Universität Ulm
Fakultät für Mathematik und Wirtschaftswissenschaften

Form Methods for Linear Evolution Problems on Hilbert Spaces

Diplomarbeit zur Erlangung des akademischen Grades "Dipl. math. oec."

vorgelegt von
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Introduction

Evolution means the change of a certain system with respect to time. In a more suggestive language, one could talk of the "motion" of a system in time. Examples could be the movement of planets, the growth of a population, ...
From a philosophical point of view (cf. G. Nickel in [**9**]) it turns out, that the right mathematical model for evolution is a one parameter (semi-)group.
By an *evolution problem*, we understand the problem of obtaining information about such a motion (i.e. the semigroup) given the information how the system changes locally. In applications, this information usually arises either from observation or theoretical reasoning and might be expressed by an *abstract Cauchy problem*:

$$(CP_A)\begin{cases} u' &= A(u) \\ u(0) &= u_0 \end{cases}$$

Here u_0 is the initial state of the system and belongs to some state space X, A expresses the local change of the system.
In order to obtain an interesting mathematical theory, one often requires additional structure on X, e.g. that X be a Banach space. Then by a *linear evolution problem* we mean a Cauchy problem, where the local change A is given by a linear operator on X.
Thus in the mathematical language of semigroup theory, a linear evolution problem is a problem of the form:

Given a linear operator A on a Banach space X, decide whether A generates a semigroup on X (and obtain information about the semigroup from A).

This problem was solved in 1948 by Hille and Yosida (in the contraction case, extendet to the general case 1952):

A is the generator of a strongly continuous semigroup if and only if A is a closed, densely defined operator such that for some ω_0 the set (ω_0, ∞) belongs to the resolvent set of A and there exists a constant M such that for all $\lambda \in (\omega_0, \infty)$ the estimate

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\operatorname{Re}\lambda - \omega_0)^n}$$

holds.

As always with mathematical theories, there is a struggle between the generality of the theorems and the applicability. The Hille-Yosida theorem is the most general theorem concerning strongly continuous semigroups. However, the applications are very limited, for only in rather special (and rare) cases these conditions can be verified.

A characterisation which is much easier to handle is given by the LUMER-PHILLIPS theorem:

A closed, densely defined, dissipative operator generates a contraction semigroup if and only if $\lambda - A$ is surjective for some $\lambda > 0$.

On Hilbertspaces, the requirement that A be dissipative means that

$$\operatorname{Re}(Ax \,|\, x) \le 0 \quad \forall\, x \in H$$

A possible interpretation for this is, that we may obtain information about A by looking at the map

$$(x, y) \mapsto (Ax \,|\, y)$$

which will be called the *form* associated with A.
The idea behind all form methods for evolution equations is the following:

Instead of working with operators, work with the associated forms.

In this thesis, we will carry out this idea:
We start by investigating not operators, but closed sectorial forms. Only after this, we will establish a one-to-one correspondence between densely defined forms and a special class of operators, which are generators of analytic semigroups. However, we will *not* require our forms to be densely defined. We will then associate not semigroups, but degenerate semigroups directly with the forms.

This procedure is justified in chapters 2 and 3, where we will see, that operations on the forms carry over appropriately to the semigroups:

- The degenerate semigroup associated to the sum of two forms is obtained by the degenerate semigroups associated to the summands by Trotter's formula.
- Corresponding to the convergence of forms, there is a convergence of the associated semigroups under certain additional hypotheses.

This also shows, that forms are much easier to handle, than operators. In particular, it is possible to construct closed, sectorial forms by adding up forms or using perturbation results. This is illustrated in the last chapter on the example of elliptic forms.

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CHAPTER 1

Forms, Operators, Semigroups

In the following H always denotes a Hilbert space with inner product $(\,\cdot\,|\,\cdot\,)$ and norm $\|\cdot\|$. We start by recalling some basic notions and properties concerning sectorial forms.

1. Definitions

A *sesquilinear form on* H is a mapping $\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \to \mathbb{C}$ which is linear in the first component and antilinear in the second. $D(\mathfrak{a})$ is a subspace of H and is called the *domain* of $\mathfrak{a}$. We say that $\mathfrak{a}$ is *densely defined* if the domain of $\mathfrak{a}$ is normdense in H.

