph, since in G the edge (v, z)

can be subdivided into a path beginning with $e = (v, w)$ and contained in G_2. This is a contradiction to the assumptions and shows that both G_1 and G_2 are planar, by minimality of G among those graphs that are non-planar and do not contains a subdivision of K^5 or $\mathsf{K}^{3,3}$. By Lemma 10.26.(2), both G_1 and G_2 can be (individually) embedded in R^2 in such a way that (v, z) *and* (w, z) are in the boundary of the unbounded face, hence the same holds for the graph G'_p. Since G'_p consists of two planar blocks only connected by the cutvertex z, its planarity remains invariant upon adding to the graph the edge $e = (v, w)$. Since G is a subgraph of $\mathsf{G}'_p \cup \{e\}$, it is planar, too – a contradiction to the construction of G. This yields the assertion.

(3) Since G_p is connected, each external piece is incident in C. Since G_p is a block, hence 2-connected and therefore not containing any cutvertex, each external piece is in fact incident in at least *two* different nodes of C. However, no external piece can be incident in more than one node of $\mathsf{C}^\circ_{1i_0}$, since otherwise this would define a cycle containing the nodes v, w and surrounding more faces than C, a contradiction. For the same reason, no external piece can be incident in more than one node of $\mathsf{C}^\circ_{i_0 1}$. Similarly, no external piece may be incident in either v or w. This completes the proof of (3).

(4) The graph $\mathsf{G} = \mathsf{G}_p \cup \{(v, w)\} = \mathsf{G}_p \cup \{(v_1, v_{i_0})\}$ is not planar, by construction. This depends on (v, w) and yields that there exists at least an internal piece I_1 that is incident in $\mathsf{C}^\circ_{1i_0}$ as well as in $\mathsf{C}^\circ_{i_0 1}$ – possibly in several nodes. Among those internal pieces with this property we consider the "first one", i.e., the one with the property that it incides in $\mathsf{C}^\circ_{1i_0}$ in the node v_i with lowest possible index i. As observed above, I_1 may be incident in several nodes of both $\mathsf{C}^\circ_{1i_0}$ and $\mathsf{C}^\circ_{i_0 1}$: we denote by v_1^*, v_2^* the nodes v_i with smallest and largest index i, respectively, among those in $\mathsf{C}^\circ_{1i_0} \cap I_1$ (they are not necessarily distinct). Similarly, we denote by w_1^*, w_2^* the nodes v_i with largest and largest index i, respectively, among those in $\mathsf{C}^\circ_{i_0 1} \cap I_1$ (they are not necessarily distinct, either).

We complete the proof by contradiction. Assume the assertion in (4) not to hold. By (3), each external piece has to be incident in exactly two points of $\mathsf{C}_{v,w}$. In fact, each external piece has to be incident either only in two nodes from $\overline{C}_{w_1^*, v_1^*}$, or in two nodes from $\overline{C}_{v_2^*, w_2^*}$: if this would not be the case, i.e., if an external piece would be incident in one node $v_\ell \in \overline{C}_{w_1^*, v_1^*} \subset \mathsf{C}^\circ_{1i_0}$ and one node $v_k \in \overline{C}_{v_2^*, w_2^*} \subset \mathsf{C}^\circ_{i_0 1}$, then it would follow that I_1 is incident in $\mathsf{C}^\circ_{1i_0}$, $\mathsf{C}^\circ_{i_0 1}$, $\mathsf{C}^\circ_{\ell k}$, $\mathsf{C}^\circ_{k\ell}$, a contradiction. Accordingly, it is possible to draw an (imaginary) simple arc s connecting $v_1^{(1)}$ and $w_1^{(1)}$ in such a way that its interior is disjoint from any edge of G_p: $s[0, 1] \cup \mathsf{C}_{v,w}$ define a Jordan curve, therefore also (by the Jordan curve theorem) a bounded and an unbounded connected domain. Then, it is possible to re-draw G_p by deleting and then inserting I_1 in the unbounded connected domain: this does not affect the property of planarity.

In case I_1 is not the only internal piece with this property, the same procedure as above can be performed for the further internal pieces $I_2, I_3, \ldots$. Eventually, there will be no internal pieces any more; accordingly, it is possible to insert (v, w) without affecting planarity of G_p. This yields that G is planar, a contradiction: the assertion is fully proved.

(5) – (i) It is possible to choose $w_3 = w_1$ and $w_4 = w_2$. Then, the nodes v_1, v_k, w_1 and v_{i_0}, v_ℓ, w_2 are connected in such a way that they define a $\mathsf{K}^{3,3}$, hence $\mathsf{K}^{3,3} \subset \mathsf{G}$. This is a contradiction to the defining properties of G.

– (ii) Assume without loss of generality that $w_3, w_4 \in C^\circ_{1i_0}$. We further consider two cases: either $v_k = w_2$ or $v_k \neq w_2$. In the former case there exists a node r in I such that three disjoint paths connect r with w_2, w_3, w_4, respectively. Then, the nodes w_2, w_3, w_4 and v_{i0}, v_ℓ, r are connected in such a way that they define a $\mathsf{K}^{3,3}$, hence $\mathsf{K}^{3,3} \subset \mathsf{G}$. This is a contradiction to the defining properties of G. If else $v_k \neq w_2$, then either w_2 belongs to $C^\circ_{i_0k}$, and in this case the nodes v_1, v_k, w_2 and v_{i_0}, v_k, w_4 are connected in such a way that they define a $\mathsf{K}^{3,3}$; or w_2 belongs to C°_{k1}, and in this case the nodes defining a $\mathsf{K}^{3,3}$ are v_1, v_k, w_3 and v_{i_0}, v_ℓ, w_2, respectively.

– (iii) Assume without loss of generality that $w_4 \in C^\circ_{1i_0}$. If w_2 belongs to $\mathsf{C}^\circ_{i_0k}$, then the nodes v_1, v_ℓ, w_2 and v_{i_0}, v_k, w_4 are connected in such a way that they define a $\mathsf{K}^{3,3}$; if else w_2 belongs to C°_{k1}, then there exists a node r in I such that three disjoint paths connect r with w_2, w_3, w_4, respectively. It follows that w_2, w_3, w_4 and v_1, v_k, r are connected in such a way that they define a $\mathsf{K}^{3,3}$. It is possible to see that a similar reasoning holds in the case $w_2 = v_k$, too.

– (iv) Finally, consider shortest paths $\mathsf{P}_1, \mathsf{P}_2$ in the internal piece I connecting v_1, v_{i_0} and v_ℓ, v_k, respectively. Then $\mathsf{P}_1, \mathsf{P}_2$ are necessarily non-disjoint: if their intersection is a singleton, say $\mathsf{P}_1 \cap \mathsf{P}_2 = \{r\}$, then the nodes $v_1, v_{i_0}, v_\ell, v_k, r$ define a K^5. If else $r, s \in \mathsf{P}_1 \cap \mathsf{P}_2$ with $r \neq s$, then the nodes v_1, r, v_k and v_{i_0}, v_ℓ, s are connected in such a way that they define a $\mathsf{K}^{3,3}$.

This concludes the proof. □

10.2. Colourability and dual graphs

Definition 10.30. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a planar graph with face set F. Its **dual graph** $\mathsf{G}^* = (\mathsf{V}^*, \mathsf{E}^*, g^*)$ is the graph given by the following construction: $\mathsf{V}' = \mathsf{F}$ and two nodes $F_1, F_2 \in \mathsf{V}'$ are connected by k edges if and only if (the corresponding faces are neighboring and) $\partial F_1 \cap \partial F_2$ consists of k edges, whereas a loop is inserted around each face on both sides of an edge (i.e., for each bridge on the face's boundary).*

Exercise 10.31. *Prove the following properties of the dual graph G^* of a planar graph G.*

(1) G^* is also planar.
(2) If G is connected, then $\mathsf{G}^{**} = \mathsf{G}$.
(3) If G contains neither loops nor bridges, then G^* contains no loops.

Exercise 10.32. *Determine the dual graph of a triangle or, more generally, of a polygon with m edges. Determine the dual graph of an n-wheel.*

Remark 10.33. *Both a tessellation G of a domain Ω by squares and a tessellation by hexagons are* locally *self-dual, i.e., their dual agree with G – with the exceptions of the cells covering $\partial\Omega$ or, equivalently, in an asymptotic sense if we let $|\mathsf{F}| \to \infty$, i.e., if we make the tessellation finer and finer.*

It is worth to observe that a planar graph can be embedded in R^2 in several ways, cf. [**8**, § 3.3]*. Accordingly, its dual graph may possibly depend on the chosen embedding. In fact, it is possible to provide an example of (two embeddings of) the same planar graph exhibiting two non-isomorphic dual graphs, cf.* `http://en.wikipedia.org/wiki/Dual_graph#Properties`.

Example 10.34. *Consider a domain $\Omega \subset \mathrm{R}^2$ and a finite set of points $S \subset \Omega$. Take $s \in S$. The set of all $x \in \Omega$ such that $|s - x| \leq |s' - x|$ for all $s' \in S$, i.e., of all points closer to s than to any other $s' \in S$ is called the* ***Voronoi cell for*** *s, which we denote by V_s. Each Voronoi cell is a polytope, i.e., a closed subset of Ω whose boundary ∂V_s is a polygon. Repeat this procedure, so that a Voronoi cell is assigned to each $s \in S$. This defines a so-called* **Voronoi diagram** *of Ω, i.e., $\Omega = \bigcup_{s \in S} V_s$ and in fact $\Omega \setminus \bigcup_{s \in S} V_s$ is the disjoint union of the interiors of the cells V_s. Voronoi tessellations are common in applied mathematics, statistics, physics, meteorology. For a brief overview of its applications, check* `http://en.wikipedia.org/wiki/Voronoi_diagram#Application`.

(Observe that an analogous construction can also be performed in any Euclidean space R^n, $n \geq 3$.)

Now, such a diagram clearly defines a planar graph. Its edges are the points of Ω that are equally close to two different points of S and its vertices are those points of Ω that are equally close to[1] *three different points of S.*

Example 10.35. *As we already know, the dual graph of a Voronoi tessellation is another planar graph. In applied sciences and numerical analysis, such a dual graph is often called* ***Delaunay triangulation*** *of the point set S. Observe that a Delaunay triangulation* actually is a triangulation*, if one does assume that there are no points equally close to four or more elements of S.*

In fact, a common heuristics to triangulate a surface consists in considering a certain number of (random) points on the surface, then determine its Voronoi diagram and subsequently its Delaunay triangulation. This is a favorite procedure, due to its optimality. To explain this, consider for any triangulation $\mathcal{T}$ the number $\gamma(\mathcal{T})$, the smallest of the angles between any two adjacent edges of $\mathcal{T}$. Then it can be proved that if $\mathcal{T}$ is a Delaunay triangulation, then $\gamma(\mathcal{T})$ is the largest possible. This is good in numerical analysis, where functions on Ω are approximated by affine functions on the (Delaunay) triangles with an error that depends linearly on steepness of functions' graphs. Triangles with small angles are "skinny" triangles, where affine functions have typically steep graphs.

More generally, we introduce the following.

Definition 10.36. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a simple planar graph.*

(1) G is called ***maximal planar*** *if it cannot be augmented by an edge in such a way that also the resulting graph is simple and planar – in other words: if G spans a simple planar graph G', then $\mathsf{G} = \mathsf{G}'$.*

(2) G is called a ***plane triangulation*** *if each of its faces (also the unbounded face!) is a triangle, i.e., if the boundary of each of its faces is a K^3.*

Proposition 10.37. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a simple planar graph with $|\mathsf{V}| \geq 3$. Then G is maximal planar if and only if G is a plane triangulation.*

Proof. Let us first assume G to be a plane triangulation and consider a new edge e connecting two existing nodes. Then the interior of e is contained in a face F of G and its

[1] *At least* three, but in applications one usually assumes that there are no points equally close to four or more elements of S. This implies in particular that each vertex has degree three, if one neglects the boundary of Ω.

endpoints lie on the boundary of F, which is a triangle by assumption. Accordingly, they lie on adjacent edges, hence the new graph obtained by adding e to G cannot possibly be simple.

Let now G be maximal planar. Consider a face F, the subgraph G' of G associated with ∂F, and the subgraph G'' of G containing all the edges whose endpoints are nodes in G'. Due to maximality of G as a planar simple graph, G'' is necessarily complete, i.e., $\mathsf{G}'' = \mathsf{K}^n$ for some n. In order to show that in fact $\mathsf{G}' = \mathsf{G}'' = \mathsf{K}^3$ we are going to prove that G' contains a cycle and has at most 3 vertices.

If G' were acyclic, i.e., a forest, then by Example 10.7 G' defines only one face – namely F – so that

$$\mathsf{G}' \cup F = \mathrm{R}^2. \tag{10.2}$$

On the other hand, $\mathsf{G} - \mathsf{G}' \neq \emptyset$: this is due to the fact that $\mathsf{G}'' = \mathsf{K}^n \subset \mathsf{G}$ (so that G does contain a cycle, which is not possible if $\mathsf{G} = \mathsf{G}'$) if $n \geq 3$; and due to the assumption that $|\mathsf{V}| \geq 3$ if $n < 3$, so that at least one node and all the incidente edges belong to G but not to G'. However, the fact that G contains edges not belonging to G' contradicts (10.2).

Finally, we prove that $n \leq 3$. If it were $n \geq 4$, then we could pick a cycle $\mathsf{C}^4 \subset \mathsf{G}'' = \mathsf{K}^n$. Since $\mathsf{C}^4 \subset \mathsf{G}$, the face F is contained in any of the two faces of C^4 – let us call it F' (and since C^4 only defines two faces, let us call F'' such a second face). Consider two nodes v, w in C^4 that are *not* adjacent in C^4. Since v, w lie on ∂F, the boundary of a face, there exists a simple arc whose endpoints are v, w and which is disjoint from G. Apply Corollary 10.9.(ii) and deduce that the edge connecting in $\mathsf{G}'' = \mathsf{K}^n$ the remaining nodes of C^4 – we call them v', w' – is necessarily contained in the face F'' instead of F'. For the same reason the edge connecting v, w in $\mathsf{G}'' = \mathsf{K}^n$ is contained in F'' rather than F'. However, again by Corollary 10.9.(ii) this contradicts the fact that the simple arcs corresponding to these edges are disjoint by planarity. □

Remark 10.38. *Accordingly, if the planar configuration of a graph G with the largest number of edges is attained, then each face is a triangle, and in particular $g(\mathsf{G}) = 3$. Taking into account Proposition 10.19, we conclude that a simple and connected planar graph with $|\mathsf{V}| \geq 3$ has at most $3|\mathsf{V}| - 6$ edges. In particular, a simple and connected planar graph with $|\mathsf{V}| \geq 3$ is maximally planar if and only if it has $3|\mathsf{V}| - 6$ edges.*

Corollary 10.39. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a simple planar graph. Then*

$$\delta(\mathsf{G}) \leq \max\left\{1, 6 - \frac{12}{|\mathsf{V}|}\right\}.$$

In particular, $\delta(\mathsf{G}) \leq 5$.

Proof. The claim clearly holds if $|\mathsf{V}| \leq 2$. If $|\mathsf{V}| \geq 3$, then we can apply Remark 10.38 and deduce from Euler's formula that

$$3|\mathsf{V}| - 6 \geq |\mathsf{E}| = \frac{1}{2}\sum_{v \in \mathsf{V}} d(v, \mathsf{G}) \geq \frac{1}{2}\delta(\mathsf{G})|\mathsf{V}|.$$

This completes the proof. □

Remark 10.40. *Planar graphs play a relevant role in the theory of Hamiltonian graphs. An important sufficient condition implying that a graph is Hamiltonian has been found by Tutte. His result can be stated as follows, but we do not prove it.*

Theorem 10.41 (W.T. Tutte, 1956)**.** *Each planar and 4-connected graph is Hamiltonian.*

The following definition refines in some sense the notion of multipartite graph.

Definition 10.42. *A graph without loops $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ is called n-**node-colourable** if there exists an n-**colouring** of G, i.e., a mapping $c : \mathsf{V} \to \{1, 2, \ldots, n\}$ such that $c(v) \neq c(w)$ for any two adjacent nodes $v, w \in \mathsf{V}$. The value $c(v)$ is called **colour of** v. The smallest n such that G is n-colourable is called **chromatic number of** G and is denoted by $\chi(\mathsf{G})$.*

Exercise 10.43. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a simple, n-edge-colourable graph with colouring mapping c.*

(1) Show that also $c' := \pi_n \circ c$ is an n-edge-colouring of G, where π_n is a permutation on $\{1, \ldots, n\}$

(2) Show that $c'' := c \circ \pi_{\mathsf{V}}$ is an n-edge-colouring of G, where π_{V} is an automorphism of the graph.

In other words, new n-edge-colourings can be obtained by permutating the colours and/or the nodes (provided that the permutation respects the adjacency).

Remark 10.44. *Since the definition of colouring does not distinguish between simple and parallel edges, it is clear that a graph G without loops is n-edge-colourable if and only if the simple version of G^2 is n-edge-colourable.*

Also observe that if a graph is unconnected, then a new colouring can be obtained by applying any of the operations described in the Exercise 10.43 in some of the individual connected components.

Definition 10.45. *A planar graph $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ is called n-**colourable** if its dual graph $\mathsf{G}^* = (\mathsf{V}^*, \mathsf{E}^*, g^*)$ is n-node-colourable, i.e., if there exists a mapping $c : \mathsf{V}^* \to \{1, 2, \ldots, n\}$ such that $c(F) \neq c(F')$ for any two adjacent nodes $F, F' \in \mathsf{V}^* = \mathsf{F}$.*

By Remark 10.44, a planar graph is therefore n-colourable if and only if the *simple version* of G^* is n-node colourable.

In other words, a planar graph is n-colourable if and only if a colour can be assigned to each face in such a way that adjacent faces are differently coloured: intuitively, this means that a political map can be coloured in such a way that neighbouring countries are given different colours.

Remark 10.46. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ a graph without loops. If G is n-node-colourable with colouring mapping c, then $c^{-1}(\{i\}) =: \mathsf{V}_i$, $i = 1, \ldots, n$ defines a partition of V into disjoint node sets, each of which is independent by definition of colourability. Accordingly, each graph without loops is $\chi(\mathsf{G})$-partite. Conversely, each partition of V into n pairwise disjoint, independent sets yields an n-node-colouring of G.*

[2] I.e., the graph $\tilde{\mathsf{G}}$ obtained by deleting from G all but one edges between any two adjacent nodes.

Example 10.47. *By definition, a bipartite graph is* 2*-node-colourable. More generally, an* n*-partite graph is* n*-node-colourable – simply assign colour* i *to the nodes in the* i^{th}*-set of the multipartition of* V*. The Petersen graph is 3-colourable.*

What is the minimal number of colours that is necessary in order to colour *any* planar map? It is clear that there are maps that cannot be coloured with less than 4 colours, i.e., that their chromatic number is at least 4: one example is the common drawing of the K^4 as a wheel. The **five colour theorem** states that each planar graph is 5-colourable. That in fact each planar graph 4-colourable is the assertion of the **four colour theorem**/conjecture.

We are only going to discuss in detail the five colour theorem. The *four* colour theorem is much, *much* harder to prove – in fact, its proof could only be completed by a smart computer-aided technique 120 years after its assertion was formulated as a conjecture. Such a proof is due to K. Appel and W. Haken, 1976. Still, both due to mathematical and philosophical concerns, several graph theorists still maintain that the proof is not satisfactory, cf. [**6**, § IX.6]. With the *five* colour theorem we are on the safe side, though. Our approach follows that presented in [**2**, §V.3]. See `http://mathworld.wolfram.com/Four-ColorTheorem.html` for a brief history of the four colour theorem/conjecture and of its incorrect proofs and disproofs.

Theorem 10.48 (P. Heawood, 1890)**.** *Each planar graph is* 5*-colourable.*

Proof. We perform the proof by contradiction. Consider a planar graph G such that its dual graph G^* is 6-node-colourable but not 5-node-colourable, and with minimal number of nodes (i.e., each proper subgraph of G is 5-colourable). By Exercise 10.31, G^* is planar. By Remark 10.44 we can assume G^* to be simple.

Thus, by Corollary 10.39 there exists a face F of G with $d(F, \mathsf{G}^*) \leq 5$. We consider the graph $\mathsf{G}^* - \{F\}$, which is 5-colourable by construction of G^* – so, consider a particular 5-colouring of it such that each colour is assigned to any of the edges in $N_{\mathsf{G}^*}(\{F\})$. In other words, we ask that the restriction of the colouring function c to $N_{\mathsf{G}^*}(\{F\})$ is surjective onto $\{1, 2, 3, 4, 5\}$: this is possible because otherwise F could be coloured with any of the unused colours and G^* would be 5-colourable. Accordingly, F has exactly five neighbours – say F_1, F_2, F_3, F_4, F_5, counted clockwise, and coloured as $1, 2, 3, 4, 5$, respectively. Denote by $\mathsf{G}^*_{1,3}$ the subgraph of $\mathsf{G}^* - \{F\}$ spanned by those nodes coloured as $1, 3$. We are going to consider the cases

(i) F_1, F_3 belong to different connected components of $\mathsf{G}^*_{1,3}$ and
(ii) F_1, F_3 belong to the same connected component of $\mathsf{G}^*_{1,3}$,

and show that each of them leads to a contradiction, thus proving the claim.

(i) If F_1, F_3 belong to different connected components of $\mathsf{G}^*_{1,3}$, then it is possible to consider a new 5-node-colouring of $\mathsf{G}^* - \{F\}$ obtained by switching each node coloured as 1 to 3, and viceversa, in the connected component containing F_3. However, in this way the restriction of the colouring function c to $N_{\mathsf{G}^*}(\{F\})$ is surjective onto $\{1, 2, 4, 5\}$, and colouring v as 3 we would obtain a 5-node-colouring of G^*, a contradiction to the construction of G^*.

(ii) If F_1, F_3 belong to the same connected component of $\mathsf{G}^*_{1,3}$, it is possible to consider a path P connecting them and only consisting of nodes coloured as 1 or 3. Consider the cycle in G^* defined connecting F with the endpoints of P. By the Jordan curve theorem, either F_2 or F_4 lie in the unbounded domain. Reasoning as above, it is possible to consider a cycle containing F, F_2, F_4 and only consisting of nodes coloured as 2 or 4. Due to planarity, these both cycles necessarily share at least one node apart from F: this contradicts the fact that they consist of nodes that have different colours and leads to the final contradiction. □

Remark 10.49. *Several other proofs are known, see e.g.* [**22**] *and* [**1**, Chapt. 30].

A concrete 5*-colouring of a given graph can be found algorithmically in an efficient way – this has been shown by N. Robertson, D. Sanders, P. Seymour, and R. Thomas in 1996.*

Although we do not discuss the proper four colour theorem, we present another weakened version of it.

Proposition 10.50. *A planar Hamiltonian graph is* 4*-colourable.*

Proof. Embed the graph in S^2 in such a way that a Hamiltonian cycle coincides with the equator. Accordingly, all the nodes lay on the equator and each hemisphere only contains edges that do not cross. Thus, each hemisphere is 2-colourable, and the whole graph is 4-colourable. □

Exercise 10.51. *Let* G *be a planar graph without loops or bridges. Show that if* G *is* 2*-colourable, then the connected components of* G *are Eulerian or isolated nodes.*
(Hint: Show that no node can have odd degree.)

Before concluding this section, we consider a nice application of the theory of planar graphs. The following problem and its solution are taken from [**1**, Chapt. 31].