We say that $\mathfrak{b}$ is an *extension* of $\mathfrak{a}$ (and write $\mathfrak{a} \subset \mathfrak{b}$ for this) if $D(\mathfrak{a}) \subset D(\mathfrak{b})$ and $\mathfrak{a}\,[x,y] = \mathfrak{b}\,[x,y]$ for all $x, y \in D(\mathfrak{a})$.

The *sum* of two forms $\mathfrak{a}$ and $\mathfrak{b}$ is defined by

$$(\mathfrak{a}+\mathfrak{b})\,[x,y] = \mathfrak{a}\,[x,y] + \mathfrak{b}\,[x,y] \quad , \quad D(\mathfrak{a}+\mathfrak{b}) = D(\mathfrak{a}) \cap D(\mathfrak{b})\,.$$

If α is a scalar we define the form $\alpha\mathfrak{a}$ by $(\alpha\mathfrak{a})\,[x,y] = \alpha \cdot \mathfrak{a}\,[x,y]$ and $D(\alpha\mathfrak{a}) = D(\mathfrak{a})$. We write $\alpha\,[x,y]$ for $\alpha \cdot (x\,|\,y)$ defined on H.

The *adjoint form* $\mathfrak{a}^*$ of $\mathfrak{a}$ is given by

$$\mathfrak{a}^*\,[x,y] = \overline{\mathfrak{a}\,[y,x]} \quad , \quad D(\mathfrak{a}^*) = D(\mathfrak{a})\,.$$

We easily obtain that $(\alpha \cdot \mathfrak{a} + \beta \cdot \mathfrak{b})^* = \bar{\alpha} \cdot \mathfrak{a}^* + \bar{\beta} \cdot \mathfrak{b}^*$ for scalars α, β and forms $\mathfrak{a}, \mathfrak{b}$ and also $\mathfrak{a}^{**} = \mathfrak{a}$. A form is called *symmetric* if $\mathfrak{a}^* = \mathfrak{a}$.

We define the *real* and *imaginary part* of a form $\mathfrak{a}$ by

$$\operatorname{Re}\mathfrak{a} = \frac{1}{2}\,(\mathfrak{a}+\mathfrak{a}^*) \quad \text{and} \quad \operatorname{Im}\mathfrak{a} = \frac{1}{2i}\,(\mathfrak{a}-\mathfrak{a}^*)\ .$$

It is not hard to see, that for any form $\mathfrak{a}$ the forms $\operatorname{Re}\mathfrak{a}$ and $\operatorname{Im}\mathfrak{a}$ are symmetric. If $\mathfrak{a}$ is a sesquilinear form the associated *quadratic form* (which we will still denote by $\mathfrak{a}$) is given by $\mathfrak{a}\,[x] = \mathfrak{a}\,[x,x]$. The *numerical range* $\Theta\,(\mathfrak{a})$ of a form $\mathfrak{a}$ is given by

$$\Theta\,(\mathfrak{a}) := \{\mathfrak{a}\,[x] \,:\, x \in D(\mathfrak{a})\,,\, \|x\| = 1\}$$

Clearly the numerical range of a symmetric form is real.
The following proposition shows, that we can reconstruct the values of a sesquilinear form if we know the values of the associated quadratic form.

Proposition 1.1. *(Polarization)*
If H is a complex Hilbert space and $\mathfrak{a}$ a sesquilinear form on H then

$$\mathfrak{a}\,[x,y] = \frac{1}{4}\left(\mathfrak{a}\,[x+y] - \mathfrak{a}\,[x-y] + i\mathfrak{a}\,[x+iy] - i\mathfrak{a}\,[x-iy]\right) \quad \forall x,y \in D(\mathfrak{a})\,.$$

Proof. This is the same computation as for inner products. □

Now we impose some restrictions on the numerical range of the form. A *sectorial form* is a form whose numerical range is contained in some sector $\Sigma_\gamma(\theta)$. Here is $\gamma \in \mathbb{R}, \theta \in [0, \frac{\pi}{2})$ and

$$\Sigma_\gamma(\theta) := \{z \in \mathbb{C} \,:\, |\arg(z-\gamma)| \leq \theta\}\ .$$

We write $\Sigma(\theta)$ for $\Sigma_0(\theta)$. γ is called a *vertex* of $\mathfrak{a}$ and θ a *corresponding semiangle.* Note that γ and θ are not uniquely determined by $\mathfrak{a}$.