Consider the problem of guarding a museum, i.e., of placing a minimal number of guards in the museum so that each point of the museum can be observed by at least one guard.

Definition 10.52. *Let* $\Omega \subset \mathrm{R}^2$ *be a polygon – i.e., let* $\partial\Omega$ *consist of straight lines. A finite subset* $S \subset \Omega$ *is said to* ***guard*** Ω *if each point* $x \in \Omega$ *is connected to at least one* $v \in S$ *by a segment that does not intersect the boundary* $\partial\Omega$*.*

If the museum Ω is star-shaped, by definition one guard placed in its center suffice. The general case of a museum with n walls is treated by the following **art gallery museum**.

Theorem 10.53 (V. Chvátal, 1975)**.** *Let* $\Omega \subset \mathrm{R}^2$ *be a (possibly non-convex) polygon with* n *sides. Then* Ω *can be guarded by a set* S *with* $|S| = \lfloor n/3 \rfloor$*.*

Here $\lfloor \cdot \rfloor$ denotes the floor function, i.e., $\lfloor x \rfloor := \max\{n \in \mathrm{N} : n \leq x\}$.

Proof. To begin with we consider a triangulation of Ω, i.e., we consider finitely many triangles $T_1, \ldots, T_N \subset \mathrm{R}^2$ with disjoint interiors in such a way that $\bigcup_{i=1}^N T_i = \overline{\Omega}$. Showing that this is always possible – and that drawing $n-3$ non-crossing diagonals between corners of Ω will do – is a not completely trivial exercise in Euclidean geometry, see [**1**, p. 204]. We only emphasize that all the nodes of the triangulation lie on the boundary $\partial\Omega$.

Such a triangulation clearly defines a planar graph G, whose edges are the sides of the original polygon and the newly drawn diagonals.[3] We are going to prove that G is 3-node-colourable. Once proved this, we deduce that in each triangle the three nodes are coloured with three different colours, so that each triangle is guarded by the node coloured, say, as 1. Accordingly, the whole polygon Ω is guarded by each of the three sets $\mathsf{V}_i = \{v \in \mathsf{V} : c(v) = i\}$, $i = 1, 2, 3$. The claim follows by considering the set V_{i_0} of smallest cardinality.

Thus, it remains to check the 3-colourability of Ω: we proceed by induction on $|\mathsf{V}|$. If $|\mathsf{V}| = 3$, this is clear since the triangulation agrees with Ω and therefore has exactly three nodes. Let now $|\mathsf{V}| > 3$ and assume that each domain triangulating as above but presenting less than $|\mathsf{V}|$ nodes is 3-colourable, too. Pick a diagonal $e = (v, w)$ and observe that it defines, together with the boundary $\partial\Omega$, two cycles in G. Choose any of them and consider the graph G' and G'' consisting of the internal and external piece, respectively (cf. Definition 10.25). Both $\mathsf{G}', \mathsf{G}''$ have less nodes than G, so that we can apply the induction hypothesis and conclude that each of them is 3-colourable with colouring functions c', c''. Upon suitably switching colours, it is possible to colour both $\mathsf{G}', \mathsf{G}''$ in such a way that $c'(v) = 1 = c''(v)$ and $c'(w) = 2 = c''(w)$. Accordingly, the whole graph is 3-colourable. $\square$

[3] Observe that it is in general not a plane triangulation in the sense of Definition 10.36 since the boundary of the unbounded domain is in general not a triangle.

CHAPTER 11

Cliques and independent node sets

In many real-world graph-like structures, and most notably in the brain, complete (or "almost complete") subgraphs appear and are often functional subunits of the whole system. In this section we collect some basic results, moslty taken from [**22**, Chapt. 9].

Definition 11.1. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph. To avoid trivial cases, we assume* G *not to contain loops.*

(1) Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a simple graph. A complete subgraph* $\mathsf{K}^p \subset \mathsf{G}$ *is called a* ***p-clique****. The largest* $p \in \mathrm{N}$ *such that a p-clique is contained in* G *is called* ***clique number*** *of* G *and is denoted by* $\omega(\mathsf{G})$*.*

(2) A node set $\mathsf{V}_0 \subset \mathsf{V}$ *is called* ***independent*** *if all its elements are pairwise non-adjacent. The maximal cardinality of an independent node set of* G *is called* ***independence number*** *of* G *and denoted by* $\alpha(\mathsf{G})$*.*

Remark 11.2. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph. Estimating the independence number of certain large graphs* G_n *related to* G*,* $n \in \mathrm{N}$*, is an important issue in the theory of information. In particular, it is relevant to compute* $\Theta(\mathsf{G}) := \sup_{n\in\mathrm{N}} \sqrt[n]{\alpha(\mathsf{G}_n)}$*, so-called* ***zero-error capacity of*** G*, cf.* [**1**, Chapt. 33]*.*

Remark 11.3. *Of course, a node set is a clique if its elements are pairwise adjacent. In this sense, the notion of clique is complementary to that of indepenent set.*

Exercise 11.4. *Let* $n \in \mathrm{N}$*,* $n \geq 3$*. Then* $\alpha(\mathsf{C}^n) = \lfloor n/2 \rfloor$*, while* $\omega(\mathsf{C}^n) = 3$ *if and only if* $n = 3$*, otherwise* $\omega(\mathsf{C}^n) = 2$*.*

Exercise 11.5. *Exhibit a graph* G *such that* $\omega(\mathsf{G}) = \chi(\mathsf{G})$*; a graph* G' *such that* $\omega(\mathsf{G}') = \chi(\mathsf{G}') - 1$*; and a graph* G'' *such that* $\omega(\mathsf{G}'') = \chi(\mathsf{G}'') - 2$*.*

Remark 11.6. *A complete graph* K^p *clearly has a chromatic number p, hence a graph containing a p-clique has chromatic number at least p – but it can happen that* $\omega(\mathsf{G}) = \chi(\mathsf{G})$*. If a graph* G *satisfies* $\omega(\mathsf{H}) = \chi(\mathsf{H})$ *for all subsets* $\mathsf{H} \subset \mathsf{G}$*, the graph is called* ***perfect****. There is a vast literature on perfect graphs – as well as many open problems, cf.* `http://www.aimath.org/WWN/perfectgraph`*.*

Proposition 11.7 (T. Gallai, 1959)**.** *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a graph without loops. A node set* V_0 *is independent in* G *if and only if* $\mathsf{V} \setminus \mathsf{V}_0$ *is a covering of* G*. Moreover,* $\alpha(\mathsf{G}) + \beta(\mathsf{G}) = |\mathsf{V}|$*.*

Proof. A node set V_0 is independent if and only if no edge has both endpoints in V_0, that is, if and only if each edge has at least one endpoint in $\mathsf{V} - \mathsf{V}_0$ – by definition, this amounts to saying that $\mathsf{V} - \mathsf{V}_0$ is a covering of G. In particular, V_0 is an independent set

of maximal cardinality if and only if $\mathsf{V} \setminus \mathsf{V}_0$ is a covering of G of minimal cardinality, i.e., if and only if

$$\alpha(\mathsf{G}) = |\mathsf{V} - \mathsf{V}_0| = |\mathsf{V}| - \beta(\mathsf{G}),$$

since $\mathsf{V}_0 \subset \mathsf{V}$. □

Definition 11.8. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a graph without loops. A **clique decomposition** (or **clustering**) of G is a collection of pairwise disjoint node sets $(\mathsf{V}_1, \ldots, \mathsf{V}_n)$ such that each $\mathsf{G}[\mathsf{V}_i]$ is a clique, $i = 1, \ldots, n$. We call **cluster number** the minimal cardinality of a clique decomposition and denote it by $\theta(\mathsf{G})$.*

Observe that a clique decomposition always exists – simply consider a clique K^1 for each node.

Example 11.9. *Consider a world divided in villages whose inhabitants surely know each other. Then each village represents a clique in the graph whose adjacency relation is defined by acquaintance of two persons (the nodes). However, it might be that neighboring villages gather on festivities, so that in fact less cliques are really needed in order to decompose the graph – in a sense, different villages then can be simply considered as different quarters of the same city (the clique). The smallest number of cities the world can be decomposed into is the cluster number of the world.*

In the following we denote by $\theta(\mathsf{G})$ the minimal cardinality of a clique decomposition of G. number of node-disjoint cliques

Exercise 11.10. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a graph without loops. Prove the following assertions.*

(1) A node set is an independent set of maximal cardinality if and only if it spans a p-clique in the complement of G such that p is maximal, i.e., $p = \omega(\mathsf{K}^{|\mathsf{V}|} - \mathsf{E})$. Therefore, $\omega(\mathsf{G}) = \alpha(\mathsf{K}^{|\mathsf{V}|} - \mathsf{E})$.

(2) Let $\mathsf{V} = \mathsf{V}_1 \dot\cup \ldots \dot\cup \mathsf{V}_n$. Then $(\mathsf{V}_1, \ldots, \mathsf{V}_n)$ is a clique decomposition of G if and only if it is a decomposition of V into pairwise disjoint independent node sets of the complement of G.

(3) Each collection of pairwise disjoint node sets defines an independent set if and only if it consists of node sets that span cliques in the complement of G. Conclude that $\theta(\mathsf{G}) = \chi(\mathsf{K}^{|\mathsf{V}|} - \mathsf{E})$ (Hint: cf. Remark 10.46).

Proposition 11.11. *Let G be a simple graph. Then the following assertions hold.*

(1) $\alpha(\mathsf{G}) \leq \theta(\mathsf{G})$.

(2) $\omega(\mathsf{G}) \leq \chi(\mathsf{G})$.

(3) If moreover an independent node set and a clique decomposition have same cardinality n, then $\alpha(\mathsf{G}) = n = \theta(\mathsf{G})$, i.e., the independent node set and the clique decomposition have maximal and minimal cardinality, respectively.

Proof. (1) Let V' be an independent node set and $(\mathsf{V}_1, \ldots, \mathsf{V}_n)$ a clique decomposition of G of maximal cardinality, i.e., $n = \theta(\mathsf{G})$. Then $|\mathsf{V}' \cap \mathsf{V}_i| \leq 1$ for all $i = 1, \ldots, n$, hence

$$|\mathsf{V}'| = \sum_{i=1}^{n} |\mathsf{V}' \cap \mathsf{V}_i| = \theta(\mathsf{G}).$$

Since the estimate does not depend on the chosen independent set V', the assertion follows.

(2) Apply the inequality in (1) to the graph $\mathsf{K}^{|\mathsf{V}|} - \mathsf{E}$: since by Exercise 11.10 $\omega(\mathsf{G}) = \alpha(\mathsf{K}^{|\mathsf{V}|} - \mathsf{E})$ and $\chi(\mathsf{G}) = \theta(\mathsf{K}^{|\mathsf{V}|} - \mathsf{E})$, the assertion follows.

(3) One has $n \leq \alpha(\mathsf{G}) \leq \theta(\mathsf{G}) \leq n$: this concludes the proof. □

Definition 11.12. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a graph.*

*A node subset $\mathsf{V}' \subset \mathsf{V}$ is said to be **dominating** in G if every node either belongs to V' or is adjacent to at least one node from V', i.e., if $\mathsf{V}' \cup N_{\mathsf{G}}(\mathsf{V}') = \mathsf{V}$. The minimal cardinality of a dominating node subset is called **domination number** of G and denoted by $\gamma(\mathsf{G})$.*

Example 11.13. *Each star with center v_0 has $\{v_0\}$ as dominating set, i.e., $\gamma(\mathsf{G}) = 1$. More generally, in each bipartite graph with respect to $\mathsf{V} = \mathsf{V}_1 \dot\cup \mathsf{V}_2$, both sets $\mathsf{V}_1, \mathsf{V}_2$ are dominating and $\gamma(\mathsf{G}) = \min\{|\mathsf{V}_1|, |\mathsf{V}_2|\}$. If $\Delta(\mathsf{G}) = 0$, i.e., if G consists of isolated nodes, then $\gamma(\mathsf{G}) = |\mathsf{V}|$.*

Proposition 11.14. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a graph without loops. Then*

(1) if a node set $\mathsf{V}' \subset \mathsf{V}$ has maximal cardinality among those independent in G, then V' is dominating in G. Accordingly, $\gamma(\mathsf{G}) \leq \alpha(\mathsf{G})$.