For symmetric $\mathfrak{a}$ any $w \in \Theta\,(\mathfrak{a})$ is real so if γ is a vertex of $\mathfrak{a}$ then $\Theta\,(\mathfrak{a})$ is contained in $\Sigma_\gamma(\theta)$ for *any* $\theta \in [0, \frac{\pi}{2})$. So being sectorial reduces to the condition $\mathfrak{a}\,[x] \geq \gamma\|x\|^2$ for all $x \in D(\mathfrak{a})$. We write $\mathfrak{a} \geq \gamma$ for this and say that $\mathfrak{a}$ is *semibounded.* The largest number γ with this property is called the *lower bound* of $\mathfrak{a}$ and is denoted by $\gamma_{\mathfrak{a}}$. We say that $\mathfrak{a}$ is *positive* if $\mathfrak{a} \geq 0$.

For a sectorial form $\mathfrak{a}$ the condition $\Theta\,(\mathfrak{a}) \subset \Sigma_\gamma(\theta)$ is equivalent to $\operatorname{Re}\mathfrak{a} \geq \gamma$ and $|\operatorname{Im}\mathfrak{a}\,[x]| \leq \tan\theta(\operatorname{Re}\mathfrak{a} - \gamma)\,[x]$ for all $x \in D(\mathfrak{a})$.

Proposition 1.2. *Let $\mathfrak{a}$ be a sectorial form on H with $\Theta\,(\mathfrak{a}) \subset \Sigma_\gamma(\theta)$. Then for any $x, y \in D(\mathfrak{a})$ the following hold:*

a) $|(\operatorname{Re}\mathfrak{a} - \gamma)\,[x,y]| \leq (\operatorname{Re}\mathfrak{a} - \gamma)\,[x]^{\frac{1}{2}}(\operatorname{Re}\mathfrak{a} - \gamma)\,[y]^{\frac{1}{2}}$
b) $|\operatorname{Im}\mathfrak{a}\,[x,y]| \leq \tan\theta(\operatorname{Re}\mathfrak{a} - \gamma)\,[x]^{\frac{1}{2}}(\operatorname{Re}\mathfrak{a} - \gamma)\,[y]^{\frac{1}{2}}$
c) $|(\mathfrak{a} - \gamma)\,[x,y]| \leq (1+\tan\theta)(\operatorname{Re}\mathfrak{a} - \gamma)\,[x]^{\frac{1}{2}}(\operatorname{Re}\mathfrak{a} - \gamma)\,[y]^{\frac{1}{2}}$

In addition

$$(x\,|\,y)_{\mathfrak{a}} := (x\,|\,y) + (\operatorname{Re}\mathfrak{a} - \gamma)\,[x,y]$$

is an inner product on $D(\mathfrak{a})$.

Proof. Without loss of generality we may assume that $\gamma = 0$ (otherwise, we replace $\mathfrak{a}$ by $\mathfrak{a}+\gamma$). $\operatorname{Re}\mathfrak{a}$ has all the properties of an inner product except that $\operatorname{Re}\mathfrak{a}\,[x] = 0$ does not imply $x = 0$. Thus a) follows as in the proof of the Cauchy-Schwarz inequality where this property is not needed (see [**18**]). We clearly have that $(x\,|\,x) \leq (x\,|\,x)_{\mathfrak{a}}$

so that $(x\,|\,x)_{\mathfrak{a}} = 0$ implies $\|x\| = 0$ and thus $x = 0$. So that $(\,\cdot\,|\,\cdot\,)_{\mathfrak{a}}$ is an inner product.

To prove b) choose $\psi \in \mathbb{R}$ such that $\operatorname{Im}\mathfrak{a}\left[e^{i\psi}x, y\right] = e^{i\psi}\operatorname{Im}\mathfrak{a}\,[x,y] \in \mathbb{R}$. Since $\operatorname{Im}\mathfrak{a}\,[\cdot]$ is realvalued it follows from 1.1 that

$$\begin{aligned}
|\operatorname{Im}\mathfrak{a}\,[x,y]| &= |\frac{1}{4}(\operatorname{Im}\mathfrak{a}\left[e^{i\psi}x+y\right] - \operatorname{Im}\mathfrak{a}\left[e^{i\psi}x-y\right])| \\
&\le \frac{1}{4}\tan\theta(\operatorname{Re}\mathfrak{a}\left[e^{i\psi}x+y\right] + \operatorname{Re}\mathfrak{a}\left[e^{i\psi}x-y\right]) \\
&= \frac{1}{2}\tan\theta(\operatorname{Re}\mathfrak{a}\left[e^{i\psi}x\right] + \operatorname{Re}\mathfrak{a}\,[y]) \\
&= \frac{1}{2}\tan\theta(\operatorname{Re}\mathfrak{a}\,[x] + \operatorname{Re}\mathfrak{a}\,[y])
\end{aligned}$$

Where we have used the fundamental estimate above in the second step. The third equality uses the parrallelogram law, the last one $|e^{i\psi}| = 1$.
Now if neither $\operatorname{Re}\mathfrak{a}\,[x]$ nor $\operatorname{Re}\mathfrak{a}\,[y]$ is equal to 0, we can replace x by $\sqrt{\alpha}x$, y by $\frac{1}{\sqrt{\alpha}}y$ where $\alpha = \operatorname{Re}\mathfrak{a}\,[y]^{\frac{1}{2}}\operatorname{Re}\mathfrak{a}\,[x]^{-\frac{1}{2}}$ and obtain b).
If both $\operatorname{Re}\mathfrak{a}\,[x]$ and $\operatorname{Re}\mathfrak{a}\,[y]$ are zero, we are also done. So now suppose $\operatorname{Re}\mathfrak{a}\,[x] = 0$ and $\operatorname{Re}\mathfrak{a}\,[y] \neq 0$. We have

$$|\operatorname{Im}\mathfrak{a}\,[x,y]| \le \frac{1}{2}\tan\theta\operatorname{Re}\mathfrak{a}\,[y]$$

If we replace y by ty for some $t > 0$ we obtain

$$t|\operatorname{Im}\mathfrak{a}\,[x,y]| \le \frac{1}{2}t^2\tan\theta\operatorname{Re}\mathfrak{a}\,[y]$$

If we now divide by $t > 0$ and let $t \to 0$ we see that $\operatorname{Im}\mathfrak{a}\,[x,y] = 0$ so we proved b) also in this case.

c) follows directly form a) and b). □

2. Closed and Closable Forms

Look again at the inner product $(x\,|\,y)_{\mathfrak{a}} = (x\,|\,y) + (\operatorname{Re}\mathfrak{a} - \gamma)\,[x,y]$ defined in the last proposition. It still depends on the vertex γ which is not uniquely determined by $\mathfrak{a}$. If γ and δ are two possible vertices for $\mathfrak{a}$ we denote the associated inner products by $(\,\cdot\,|\,\cdot\,)_{\mathfrak{a}}^{\gamma}$ and $(\,\cdot\,|\,\cdot\,)_{\mathfrak{a}}^{\delta}$.

Let us assume that $\gamma \le \delta$. Then we have:

$$(x\,|\,x)_{\mathfrak{a}}^{\delta} = \|x\|^2 + (\operatorname{Re}\mathfrak{a} - \delta)\,[x] \le \|x\|^2 + (\operatorname{Re}\mathfrak{a} - \gamma)\,[x] = (x\,|\,x)_{\mathfrak{a}}^{\gamma}$$

and on the other hand

$$(x\,|\,x)_{\mathfrak{a}}^{\gamma} = \|x\|^2 + (\operatorname{Re}\mathfrak{a} - \delta)\,[x] + (\delta - \gamma)\|x\|^2 \le (1 + \delta - \gamma)(x\,|\,x)_{\mathfrak{a}}^{\delta},$$

where we used that $\operatorname{Re}\mathfrak{a} - \delta \geq 0$. Thus for two different choices of vertices, the corresponding inner products are equivalent and the following definition makes sense.

DEFINITION. Let $\mathfrak{a}$ be a sectorial form and $(\cdot\,|\,\cdot)_{\mathfrak{a}}$ be the associated inner product defined as above. We denote the associated norm by $\|\cdot\|_{\mathfrak{a}}$. Also, we often write $H_{\mathfrak{a}}$ instead of $(D(\mathfrak{a}), (\cdot\,|\,\cdot)_{\mathfrak{a}})$. We say that $\mathfrak{a}$ is a *closed form* if $H_{\mathfrak{a}}$ is a Hilbert space.