Let moreover $\delta(\mathsf{G}) \geq 1$. Then

(2) if a node set $\mathsf{V}' \subset \mathsf{V}$ is a covering of G, then V' is dominating in G. Accordingly, $\gamma(\mathsf{G}) \leq \beta(\mathsf{G})$;

(3) if a node set $\mathsf{V}' \subset \mathsf{V}$ is independent in G, then $\mathsf{V} \setminus \mathsf{V}'$ is dominating in G;

(4) $2\gamma(\mathsf{G}) \leq |\mathsf{V}|$;

(5) $\gamma(\mathsf{G}) \leq \alpha_0(\mathsf{G})$.

Proof. (1) Assume that $\mathsf{V}' \subset \mathsf{V}$ has maximal cardinality among those independent in G, but is not dominating. Then there is a node $v \notin N_{\mathsf{G}}(\mathsf{V}')$, so that $\mathsf{V}' \cup \{v\}$ is independent – a contradiction to maximality of V'.

(2) Let V' be a covering of G. Since G contains neither loops nor isolated edges, each node $v \notin \mathsf{V}'$ is adjacent to another node, which necessarily belongs to V' (since V' is a covering). Accordingly, $v \in N_{\mathsf{G}}(\mathsf{V}')$.

(3) Let $\mathsf{V}' \subset \mathsf{V}$ be independent in G. By Proposition 11.7, $\mathsf{V} \setminus \mathsf{V}'$ is a covering in G, hence by (2) a dominating set.

(4) By Proposition 11.7 and by (1), (2) one obtains

$$2\gamma(\mathsf{G}) \leq \alpha(\mathsf{G}) + \beta(\mathsf{G}) = |\mathsf{V}|.$$

(5) Let T be a forest spanning G. Clearly $\gamma(\mathsf{G}) \leq \gamma(\mathsf{T})$, since G has more edges and hence each node set has a larger neighbourhood. Similarly, a matching in G can have larger cardinality than a matching in T, hence $\alpha_0(\mathsf{G}) \geq \alpha(\mathsf{T})$. Combining with (2) and Theorem 8.17 we obtain

$$\gamma(\mathsf{T}) \leq \beta(\mathsf{T}) \leq \alpha_0(\mathsf{T})$$

and the claim follows. □

Definition 11.15. *Let $p \in \mathrm{N}$ and V be a nonempty set with $n \in \mathrm{N}$ elements. Then the **Turán graph** $\mathsf{T}(n, p)$ is the complete p-partite graph $\mathsf{K}^{|\mathsf{V}_1|,\ldots,|\mathsf{V}_p|}$ such that $||\mathsf{V}_i| - |\mathsf{V}_j|| \leq 1$ for all $1 \leq i, j \leq p$.*

In other words, $\mathsf{T}(n,p)$ is the complete p-partite graph whose node distribution among the node subsets $\mathsf{V}_1,\ldots,\mathsf{V}_p$ is as even as possible. Applying the usual algorithm for the division by p in N yields that there exist natural numbers k,m, $0 \leq k < p$, such that $n = p(m-1)+k = km+(p-k)(m-1)$: then $\mathsf{T}(n,p)$ is the complete p-partite graph such that k node sets consist of m elements each and the remaining $p-k$ node sets consist of $m-1$ elements. Then by definition

$$k = n \bmod p \qquad \text{and} \qquad m = \left\lceil \frac{n}{p} \right\rceil \qquad \text{and} \qquad m-1 = \left\lfloor \frac{n}{p} \right\rfloor . \tag{11.1}$$

Observe that Turán graphs generalize the notion of clique, in the sense that $\mathsf{T}(n,n) = \mathsf{K}^n$ for all $n \in \mathrm{N}$.

Theorem 11.16 (P. Turán, 1941)**.** *Let $p \in \mathrm{N}$, $p \geq 2$. Let $\mathsf{G} = (\mathsf{V},\mathsf{E},g)$ be a simple graph containing no p-cliques. Then*

$$|\mathsf{E}| \leq \left(1 - \frac{1}{p-1}\right) \frac{|\mathsf{V}|^2}{2}. \tag{11.2}$$

If in particular (11.2) *holds with an equality, then G is isomorphic to the Turán graph $\mathsf{T}(|\mathsf{V}|,p-1)$.*

Proof. Validity of the assertion in the case $p = 2$ is clear, since then the graph consists of isolated nodes.

Let now $p \geq 3$ and G be a simple graph with node set V without any p-clique and with a maximal number of edges among those with this property. We are going to show that

(1) G is a complete $(p-1)$-partite graph,
(2) Turán graphs are optimal among such graphs, i.e., any further complete $(p-1)$-partite graph with node set V has less edges than $\mathsf{T}(|\mathsf{V}|,p-1)$, and finally that
(3) Turán graphs have exactly

$$|\mathsf{E}| = \left(1 - \frac{1}{p-1}\right) \frac{|\mathsf{V}|^2}{2}$$

edges.

This will conclude the proof.

(1) We are going to show that G does not contain nodes v,w,z such thar $(v,w) \in \mathsf{E}$ but $(v,z),(w,z) \notin \mathsf{E}$. Once this has been proved, we conclude that an equivalence relation R is defined via $R := \{(v,w) \in \mathsf{V}\times\mathsf{V} : (v,w) \notin \mathsf{E}\}$. It follows that G is a complete multipartite graph – i.e., since G is K^p-free, a complete $(p-1)$-partite graph.

We consider two cases: (i) $d(z,\mathsf{G}) < d(v,\mathsf{G})$ or $d(z,\mathsf{G}) < d(w,\mathsf{G})$ and (ii) $d(z,\mathsf{G}) \geq d(v,\mathsf{G})$ as well as $d(z,\mathsf{G}) \geq d(w,\mathsf{G})$.

If (i) holds – say, without loss of generality, $d(z,\mathsf{G}) < d(v,\mathsf{G})$ – then consider a new graph $\mathsf{G}' = (\mathsf{V}',\mathsf{E}',g')$ constructed as follows:

- its node set V' is $\mathsf{V} \cup \{v'\}$, where v' is a new node, and $(u,v') \in \mathsf{E}$ if and only if $(u,v) \in \mathsf{E}$ for any $u \in \mathsf{V}$ (in particular, $(v,v') \notin \mathsf{E}$); and
- z is deleted.

In other words, in order to consider a graph with the same number of nodes but more edges, we are replacing z by "cloning" the node v. Since G does not contain any p-clique, the same is true for G' (this is due to the fact that $(v, v') \notin \mathsf{E}$). However, G' has $d(v, \mathsf{G}) - d(z, \mathsf{G})$ more edges than G, i.e.,

$$|\mathsf{E}'| = |\mathsf{E}| + d(v, \mathsf{G}) - d(u, \mathsf{G}) > |\mathsf{E}|,$$

a contradiction to the maximality of G.

Let now (ii) hold. Similarly to what done in (i), consider a new graph $\mathsf{G}'' = (\mathsf{V}'', \mathsf{E}'', g'')$ constructed as follows:

- its node set V' is $\mathsf{V} \cup \{z', z''\}$, where z' is a new node, and $(u, z') \in \mathsf{E}$ if and only if $(u, z'') \in \mathsf{E}$ if and only if $(u, z) \in \mathsf{E}$ for any $u \in \mathsf{V}$ (in particular, $(z, z'), (z, z''), (z', z'') \notin \mathsf{E}$) and
- v, w are deleted, and in particular we also delete (v, w).

Since G does not contain any p-clique, the same is true for G' (this is due to the fact that the graph does not contain the K^3 with nodes z, z', z''). However, G' has $2d(z, \mathsf{G}) - (d(v, \mathsf{G}) + d(w, \mathsf{G}) - 1)$ more edges than G, i.e.,

$$|\mathsf{E}'| = |\mathsf{E}| - d(v, \mathsf{G}) - d(w, \mathsf{G}) + 2d(z, \mathsf{G}) + 1 > |\mathsf{E}|,$$

again a contradiction to the maximality of G. This completes the proof.

(2) In order to see that $\mathsf{T}(|\mathsf{V}|, p-1)$ graphs have maximal number of edges among complete $(p-1)$-partite graphs with $|\mathsf{V}|$ nodes, consider another complete $(p-1)$-multipartite graph and shift a node from, say, V_1 to V_2. Since by definition of Turán graph $||\mathsf{V}_1| - |\mathsf{V}_2|| \leq 1$, by Exercise 7.26 this decreases the number of total edges by $|\mathsf{V}_1||\mathsf{V}_2| - (|\mathsf{V}_1| - 1)(|\mathsf{V}_2| + 1) = -|\mathsf{V}_1| + |\mathsf{V}_2| + 1 \geq 0$, with the equality sign holding if and only if $|\mathsf{V}_1| = |\mathsf{V}_2| + 1$, i.e., if we have simply swapped V_1 and V_2, and a strict inequality sign holding otherwise.

(3) The assertion follows directly from Exercise 7.26 and (11.1). □

Remark 11.17. *Turán graphs are also useful in applications. For example, if a graph (say, representing genes in genome) is known to be K^p-free, counting the $(p-1)$ cliques may be difficult due to their large number, whereas "merging" them into a suitable complete $(p-1)$-partite graphs allows for a more efficient enumeration.*

Remark 11.18. *Turán's theorem can be generalized. In fact, in 1946 Erdős and Stone have proved an upper bound on the number of edges of any graph not containing as a subgraph a Turán graph, rather than a clique. It is a consequence of this Erdős–Stone Theorem that similar bounds also hold for graphs not containing any graph with given chromatic number. This and related results are usually related to as* extremal graph theory*.*

Exercise 11.19. *Work out the details of point (3) in the proof of Turán's theorem.*

CHAPTER 12

Spectral graph theory

The material in this chapter is only a very limited collection of results in *spectral graph theory*, one of the liveliest branches of graph theory. We refer to [**16**] for a brief introduction and to [**5, 7**] for thorough monographs. A crash course in spectral graph theory and applications can be found in [**17**].

In the following we consider the *node space*, i.e., the vector space $\mathrm{C}^{|\mathsf{V}|}$ whose elements are vectors each of whose entries is associated with a single node. Observe that $\mathrm{C}^{|\mathsf{V}|}$ can be identified with C^{V}, the space of all functions from V to C. Accordingly, we fix an arbitrary numeration $\{v_1, \dots, v_{|\mathsf{V}|}\}$ of V and write x_v instead of x_i if $v = v_i$. In particular, $\mathrm{C}^{|\mathsf{V}|}$ is an euclidean space with respect to the inner product

$$x^T y := \sum_{v \in \mathsf{V}} x_v \overline{y_v}.$$

Recall the notion of admittance matrix of a graph G, see Definition 4.8. In the following we list a few important properties of this matrix, which we denote by $\mathcal{B}(\mathsf{G})$ or $\mathcal{B}$.

Proposition 12.1. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a simple graph. Then the following assertions hold.*

(1) $\mathcal{B}$ is a symmetric square matrix.
(2) $\mathcal{B}$ is positive definite.
(3) 0 is always an eigenvalue of $\mathcal{B}$ and its multiplicity agrees with the number of connected components of G.

In particular, it follows from (3) that the second-smallest eigenvalue of $\mathcal{B}$ vanishes if and only if $\mathcal{G}$ is unconnected: for this reason, $\lambda_2(\mathsf{G})$ has been called by M. Fiedler **algebraic connectivity of** G in 1973, and this definition is now quite usual. The larger λ_2, the more connected is the graph.