Closedness behaves well with respect to summation. In fact we have:

PROPOSITION 1.3. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be sectorial forms. Then $\mathfrak{a}+\mathfrak{b}$ is sectorial. If $\mathfrak{a}$ and $\mathfrak{b}$ are closed, then so is $\mathfrak{a}+\mathfrak{b}$.*

PROOF. Without loss of generaliy, we may assume that both forms have a vertex 0. Let $\theta_{\mathfrak{a}}$ and $\theta_{\mathfrak{b}}$ be corresponding semiangles of $\mathfrak{a}$ resp. $\mathfrak{b}$. And let $\theta = \max\{\theta_{\mathfrak{a}}, \theta_{\mathfrak{b}}\}$. Then for $x \in D(\mathfrak{a}+\mathfrak{b}) = D(\mathfrak{a}) \cap D(\mathfrak{b})$ we have:

$$\begin{aligned} |\operatorname{Im}(\mathfrak{a}+\mathfrak{b})[x]| &= |\operatorname{Im}\mathfrak{a}[x] + \operatorname{Im}\mathfrak{b}[x]| \\ &\leq \tan\theta_{\mathfrak{a}}\operatorname{Re}\mathfrak{a}[x] + \tan\theta_{\mathfrak{b}}\operatorname{Re}\mathfrak{b}[x] \\ &\leq \tan\theta\operatorname{Re}(\mathfrak{a}+\mathfrak{b})[x] \end{aligned}$$

proving that $\mathfrak{a}+\mathfrak{b}$ is sectorial.

Now suppose that $\mathfrak{a}$ and $\mathfrak{b}$ are closed. We have

$$\|x\|_{\mathfrak{a}+\mathfrak{b}}^2 = \|x\|^2 + \operatorname{Re}\mathfrak{a}[x] + \operatorname{Re}\mathfrak{b}[x]$$

which shows that both $\|\cdot\|_{\mathfrak{a}}$ and $\|\cdot\|_{\mathfrak{b}}$ are dominated by $\|\cdot\|_{\mathfrak{a}+\mathfrak{b}}$. Thus if x_n is a $\|\cdot\|_{\mathfrak{a}+\mathfrak{b}}$ - Cauchy sequence it is as well a $\|\cdot\|_{\mathfrak{a}}$ and a $\|\cdot\|_{\mathfrak{b}}$ - Cauchy sequence and hence by the closedness of $\mathfrak{a}$ and $\mathfrak{b}$ convergent. So suppose that $\|x_n - x\|_{\mathfrak{a}} \to 0$ and $\|x_n - y\|_{\mathfrak{b}} \to 0$. Since our Hilbertspace norm $\|\cdot\|$ is dominated by both the $\mathfrak{a}$ and the $\mathfrak{b}$ - norm, $x = y$ follows and we conclude that $x = y \in D(\mathfrak{a}) \cap D(\mathfrak{b})$ and that $\|x_n - x\|_{\mathfrak{a}+\mathfrak{b}} \to 0$ proving that $\mathfrak{a}+\mathfrak{b}$ is closed. □

Now it is time to give some examples of closed forms.

PROPOSITION 1.4. *Let H be a Hilbertspace and V a closed subspace of H.*

a) *If B is a bounded operator on V and $\mathfrak{b}[x,y] = (Bx\,|\,y)$ with $D(\mathfrak{b}) = V$, then $\mathfrak{b}$ is a closed sectorial form on H. More pecicely:*
We may chose any $\gamma < -\|B\|$ as a vertex for $\mathfrak{b}$ then a corresponding semiangle is $\arcsin(|\gamma|^{-1}\cdot\|B\|)$.
b) *If $C : V \to H$ is a closed operator and $\mathfrak{c}[x,y] = (Cx\,|\,Cy)$ with $D(\mathfrak{c}) = D(C)$, then $\mathfrak{c}$ is a closed, positive form on H.*

PROOF. a) We have that $|\mathfrak{b}[x]| \leq \|B\|\cdot\|x\|^2$ and thus $\Theta(\mathfrak{b})$ is contained in the disk around the origin with radius $\|B\|$. That the numerical range lies in the sector as claimed follows from elementary geometric considerations.