Proof. 1) By definition of $\mathcal{B}$, symmetry follows from symmetry of $\mathcal{A}$.
2) The assertion can be proved considering the quadratic form associated with $\mathcal{B}$, i.e.,

$$\begin{aligned} x^T \mathcal{B} x &= \sum_{v,w \in \mathsf{V}} \beta_{vw} x_w \overline{x_v} = \sum_{v \in \mathsf{V}} d(v, \mathsf{G}) |x_v|^2 - \sum_{v \sim w} x_w \overline{x_v} \\ &= \sum_{(v,w) \in \mathsf{E}} |x_v - x_w|^2 \geq 0, \end{aligned}$$

where the last equality holds because when summing over $v, w \in \mathsf{V}$ each edge is being counted twice.

3) It follows from Remark 1.15.(i) that 0 is an eigenvalue, considering the unit vector as eigenvector. More generally, let G' be a connected component of G with node set V'. Let $x^{\mathsf{G}'} \in \{0,1\}^{|\mathsf{V}|}$ be the characteristic vector of G', i.e.,

$$x_v^{\mathsf{G}'} = \begin{cases} 1 & \text{if } v \in \mathsf{V}', \\ 0 & \text{otherwise.} \end{cases}$$

Now, $x^{\mathsf{G}'} \in \ker(\mathcal{B})$. Since this holds for each connected component G', and since characteristic vectors of different connected components are linearly independent, the multiplicity of the eigenvalue 0 – i.e., dim $\ker(\mathcal{B})$ – is at least as large as the number of connected components.

Conversely, take $x \in \ker(\mathcal{B})$ and observe that

$$0 = x^T \mathcal{B} x = \sum_{(v,w)\in\mathsf{E}} |x_v - x_w|^2,$$

so that the vector x has same entries in the node corresponding to the vertices that belong to connected components. Accordingly, x is linear combination of characteristic vectors of connected components. □

Remark 12.2. *It follows from Remark 1.4 and Proposition 12.1 that both $\mathcal{B}, \mathcal{A}$ have $|\mathsf{V}|$ real eigenvalues. We denote them by λ_i, μ_i, $i = 1, \ldots, |\mathsf{V}|$, respectively, counted with their multiplicities and such that $\lambda_{min} := \lambda_1 \leq \lambda_2 \leq \ldots \lambda_{|\mathsf{V}|} =: \lambda_{max}$ and $\mu_{min} := \mu_1 \leq \mu_2 \leq \ldots \mu_{|\mathsf{V}|} =: \mu_{max}$.*

*In particular, as already observed one has $0 = \lambda_1$, i.e., all the eigenvalues of $\mathcal{B}$ are positive. With an abuse of terminology, the numbers λ_i are often called **eigenvalues of** G, and the spectrum of $\mathcal{B}$ is accordingly called **spectrum of** G.*

Observe that in particular we can consider an orthonormal basis of eigenvectors of $\mathcal{B}$; it is natural to choose

$$x_1 = \frac{1}{\sqrt{|\mathsf{V}|}}(1, \ldots, 1)$$

as the first (eigen)vector of the basis.

Remark 12.3. *One of the basic goals of spectral graph theory is (trying) to distinguish non-isomorphic graphs by computing the eigenvalues of their admittance and/or adjacency matrix. This is in general not possible, as it is known that there exist isomorphic graphs with same spectrum – so-called **cospectral** or **isospectral graphs**, see e.g.* [**7**, § 5.5]. *E.g., the easiest known pair of non-isomorphic graphs with same admittance spectrum is given by the union of a 4-path and an isolated points vs. a 5-star, i.e., $\mathsf{C}^4 \cup \mathsf{K}^1$ vs. S^5, as for both these graphs $\mu_1 = -2$, $\mu_2 = \mu_3 = \mu_4 = 0$, $\mu_5 = 2$.*

Lemma 12.4. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a simple graph. Then the eigenvalues $\lambda_i(\mathsf{G})$ of G and $\lambda_i(\mathsf{K}^{|\mathsf{V}|} - \mathsf{E})$ of its complement satisfy*

$$\lambda_i(\mathsf{K}^{|\mathsf{V}|} - \mathsf{E}) = |\mathsf{V}| - \lambda_{n-i+2}(\mathsf{G}), \qquad i = 2, \ldots, |\mathsf{V}|.$$

PROOF. One has

$$\mathcal{A}(\mathsf{G}) + \mathcal{A}(\mathsf{K}^{|\mathsf{V}|} - \mathsf{E}) = \langle 1 \rangle - Id = \mathcal{A}(\mathsf{K}^{|\mathsf{V}|}) = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \dots & 0 \end{pmatrix},$$

where $\langle 1 \rangle$ is the matrix whose entries are all 1 and Id is the identity matrix.

Accordingly,

$$\begin{aligned} \mathcal{B}(\mathsf{G}) + \mathcal{B}(\mathsf{K}^{|\mathsf{V}|} - \mathsf{E}) &= \mathcal{D}(\mathsf{G}) + \mathcal{D}(\mathsf{K}^{|\mathsf{V}|} - \mathsf{E}) - \mathcal{A}(\mathsf{G}) - \mathcal{A}(\mathsf{K}^{|\mathsf{V}|} - \mathsf{E}) \\ &= \mathcal{D}(\mathsf{G}) + \mathcal{D}(\mathsf{K}^{|\mathsf{V}|} - \mathsf{E}) - \langle 1 \rangle + Id \\ &= |\mathsf{V}|\ Id - \langle 1 \rangle. \end{aligned}$$

Consider an orthonormal basis $x_1, \dots, x_{|\mathsf{V}|}$ of eigenvectors of $\mathcal{B}(\mathsf{G})$. By $\mathcal{B}(\mathsf{K}^{|\mathsf{V}|} - \mathsf{E}) = |\mathsf{V}|\ Id - \langle 1 \rangle - \mathcal{B}(\mathsf{G})$, one obtains

$$\mathcal{B}(\mathsf{K}^{|\mathsf{V}|} - \mathsf{E})x_i = |\mathsf{V}|x_i - \langle 1 \rangle x_i - \lambda_i x_i = (n - \lambda_i)x_i, \qquad i = 2, \dots, |\mathsf{V}|.$$

This is due to the fact that all x_i are orthogonal to x_1 and hence $\langle 1 \rangle x_i = 0$, $i = 2, \dots, |\mathsf{V}|$. □

Exercise 12.5. *(1) Prove that the spectrum of a simple graph consists of the union of the spectra of its connected components.*

(2) Use (1) and Lemma 12.4 in order to determine the spectrum of a complete bipartite graph $\mathsf{K}^{m,n}$.

Exercise 12.6. *Determine the spectrum of the complete graph* K^n*, of the cycle* C^n*, and of the path* P^n*,* $n \in \mathrm{N}$.

In order to find bounds for the eigenvalues of a graph, the following so-called **minimax formula** is useful. Whenever we write $U \leq \mathrm{C}^n$, we mean that U is a linear subspace of C^n.

Proposition 12.7 (R. Courant, 1937)**.** *Let A be an $n \times n$ real symmetric matrix. Then its k^{th} eigenvalue is given by*

$$\min_{\substack{U \leq \mathrm{C}^n \\ \dim U = k}} \max_{\substack{x \in U \\ x \neq 0}} \frac{x^T A x}{\|x\|^2} = \min_{\substack{x \neq 0 \\ x^T x_i = 0\ \forall i \leq k}} \frac{x^T A x}{\|x\|^2},$$

where x_i denotes the i^{th} (eigen)vector of the orthonormal basis.

Observe that on one hand the first eigenvalue of a graph is always 0; on the other hand, for many applications – to e.g. random walks, ergodic theory, asymptotics of differential equations, linear programming (see e.g. [**9, 16, 7**]) – the relevant parameter is the second least eigenvalue. Relating it to the graph theoretical properties of the graph is particularly interesting.

Proposition 12.8. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a simple graph with $|\mathsf{V}| \geq 2$. Then*

$$\lambda_2 \leq \frac{1}{2} \min\{d(v, \mathsf{G}) + d(w, \mathsf{G}) : (v, w) \notin \mathsf{E}\}.$$

PROOF. Pick $v, w \in \mathsf{V}$. Consider $y \in \mathrm{C}^{|\mathsf{V}|}$ defined as the vector containing only 0-entries, with the exception of one entry $y_v = 1$ corresponding to v and one entry $y_w = -1$ corresponding to w. Clearly, y is orthogonal to the 1-vector x_1 and therefore we can apply the minimax formula and deduce that

$$\begin{aligned} \lambda_2 &= \min_{\substack{x \neq 0 \\ x^T x_1 = 0}} \frac{x^T \mathcal{B} x}{\|x\|^2} \\ &\leq \frac{y^T \mathcal{B} y}{\|y\|^2} \\ &= \frac{\sum_{(v,w)\in \mathsf{E}} \beta_{vw} y_w \overline{y_v}}{\sum_{v \in \mathsf{V}} |y_v|^2} \\ &= \frac{d(v, \mathsf{G}) + d(w, \mathsf{G})}{2}. \end{aligned}$$

This concludes the proof. □

Another estimate on λ_2 is provided in the following in terms of the edge-boundary of a node set, cf. Definition 8.6.

Proposition 12.9. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a simple graph. Then its eigenvalue λ_2 satisfies*

$$|\mathsf{V}||\partial \mathsf{V}'| \geq \lambda_2 \min_{\mathsf{V}' \subset \mathsf{V}} |\mathsf{V}'||\mathsf{V} \setminus \mathsf{V}'|.$$

PROOF. Fix $\mathsf{V}' \subset \mathsf{V}$. In order to avoid trivial cases we can assume that V' does not agree with either $\emptyset$ or V (in these cases the claimed inequality reduces to $0 = 0$). Consider a vector $x^{\mathsf{V}'} \in \mathrm{C}^{\mathsf{V}}$ defined by

$$x_v^{\mathsf{V}'} = \begin{cases} |\mathsf{V} - \mathsf{V}'| & \text{if } v \in \mathsf{V}', \\ -|\mathsf{V}'| & \text{otherwise.} \end{cases}$$

Clearly, $x^{\mathsf{V}'}$ is orthogonal to the 1-vector x_1 and $\|x\|^2 = |\mathsf{V}||\mathsf{V}'||\mathsf{V} \setminus \mathsf{V}'|$. Computing the quadratic form of $\mathcal{B}$ yields

$$x^{\mathsf{V}'\,T} \mathcal{B} x^{\mathsf{V}'} = |\partial V'||\mathsf{V}|^2,$$

and by the minimax formula one obtains

$$\begin{aligned} \lambda_2 &= \min_{\substack{x \neq 0 \\ x^T x_1 = 0}} \frac{x^T \mathcal{B} x}{\|x\|^2} \\ &\leq \frac{x^{\mathsf{V}'\,T} \mathcal{B} x^{\mathsf{V}'}}{\|y\|^2} \\ &= \frac{\sum_{(v,w)\in \mathsf{E}} \beta_{vw} x_w^{\mathsf{V}'} \overline{x_v^{\mathsf{V}'}}}{\sum_{v \in \mathsf{V}} |x_v^{\mathsf{V}'}|^2} \\ &= \frac{|\mathsf{V}||\partial \mathsf{V}'|}{|\mathsf{V}'||\mathsf{V} \setminus \mathsf{V}'|}. \end{aligned}$$

This concludes the proof. □

Proposition 12.10 (M. Fiedler 1973)**.** *Let* $\mathsf{G}_1 = (\mathsf{V}, \mathsf{E}_1, g_1)$ *and* $\mathsf{G}_2 = (\mathsf{V}, \mathsf{E}_2, g_2)$ *be two simple graphs with same node set. Then the following assertions hold.*
(1) If $\mathsf{E}_1 \cap \mathsf{E}_2 = \emptyset$*, then their algebraic connectivities satisfy* $\lambda_2(\mathsf{G}_1) + \lambda_2(\mathsf{G}_2) \leq \lambda_2(\mathsf{G}_1 \cup \mathsf{G}_2)$.
(2) If $\mathsf{G}_1 \subset \mathsf{G}_2$*, then* $\lambda_2(\mathsf{G}_1) \leq \lambda_2(\mathsf{G}_2)$.

Proof. (1) Observe that $\mathcal{B}(\mathsf{G}_1 \cup \mathsf{G}_2) = \mathcal{B}(\mathsf{G}_1) + \mathcal{B}(\mathsf{G}_2)$. Accordingly, by the minimax formula

$$\begin{aligned} \lambda_2(\mathsf{G}_1 \cup \mathsf{G}_2) &= \min_{\substack{x \neq 0 \\ x^T x_1 = 0}} \frac{x^T \mathcal{B}(\mathsf{G}_1 \cup \mathsf{G}_2) x}{\|x\|^2} \\ &\geq \min_{\substack{x \neq 0 \\ x^T x_1 = 0}} \frac{x^T \mathcal{B}(\mathsf{G}_1) x}{\|x\|^2} + \min_{\substack{x \neq 0 \\ x^T x_1 = 0}} \frac{x^T \mathcal{B}(\mathsf{G}_2) x}{\|x\|^2} \\ &= \lambda(\mathsf{G}_1) + \lambda(\mathsf{G}_2). \end{aligned}$$

(2) Since the algebraic connectivity of a graph is always nonnegative, write $\mathsf{E}_2 = \mathsf{E}_1 \cup (\mathsf{E}_2 \setminus \mathsf{E}_1)$ and apply (1). □

Remark 12.11. *Let* G *be a simple graph. Using a generalized minimax formula as well as the Perron–Frobenius Theorem for matrices with positive entries one can also prove that*

$$\lambda_2 \leq \frac{|\mathsf{V}|}{|\mathsf{V}| - 1} \delta(\mathsf{G}) \leq \frac{|\mathsf{V}|}{|\mathsf{V}| - 1} \Delta(\mathsf{G}) \leq \lambda_{max} \leq 2\Delta(\mathsf{G}),$$

cf. [**16**, Prop. 2.8]*. On the other hand, by a result of Fiedler one can estimate the algebraic connectivity in terms of connectivity* $\lambda(\mathsf{G})$ *and edge-connectivity* $\sigma(\mathsf{G})$ *of* G *and obtain*

$$2\sigma(\mathsf{G}) \left(1 - \cos \frac{\pi}{|\mathsf{V}|}\right) \leq \lambda_2 \leq \lambda(\mathsf{G}).$$

Further bounds on the eigenvalues of G *can be obtained if* G *is symmetric in the sense of Chapter 9, cf.* [**2**, § VIII.3].

Exercise 12.12. *Apply Propositions 12.8–12.9 in order to deduce an upper bound for the algebraic connectivity of a star with* m *edges. Which criterium yields the better estimate? Which one can be applied more easily?*

Exercise 12.13. *Find an estimate for the algebraic connectivity of the Petersen graph using the results of Remark 12.11.*

Exercise 12.14. *Let* G *be an* r*-regular simple graph.*
(1) Prove that λ *is an eigenvalue of* $\mathcal{B}$ *if and only if* $r - \lambda$ *is an eigenvalue of* $\mathcal{A}$.
(2) More precisely, denote by λ_i, μ_i *the* i^{th} *eigenvalues of* $\mathcal{B}$ *and* $\mathcal{A}$*, respectively. Prove that*

$$\mu_i = r - \lambda_{n+1-i}.$$

(3) Deduce from (2) that the admittance matrix of the complete graph K^n *are* 0 *(simple eigenvalue) and* n *(with multiplicity* $n - 1$*).*

Adjacency matrices offer a rich spectral theory, too. Some connections to relevant graph theoretical values introduced in Chapter 11, like the chromatic number $\chi(\mathsf{G})$ and the independence number $\beta(\mathsf{G})$, are presented in the following.

Proposition 12.15. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a connected simple graph.*
(1) $|\mu_i| \leq \Delta(\mathsf{G})$ for all $i = 1 \ldots, |\mathsf{V}|$ and moreover $\delta(\mathsf{G}) \leq \mu_{max} \leq \Delta(\mathsf{G})$.
(2) $\Delta(\mathsf{G})$ is an eigenvalue of $\mathcal{A}$ if and only if G is regular – and in this case it is a simple eigenvalue.
(3) If $-\Delta(\mathsf{G})$ is an eigenvalue of $\mathcal{A}$, then G is regular and bipartite.
(4) Let $\mathsf{V}' \subset \mathsf{V}$ and consider the subgraph $\mathsf{G}[\mathsf{V}']$. Then all eigenvalues of the adjacency matrix of $\mathsf{G}[\mathsf{V}']$ are contained in $[\mu_{min}, \mu_{max}]$.
(5) $\chi(\mathsf{G}) \leq \mu_{max} + 1$.

Proof. (1) Let $\mathcal{A}x = \mu_i x$ for some $x \neq 0$ and some $i = 1, \ldots, |\mathsf{V}|$ and consider an entry x_v such that $|x_v| \geq |x_w|$ for all $w \in \mathsf{V}$. We can also rescale x in oder to have $x_v = 1$. It follows that

$$|\mu_i| = |\mu_i x_v| = |\sum_{w \in \mathsf{V}} \alpha_{vw} x_w| \leq |x_v| \sum_{w \in \mathsf{V}} \alpha_{vw} = d(w, \mathsf{G})|x_v| \leq \Delta(\mathsf{G}).$$

Since this implies in particular $\mu_{max} \leq \Delta(\mathsf{G})$, it remains to show that $\mu_{max} \geq \delta(\mathsf{G})$. To this aim, consider the 1-vector x_1 and observe that

$$x_1^T \mathcal{A} x_1 = \sum_{v,w \in \mathsf{V}} \alpha_{vw} = \sum_{v \in \mathsf{V}} d(v, \mathsf{G}) \geq |\mathsf{V}| \delta(\mathsf{G}).$$

Since the numerical radius of $\mathcal{A}$[1] is by definition not smaller than $x_1^T \mathcal{A} x_1 / |\mathsf{V}|$, the claim follows from the observation that

$$\|x_1\|^2 = \sum_{v \in \mathsf{V}} 1^2 = \|\mathsf{V}\|.$$

(2) Let $\Delta(\mathsf{G})$ be an eigenvalue and chose x (and in particular x_v) as in (1). Then

$$\Delta(\mathsf{G}) = \Delta(\mathsf{G}) x_v = \sum_{w \in \mathsf{V}} \alpha_{vw} x_w.$$

Since $x_w \leq x_v = 1$ for all $w \in \mathsf{V}$, it follows that

$$\Delta(\mathsf{G}) \leq \sum_{w \in \mathsf{V}} \alpha_{vw} = d(v, \mathsf{G}),$$

i.e., $d(v, \mathsf{G}) = \Delta(\mathsf{G})$. Moreover, it also follows that $x_w = x_v = 1$ for all nodes w adjacent to v. Repeating the above reasoning we deduce that $d(w, \mathsf{G}) = \Delta(\mathsf{G})$ for all w adjacent to v. Due to connectedness of G, we can continue in this way and deduce $x_w = 1$ and $d(w, \mathsf{G}) = \Delta(\mathsf{G})$ for all $w \in \mathsf{V}$. In particular, G is regular. If conversely G is Δ-regular, then the v^{th} coordinate of $\mathcal{A}x_1$ (where x_1 is the 1-vector) is given by

$$\mathcal{A}x_1 = \sum_{w \in \mathsf{V}} \alpha_{vw} = \Delta,$$

[1] Recall that the **numerical range** of a real symmetric matrix A is by definition the set

$$\{x^T A x \in \mathrm{R} : \|x\| = 1\} = \left\{ \frac{x^T A x}{\|x\|^2} \in \mathrm{R} : \|x\| \neq 0 \right\}.$$

By Proposition 12.7 its maximum (the **numerical radius** $w(A)$) and minimum agree with the largest and smallest eigenvalue of A, respectively.

showing that Δ is an eigenvalue corresponding to the eigenvector x_1.

(3) If $-\Delta$ is an eigenvalue, then following the reasoning in (2) we deduce that $d(v, \mathsf{G}) = \Delta(\mathsf{G})$ and $x_v = -x_w = -1$ for all nodes w adjacent to v. Again spreading these values around the connected graph we obtain that G is Δ-regular and such that the nodes at odd/even distance from v can be coloured as $-1/1$, respectively. It follows that G is bipartite.

(4) We only prove it for a node set V' such that $|\mathsf{V}'| = |\mathsf{V}| - 1$, i.e., $\mathsf{V}' = \mathsf{V} \setminus \{v\}$. The general case will then follow by repeating the argument.

We denote by $\mathcal{A}' = (\alpha'_{wz})$ the adjaceny matrix of $\mathsf{G}[\mathsf{V}']$. Observe that two nodes different from v are adjacent in $\mathsf{G}[\mathsf{V}']$ if and only if they are adjacent in G^2. Consider an eigenvector $y \in \mathrm{C}^{|\mathsf{V}'|}$ of norm 1 to μ'_{max}, the largest eigenvalue of $\mathcal{A}'$. We can extend it to a vector $x \in \mathrm{C}^{|\mathsf{V}|}$ by setting $x_v := 0$ and $x_w := y_w$ for all $w \in \mathsf{V}'$. Then also $\|x\| = 1$ and

$$x^T \mathcal{A} x = \sum_{w,z \in \mathsf{V}} \alpha_{wz} x_z \overline{x_w} = \sum_{w \neq v \neq z} \alpha_{wz} x_z \overline{x_w} = y^T \mathcal{A}' y = \mu'_{max}.$$

This means that μ'_{max} belongs to the numerical range of $\mathcal{A}$, hence it is not larger than μ_{max}. One shows similarly that the smallest eigenvalue of $\mathcal{A}'$ is not smaller than μ_{min}.

(5) In order to perform the proof we need an application of an algorithm: write

$$k(\mathsf{G}) := \max_{\mathsf{V}' \subset \mathsf{V}} \delta(\mathsf{G}[\mathsf{V}']),$$

consider $\mathsf{G}_n := \mathsf{G}$ and choose a node v_n with $d(v_n, \mathsf{G}_n) \leq k(\mathsf{G})$ – it exists by assumption. We now consider the smaller graph $\mathsf{G}_{n-1} := \mathsf{G}_n \setminus \{v_n\}$ and pick another node v_{n-1} with $d(v_{n-1}, \mathsf{G}_{n-1}) \leq k(\mathsf{G})$, and so on until G_1 and v_1. Since by assumption each node v_j is connected to at most k nodes v_i with $i < j$, it is possible to colour the whole graph with at most $k(\mathsf{G}) + 1$ colours, i.e.,

$$k(\mathsf{G}) \geq \chi(\mathsf{G}) - 1. \tag{12.1}$$

Let now $\mathsf{V}' \subset \mathsf{V}$ be a node set and consider the induced graph $\mathsf{G}[\mathsf{V}']$. Since $\delta(\mathsf{G}[\mathsf{V}']) \leq \mu'_{max} \leq \mu_{max}$ for all such V', we deduce that $k(\mathsf{G}) \leq \mu_{max}$ and by (12.1) the assertion follows by (1). □

Remark 12.16. *It is possible to introduce the admittance matrix of a non-simple or even weighted graph, too. More precisely, let $\rho : \mathsf{E} \to \mathrm{C}$ is a weight and extend it by 0 to all edges of $\mathsf{K}^{|\mathsf{V}|}$. With an abuse of notation we denote this extension by ρ, too. We introduce a weighted notion of node degree: for $v \in \mathsf{V}$ we define*

$$d(v, \mathsf{G}) := \sum_{e \text{ incident in } v} \rho(e).$$

Clearly, this agrees with the usual notion if the graph is simple and each edge has weight 1. Moreover, replace the usual adjacency matrix by a generalized one whose entries α_{vw} are given by the sum of the weights of all edges connecting v_i, v_j. Then we can introduce just as in Definition 4.8 a generalized admittance matrix, which enjoys most of the spectral properties of the unweighted one, cf. [**16**] *and* [**5**].