Since $|\mathrm{Re}\,\mathfrak{b}\,[x]| = |\mathrm{Re}\,(Bx\,|\,x)| \le \|B\|\|x\|^2$ we easily obtain:

$$\|x\|^2 \le \|x\|^2 + (\mathrm{Re}\,\mathfrak{b} + \|B\|)\,[x] \le \|x\|_{\mathfrak{b}}^2 \quad \text{and} \quad \|x\|_{\mathfrak{b}}^2 \le (1+2\|B\|)\|x\|^2$$

Thus the norms $\|\cdot\|$ and $\|\cdot\|_{\mathfrak{b}}$ are equivalent. Now V is $\|\cdot\|$-closed and hence $\|\cdot\|$-complete and the equivalence of the norms shows, that it is $\|\cdot\|_{\mathfrak{b}}$-complete.

b) Clearly $\mathfrak{c}\,[x] = \|Cx\|^2 \ge 0$. Thus $\mathfrak{c}$ is a positive, symmetric form. The estimate

$$\begin{aligned} \|x\|_{\mathfrak{c}} = \left(\|x\|^2 + \|Cx\|^2\right)^{\frac{1}{2}} &\le \|x\| + \|Cx\| \\ &\le 2\max\{\,\|x\|, \|Cx\|\,\} \\ &\le 2\ \left(\|x\|^2 + \|Cx\|^2\right)^{\frac{1}{2}} = 2\|x\|_{\mathfrak{c}} \end{aligned}$$

shows that $\|\cdot\|_{\mathfrak{c}}$ is equivalent to the graph norm $\|\cdot\|_C$ which is defined as $\|x\| + \|Cx\|$. Since $D(C)$ is $\|\cdot\|_C$-complete (because C is a closed operator) it follows that $H_{\mathfrak{c}}$ is complete.

□

The form in part a) of this proposition is of rather special form:

DEFINITION. A form $\mathfrak{a}$ is called *bounded* (or *continuous*) if there exits a constant $C > 0$ such that

$$|\mathfrak{a}\,[x,y]| \le C\|x\|\|y\|$$

for all $x, y \in D(\mathfrak{a})$.

REMARKS. a) Every bounded form is automatically sectorial which is seen as above.

b) Every sectorial form $\mathfrak{a}$ is a bounded form on $H_{\mathfrak{a}}$. To see this recall from 1.2 that

$$\begin{aligned} |(\mathfrak{a}-\gamma)\,[x,y]| &\le (1+\tan\theta)(\mathrm{Re}\,\mathfrak{a}-\gamma)\,[x]^{\frac{1}{2}}(\mathrm{Re}\,\mathfrak{a}-\gamma)\,[y]^{\frac{1}{2}} \\ &\le (1+\tan\theta)\|x\|_{\mathfrak{a}}\|y\|_{\mathfrak{a}} \end{aligned}$$

Thus $\mathfrak{a}-\gamma$ is a bounded form and since $\|\cdot\| \le \|\cdot\|_{\mathfrak{a}}$ γ is a bounded form so $\mathfrak{a} = (\mathfrak{a}-\gamma)+\gamma$ is bounded.

PROPOSITION 1.5. *If $\mathfrak{a}$ is a bounded form on H then $\mathfrak{a}$ has a unique bounded extension which is defined on $\overline{D(\mathfrak{a})}$.*

PROOF. Let $x \in \overline{D(\mathfrak{a})}$. Then there exists a sequence $(x_n) \subset D(\mathfrak{a})$ converging in norm to x. Since x_n is convergent it is bounded, say $\|x_n\| \le M$ for all n. Now we have

$$\begin{aligned} |\mathfrak{a}\,[x_n] - \mathfrak{a}\,[x_m]| &= |\mathfrak{a}\,[x_n, x_n - x_m] + \mathfrak{a}\,[x_n - x_m, x_m]| \\ &\le 2CM\|x_n - x_m\| \to 0 \quad \text{as} \quad m,n \to \infty \end{aligned}$$

So we may define $\bar{\mathfrak{a}}[x] = \lim \mathfrak{a}[x_n]$. (To see that this is well defined assume having two sequences $x_n, y_n \to x$ and repeat the above computation with x_m replaced by y_n.) Now define the form $\bar{\mathfrak{a}}$ by polarisation. Obviously $\bar{\mathfrak{a}}$ is a bounded form that extends $\mathfrak{a}$. $\square$

Next, we give some examples of forms that are not closed.