[2] The failure of this equivalence for the *admittance matrix*, due to the fact that in general $d(w, \mathsf{G}) \neq d(w, \mathsf{G}')$, is the reason why this proof cannot be carried over from $\mathcal{A}$ to $\mathcal{B}$.

CHAPTER 13

Random graphs

The theory of random graphs has been initiated by a series of fundamental papers by P. Erdős and A. Rényi at the end of the 1950s. Several monographs have been devoted to this topic, including [**3**]. In our brief overview of some basic methods and results we will follow [**2**, Chapt. VII], [**8**, Chapt. 11], and [**1**, Chapt. 35].

Roughly speaking, a random graph is a class of graphs arising under the action of some random process that determines which nodes and/or which edges should exist. Several such models have been introduced in the literature: we only focus on two of them. The most famous one is presented in the following and has been introduced by Erdős and Rényi in 1959.

Definition 13.1. *Let V be a nonempty set of n elements. Let $p \in [0,1]$. Then we denote by $\mathcal{G}(n,p)$ – usually called the **Erdős–Rényi random graph** – the probability space whose sample space consists of all simple graphs with node set V, and such that any possible edge between elements of V[1] actually appears with probability p, independently of each other.*

With an abuse of notation, we do not distinguish between a random graph $\mathcal{G}$ and its sample space and commonly write $\mathsf{H} \in \mathcal{G}$ to express that a graph H arises as a possible outcome of the random graph $\mathcal{G}$.

Example 13.2. *Let $\mathsf{V} := \{v, w\}$ and $p \in [0,1]$. Then of course the sample space of $\mathcal{G}(2,p)$ is $\{\mathsf{G}_0, \mathsf{G}_1\} \equiv \{0,1\}$, where G_i is the graph consisting of two points connected by i edges, $i = 0, 1$, and we assign probability p to arising of G_1 and $1-p$ to the arising of G_0.*

Remark 13.3. *(1) Thus, one can construct an Erdős–Rényi random graph on two nodes by flipping a (possibly unfair) coin. Similarly, an Erdős–Rényi random graph on three nodes can be constructed flipping the coin three times, one for each of the three possible edges $(v_1, v_2), (v_1, v_3), (v_2, v_3)$. More generally, denoting by $\mathsf{E}_{\max}$ the set of all possible edges in a graph with n nodes (i.e., the edge set of K^n), we consider $\mathcal{G}(n,p)$ as the product space*

$$\mathcal{G}(n,p) = \prod_{e \in \mathsf{E}_{\max}} \mathcal{G}(2,p),$$

i.e., a space (Ω, P) where $\Omega = 2^{\mathsf{E}_{\max}}$ and P is the product probability. In fact, the total number of edges is a random variable with expectation value $E(n) = p \binom{n}{2}$.

(2) Equivalently, we can think of a random graph as a simple graph with given node set V and whose adjacency matrix is a symmetric $|\mathsf{V}| \times |\mathsf{V}|$-matrix whose diagonal entries

[1] Of course, there are $\binom{n}{2}$ of them.

are surely 0 and whos off-diagonal entries are either 0 (with probability $1-p$) or 1 (with probability p) independently of each other, i.e., 0 or 1 according to the Bernoulli distribution. This class of random matrices is often called symmetric Bernoulli ensemble.

Another – less common – random graph is presented in the following.

Definition 13.4. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a simple graph. Let $p \in [0,1]$. Then we denote by $\mathcal{G}(\mathsf{G}, p)$ the probability space whose sample space consists of all subgraphs of G induced by a node set $\mathsf{V}_p \subset \mathsf{V}$, and such that any element of V is actually contained in V_p with probability p, independently of each other.*

Example 13.5. *Let G be a path of length 2 defined by the edges $(v,w),(w,z)$ and $p \in [0,1]$. Then the possible outcomes of V_p are $2^3 = 8$, namely $\emptyset$, $\{v\}$, $\{w\}$, $\{z\}$, $\{v,w\}$, $\{v,z\}$, $\{w,z\}$, $\{v,w,z\}$. Accordingly, the sample space of $\mathcal{G}(\mathsf{G},p)$ contains eight graphs: the empty graph (appearing with probability $(1-p)^3$), three graphs consisting of only one isolated node (each appearing with probability $(1-p)^2 p$), a graph consisting of isolated nodes v,z and the two graphs (v,w) and (w,z) (each appearing with probability $(1-p)p^2$), and finally the original graph G (appearing with probability p^3).*

Like in Remark 13.3, it is now possible to define (typically N-valued) random variables acting on the sample space associated with the random graph – say, number of connected components, connectivity, clique number, etc. – these and many more relevant graph theoretical properties of the Erdős–Rényi random graph are studied, e.g., in [**3**]. See `http://tinyurl.com/czhcdg` for some basic results on connectedness of the Erdős–Rényi random graph.

Three simple examples are given.

Example 13.6. *Let V be a set with n elements, $p \in [0,1]$, and let $k \in \mathrm{N}$, $k \leq n$. In the random graph $\mathcal{G}(n,p)$, the expected number of nodes with degree k is*

$$n \binom{n-1}{k} p^k (1-p)^{n-1-k}.$$

Fix a node $v \in \mathsf{V}$. We are going to find the probability that v has degree k. Due to independency, the probability that exactly k given edges incident in v appear in the random graph agrees with the total probability of "finding" k given nodes (the endpoints of the edges) among the remaining $n-1$, each with individual probability p, i.e., $p^k(1-p)^{n-1-k}$. Since there are $\binom{n-1}{k}$ subsets of k elements in $\mathsf{V} \setminus \{v\}$, it follows that for all G in the sample space of $\mathcal{G}(n,p)$

$$P(d(v,\mathsf{G}) = k) = \binom{n-1}{k} p^k (1-p)^{n-1-k}.$$

This defines an N-valued random variable over $\mathcal{G}(n,p)$. Since $P(d(v,\mathsf{G}) = k)$ does not depend on v, integrating over the sample space (i.e., summing over the n nodes) the claim follows.

Example 13.7. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a simple graph and $p \in [0,1]$. In $\mathcal{G}(\mathsf{G},p)$ the expected number of nodes and edges is $p|\mathsf{V}|$ and $p^2|\mathsf{E}|$, respectively.*

In fact, the counting functions n_p, m_p of nodes and edges, respectively, define two N*-valued random variables over $\mathcal{G}(n,p)$. Since each node appears with probability p, summing over V one sees that the expected number of nodes is $p|\mathsf{V}|$. Since each edge appears if and only if both its endpoints appear, hence with probability p^2, summing over E one deduces that the expected number of edges is $p^2|\mathsf{E}|$.*

Example 13.8. *Let V be a set with n elements, $p \in [0,1]$, and let $k \in \mathrm{N}$, $k \leq \binom{n}{2}$. In $\mathcal{G}(n,p)$ the probability that exactly k edges exist is*

$$p^k(1-p)^{\binom{n}{2}-k}.$$

As already remarked, $\binom{n}{2}$ edges can possibly exist in $\mathcal{G}(n,p)$. Each of those appear with probability p and the remaining ones do not appear with probability $1-p$.

Exercise 13.9. *Let V be a set with n elements, $p \in [0,1]$, and consider the random graph $\mathcal{G}(n,p)$. Prove that the probability that there exist k independent nodes, i.e., that $\alpha(\mathsf{G}) \geq k$, is not larger than*

$$\binom{n}{k}(1-p)^{\binom{k}{2}}.$$

The interest in the theory of random graphs is twofold: on one hand, if a real-life problem can be modelled by a random graph, then one can hope to discuss statistichal properties of the concrete systems by means of the random graph theory. On the other hand, another feature of random graph theory is that it allows to apply so-called *probabilistic method* to graph theory. Roughly speaking, the idea is to show that a graph satisfying a certain property Π exists because in a suitable random graph (with the choices of a specific random graph and of the parameter p typically depending on the concrete problem) the probability that a graph satisfies Π is larger than 0.

Proposition 13.10. *Let V be a set with n elements, $p \in [0,1]$, and let $k \in \mathrm{N}$. The expected number of cycles of length k in $\mathcal{G}(n,p)$ is given by*

$$\frac{n!}{(n-k)!}\frac{p^k}{2k}.$$

Proof. Fix a cycle $\mathsf{C}^k \subset \mathsf{K}^n$. More generally, we denote by $\mathcal{C}^k$ the set of all cycles of length k whose nodes belong to V. Define an $\{0,1\}$-valued random variable X_{C^k} on $\mathcal{G}(n,p)$ by

$$X_{\mathsf{C}^k}(\mathsf{G}) := \begin{cases} 1 & \text{if } \mathsf{C}^k \subset \mathsf{G}, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that

$$E(X_{\mathsf{C}^k}) = \sum_{\mathsf{G}\in\mathcal{G}(n,p)} P(\{\mathsf{G}\})X_{\mathsf{C}^k}(\{\mathsf{G}\}) = \sum_{\substack{\mathsf{G}\in\mathcal{G}(n,p)\\ \mathsf{C}^k\subset\mathsf{G}}} P(\{\mathsf{G}\}) = P[\mathsf{C}^k \subset \mathsf{G} \in \mathcal{G}(n,p)].$$

However, the given cycle C^k appears in some $\mathsf{G} \in \mathcal{G}(n,p)$ if and only if all of its edges appear in G, hence $\mathsf{C}^k \subset \mathsf{G}$ with probability p^k. We conclude that

$$E(X_{\mathsf{C}^k}) = p^k \qquad \text{for all } \mathsf{C}^k \in \mathcal{C}^k.$$

Consider a new random variable

$$X_k := \sum_{\mathsf{C}^k \in \mathcal{C}^k} X_{\mathsf{C}^k},$$

which assigns to each $\mathsf{G} \in \mathcal{G}(n,p)$ the number of cycles of length k it contains. Compute its expected value by exploiting linearity E and obtain

$$E(X_k) = E\left(\sum_{\mathsf{C}^k \in \mathcal{C}^k} X_{\mathsf{C}^k}\right) = \sum_{\mathsf{C}^k \in \mathcal{C}^k} E(X_{\mathsf{C}^k}) = \sum_{\mathsf{C}^k \in \mathcal{C}^k} p^k = |\mathcal{C}^k| p^k.$$

Thus, it remains to determine $|\mathcal{C}^k|$. To this aim, observe that there are $\frac{n!}{(n-k)!}$ sequences of V containing k (distinct) elements. Clearly, a cycle is not uniquely determined by any of this subsets, say $(v_1, \ldots, v_k)$, since also the "shifted" sets $(v_2, \ldots, v_k, v_1), \ldots, (v_k, v_1, \ldots, v_{k-1})$ and the "reversed" sets $(v_k, v_{k-1}, \ldots, v_1)$, $(v_{k-1}, \ldots, v_1, v_k)$, …, $(v_1, v_k, \ldots, v_2)$ all identify the same cycle. We conclude that there are $\frac{n!}{(n-k)!}\frac{1}{2k}$ possible cycles of length k in graphs $\mathsf{G} \in \mathcal{G}(n,p)$ – i.e.,

$$|\mathcal{C}^k| = \frac{n!}{(n-k)!}\frac{1}{2k}.$$

This yields the claim. □

In fact, the above result also serves as a lemma in order to prove an abstract result not related to random graphs. We formulate it as an existence result following [**1**, Thm. 35.3], although by now explicit constructions of graphs with the claimed properties are known.

Theorem 13.11 (P. Erdős, 1959)**.** *For every $k \in \mathbb{N}$, $k \geq 2$, there exists a simple graph with both girth $g(\mathsf{G})$ and chromatic number $\chi(\mathsf{G})$ larger than k.*

Proof. The proof can be performed by exploiting the theory of random graphs $\mathcal{G}(n,p)$, where parameters n, p will be carefully fixed later. We will prove that exhibiting a graph with high independence number is the same as exhibiting a graph with low chromatic number; and we will show that, for n large enough and a suitable $p = p_n$, the probability of finding a graph with many short cycles and high independence number in $\mathcal{G}(n,p)$ is low enough to ensure the existence of a graph with the desired properties. The proof goes in several steps.

(1) First, we observe that for any simple graph $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ one has $\alpha(\mathsf{G})\chi(\mathsf{G}) \geq |\mathsf{V}|$.
(2) We show that the for the random graph $\mathcal{G}(n,p)$ on n nodes and $2 \leq r \leq n$ one has $P(\alpha(\mathsf{G}) \geq r) \leq (ne^{-p(r-1)}2)^r$.
(3) We fix $k > 0$, set $p := n^{-\frac{k}{k+1}}$ and prove that there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ one has $P(\alpha(\mathsf{G}) \geq \frac{n}{2k}) < \frac{1}{2}$.
(4) We show that there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ the probability that G contains more than $\frac{n}{2}$ cycles of length at most k is less than $\frac{1}{2}$.

(5) Finally, take $n \geq \max\{N_1, N_2\}$ and consider a graph G' on n nodes with $\alpha(\mathsf{G}') < \frac{n}{2k}$ and less than $\frac{n}{2}$ cycles of length at most k. Deleting a node from each of these cycles we obtain the graph with desired properties.

Let us prove these points.

(1) By definition, in a $\chi(\mathsf{G})$-colouring all colours define node sets $\mathsf{V}_1, \ldots, \mathsf{V}_{\chi(\mathsf{G})}$ that are independent, i.e., $|\mathsf{V}_i| \leq \alpha(\mathsf{G})$ for all $i = 1 \ldots, \chi(\mathsf{G})$. Thus,

$$|\mathsf{V}| = \sum_{i=1}^{\chi(\mathsf{G})} |\mathsf{V}_i| \leq \sum_{i=1}^{\chi(\mathsf{G})} \alpha(\mathsf{G}) = \chi(\mathsf{G})\alpha(\mathsf{G}).$$

(2) For a fixed set $\mathsf{V}' \subset \mathsf{V}$ with r elements the probability that V' is independent agrees with the probability that no 2-element subset of V' is connected by an edge, i.e., $(1-p)^{\binom{r}{2}}$. Accordingly, the probability that *no* node set with r elements is independent, i.e., the probability that the independence number is larger than r, is

$$\begin{aligned} P(\alpha(\mathsf{G}) \geq r) &\leq \sum_{\substack{V' \subset V \\ |\mathsf{V}'|=r}} P(\alpha(\mathsf{G}) = r) = \sum_{\substack{V' \subset V \\ |\mathsf{V}'|=r}} (1-p)^{\binom{r}{2}} = \binom{n}{r} (1-p)^{\binom{r}{2}} \\ &\leq n^r (1-p)^{\binom{r}{2}} = (n(1-p)^{\frac{r-1}{2}})^r \leq (ne^{\frac{-p(r-1)}{2}})^r, \end{aligned}$$

where the last inequality follows from the fact that $1 - p \leq e^{-p}$ for all $p > 0$.

(3) We observe that

$$\lim_{m \to \infty} \frac{m^{\frac{1}{k+1}}}{6k \log m} = +\infty,$$

and accordingly there exists $N_1' \in \mathrm{N}$ such that $m^{\frac{1}{k+1}} \geq 6k \log m$, i.e., $p_m := \frac{m^{\frac{1}{k+1}}}{m} \geq \frac{6k \log m}{m}$ for all $m \geq N_1'$. We want to apply (2) with $r := \lceil \frac{m}{2k} \rceil$, so that in particular $2 \leq r \leq m$ and $p_m r \geq 3 \log m$ for all $m \geq N_1'$. For all $m \geq N_1$ we obtain

$$0 \leq me^{\frac{-p_m(\lceil \frac{m}{2k} \rceil - 1)}{2}} = me^{-\frac{p_m \lceil \frac{m}{2k} \rceil}{2}} e^{\frac{p_m}{2}} \leq me^{-\frac{3 \log m}{2}} e^{\frac{1}{2}} = mm^{-\frac{3}{2}} e^{\frac{1}{2}} = \left(\frac{e}{m}\right)^{\frac{1}{2}}.$$

Since $\lim_{m \to \infty} \left(\frac{e}{m}\right)^{\frac{1}{2}} = 0$, by the sandwich theorem we deduce that

$$\lim_{m \to \infty} me^{\frac{-p_m(\lceil \frac{m}{2k} \rceil - 1)}{2}} = \lim_{m \to \infty} \left(me^{\frac{-p_m(\lceil \frac{m}{2k} \rceil - 1)}{2}} \right)^{\lceil \frac{m}{2k} \rceil} = 0.$$

We are finally in the position to apply (2) and deduce the existence of $N_1 \in \mathrm{N}$ such that

$$P\left(\alpha(\mathsf{G}) \geq \frac{n}{2k}\right) < \frac{1}{2}$$

for all graphs G on n nodes, $n \geq N_1$.

(4) Consider the random variables X_k introduced in Proposition 13.10. Introduce the random variable

$$X_{\leq k} := \sum_{i=3}^{k} X_i,$$

which assigns to each $\mathsf{G} \in \mathcal{G}(n,p)$ the number of cycles of length at most k it contains. By Proposition 13.10 we obtain

$$E(X_{\leq k}) = \sum_{i=3}^{k} E(X_i) = \sum_{i=3}^{k} \frac{m!}{(m-i)!} \frac{p_m^i}{2i} = \frac{1}{2} \sum_{i=3}^{k} m^i p_m^i \leq \frac{(k-2)}{2} m^k p_m^k,$$

due to the fact that $mp_m = m^{\frac{1}{k+1}} > 1$ by construction of p_m. By Markov's inequality we conclude that

$$0 \leq P(X_{\leq k}(\mathsf{G}) \geq \frac{m}{2}) \leq \frac{E(X)}{\frac{m}{2}} \leq (k-2)m^{k-1}p_m^k = (k-2)m^{-\frac{1}{k+1}}.$$

Again, $\lim_{m\to\infty}(k-2)m^{-\frac{1}{k+1}} = 0$ and by the sandwich theorem $\lim_{m\to\infty} P(X_{\leq k}(\mathsf{G}) \geq \frac{m}{2}) = 0$, i.e., there exists $N_2 \in \mathrm{N}$ such that for all $m \geq N_2$

$$P(\alpha(\mathsf{G}) \geq \frac{m}{2k}) \leq P(\alpha(\mathsf{G}) \geq \frac{m}{2}) < \frac{1}{2}.$$

(5) A graph G with girth $g(\mathsf{G}) > k$ can be obtained by deleting from each cycle a node (why?). Observe that G has at least $\frac{n}{2}$ nodes. Moreover, $\alpha(\mathsf{G}) \leq \alpha(\mathsf{G}') < \frac{n}{2k}$. Accordingly,

$$\chi(\mathsf{G}) \geq \frac{n}{2}\frac{1}{\alpha(\mathsf{G})} \geq \frac{n}{2\alpha(\mathsf{G}')} > \frac{n}{\frac{n}{k}} = k.$$

This concludes the proof. □

Remark 13.12. *The theorem states that comparatively small graphs can be constructed that have arbitrarily large girth and chromatic number. Since graphs with large girth look* locally *like a tree, and hence are* locally 2*-node-colourable, this seems to imply that chromatic properties of a graph are a* global *property, instead. This appears quite surprising.*

Definition 13.13. *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ be a graph without loops. Consider a drawing of G in R^2 such that*

- *no edge crosses itself,*
- *edges with a common endpoint do not cross,*
- *at most two edges cross in a point, and*
- *two edges cross in at most one point.*

Then the ***crossing number*** *of G is the minimal number of crossings appearing in any drawing respecting the above rules. We denote it by* $\mathrm{cr}(\mathsf{G})$.

Observe that if a graph has more than $3|\mathsf{V}|-6$ edges, then by Remark 10.38 it necessarily has nonvanishing crossing number.

Example 13.14. *A graph is planar if and only if* $\mathrm{cr}(\mathsf{G}) = 0$. *We know that K^5 is not planar, hence* $\mathrm{cr}(\mathsf{K}^5) > 0$*; in fact, one can see that* $\mathrm{cr}(\mathsf{K}^5) = 1$.

The following is usually called **crossing lemma** and has been formulated as a conjecture by P. Erdős and R.K. Guy in 1973.

Theorem 13.15. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E}, g)$ *be a simple graph. Then the following assertions hold.*
(1) $\mathrm{cr}(\mathsf{G}) \geq |\mathsf{E}| - 3|\mathsf{V}| + 6$.
(2) If $|\mathsf{E}| \geq 4|\mathsf{V}|$, *then* $\mathrm{cr}(\mathsf{G}) \geq \frac{|\mathsf{E}|^3}{64|\mathsf{V}|^2}$.

The first assertion can be proved directly, while we will use a probabilistic approach to check the second one.

Proof. (1) Construct a new simple and planar graph $\tilde{\mathsf{G}} = (\tilde{\mathsf{V}}, \tilde{\mathsf{E}}, \tilde{g})$ substituting each crossing point of G by a new node[2]. All the new vertices have degree 4, by construction; hence $\tilde{\mathsf{G}}$ has $|\mathsf{V}| + \mathrm{cr}(\mathsf{G})$ nodes and $|\mathsf{E}| + 2\mathrm{cr}(\mathsf{G})$ edges. Due to planarity, by Remark 10.38 $\tilde{\mathsf{G}}$ satisfies the bound $3|\tilde{\mathsf{V}}| - 6 \geq |\tilde{\mathsf{E}}|$, i.e.,

$$3|\mathsf{V}| + 3\mathrm{cr}(\mathsf{G}) - 6 \geq |\mathsf{E}| + 2\mathrm{cr}(\mathsf{G}).$$

This yields the claimed inequality.

(2) Consider a minimal drawing realising the crossing number. For some $p \in [0, 1]$ (to be determined later) we consider the random graph $\mathcal{G}(\mathsf{G}, p)$. We consider three random variables $n_p, m_p, \mathrm{cr}_p : \mathcal{G}(\mathsf{G}, p) \to \mathrm{N}$, counting the number of nodes, the number of edges, and the crossing number, respectively. By (1), $\mathrm{cr}_p - m_p + 3n_p$ is a positive random variable, hence by monotonicity its expected value $E(\mathrm{cr}_p - m_p + 3n_p)$ is positive, too. By Example 13.7, the expected values of n_p and m_p are $p|\mathsf{V}|$ and $p^2|\mathsf{E}|$, respectively.

Like in Example 13.7, one also sees that a given crossing of G, say of $e', e'' \in \mathsf{E}$, appears in $\mathcal{G}(\mathsf{G}, p)$ with probability p^4, because this is the probability that both edges (i.e., all four endpoints) appear in $\mathcal{G}(\mathsf{G}, p)$. Summing over the set of crossings in G one obtains the expected numer of crossing in $\mathcal{G}(\mathsf{G}, p)$, i.e., $p^4\mathrm{cr}(\mathsf{G})$. By linearity of the expectation, $0 \leq E(\mathrm{cr}_p - m_p + 3n_p) = E(\mathrm{cr}_p) - E(m_p) + 3E(n_p) = p^4\mathrm{cr}(\mathsf{G}) - p^2|\mathsf{E}| + 3p|\mathsf{V}|$, whence

$$\frac{|\mathsf{E}|}{p^2} - \frac{3|\mathsf{V}|}{p^3} \leq \mathrm{cr}(\mathsf{G}).$$

We are still free to optimise this estimate: since by assumption $|\mathsf{E}| \geq 4|\mathsf{V}|$, set $p := \frac{4|\mathsf{V}|}{|\mathsf{E}|} \leq 1$ and deduce the claimed inequality. □

[2] Observe that in particular $\tilde{\mathsf{G}}$ is a subdivision of G

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