Example:

a) Let S be a positive, selfadjoint unbounded operator and define the form $\mathfrak{s}$ by

$$\mathfrak{s}[x,y] := (Sx \,|\, y) \quad , \quad D(\mathfrak{s}) = D(S)\,.$$

Then $\mathfrak{s}$ is not closed:
We have that

$$\|x\|_{\mathfrak{s}}^2 = \|x\|^2 + (Sx\,|\,x) = \|x\|^2 + \left(S^{\frac{1}{2}}x \,\middle|\, S^{\frac{1}{2}}x\right) = \|x\|^2 + \left\|S^{\frac{1}{2}}x\right\|^2,$$

which was seen to be equivalent to the graph norm of $S^{\frac{1}{2}}$ in the preceding proposition. But the domain of S is an operator core for $S^{\frac{1}{2}}$ and hence dense with respect to this graph norm. Thus if $H_{\mathfrak{s}}$ was complete (and thus closed) it would follow that $D(S) = D(S^{\frac{1}{2}})$. This would imply that S is bounded:

Because if T is a positive selfadjoint operator and $D(T) = D(T^2)$ then T maps $D(T)$ into $D(T)$. But $T : D(T) \to D(T)$ is a closed operator and hence bounded by the closed graph Theorem. Since T is selfadjoint, it has nonempty resolvent set $\rho(T)$. So let $\lambda \in \rho(T)$. Then $R(\lambda, T)$ is a bounded mapping from $D(T)$ to $D(T^2) = D(T)$ and we have the following situation:

$$\begin{array}{ccc}
(D(T), \|\cdot\|) & \xrightarrow{T} & (H, \|\cdot\|) \\
{\scriptstyle R(\lambda,T)}\downarrow & & \uparrow{\scriptstyle \lambda - T} \\
(D(T), \|\cdot\|_T) & \xrightarrow[T]{} & (D(T), \|\cdot\|_T)
\end{array}$$

So since the mappings on the sides and the bottom of this diagram are all bounded and since the diagram commutes, the mapping T on top has to be bounded as well.

b) Let $H = L^2(\mathbb{R})$ and define the form $\mathfrak{a}$ by

$$\mathfrak{a}[f,g] := f(0)\overline{g(0)} \quad , \quad D(\mathfrak{a}) = C_c^\infty(\mathbb{R})$$

Then $\mathfrak{a}$ is a positive form but $\mathfrak{a}$ is not closed. If fact we can choose f_n to be a C^∞- function which is supported in $[-\frac{2}{n}, \frac{2}{n}]$ and satisfies

$$f_n\big|_{[-\frac{1}{n},\frac{1}{n}]} \equiv 1 \quad \text{and} \quad 0 \le f_n \le 1$$

Then $f_n \to 0$ in L^2 and $\mathfrak{a}[f_n - f_m] \equiv 0$ and hence f_n is a $\|\cdot\|_{\mathfrak{a}}$ - Cauchy sequence. But f_n is not convergent with respect to $\|\cdot\|_{\mathfrak{a}}$. Since $\|\cdot\| \le \|\cdot\|_{\mathfrak{a}}$

f_n could only converge to 0. However $\|f_n\|_{\mathfrak{a}}^2 \geq \mathfrak{a}[f_n] \equiv 1 \not\to 0$. Hence $\mathfrak{a}$ is not closed. □

It turns out that these two examples are rather different. In the first example it seems that we have only chosen the wrong domain. In fact 1.4 tells us that $\mathfrak{s}_0[x,y] = \left(S^{\frac{1}{2}}x \middle| S^{\frac{1}{2}}x\right)$ is a closed form on $D(\mathfrak{s}_0) = D(S^{\frac{1}{2}})$. And we know that $\mathfrak{s}[x,y] = \mathfrak{s}_0[x,y]$ for all $x, y \in D(\mathfrak{s}) \subset D(\mathfrak{s}_0)$. This meens that $\mathfrak{s}$ has a closed extension.
On the other hand we will see in Theorem 1.9 that the form in the second example has *no* closed extension.

Definition. A sectorial form $\mathfrak{a}$ is called *closable* if it has a closed extension.

We note the following consequence of proposition 1.3:

Corollary 1.6. *If $\mathfrak{a}$ and $\mathfrak{b}$ are closable sectorial forms, then so is $\mathfrak{a}+\mathfrak{b}$.*

Proof. Let $\bar{\mathfrak{a}}$ and $\bar{\mathfrak{b}}$ be closed extensions of $\mathfrak{a}$ resp. $\mathfrak{b}$. Then $\bar{\mathfrak{a}}+\bar{\mathfrak{b}}$ is closed by 1.3 and extends $\mathfrak{a}+\mathfrak{b}$. □

We also have the following result concerning everywhere defined forms:

Proposition 1.7. *A closable form $\mathfrak{a}$ on H with domain H is bounded.*

Proof. Since $\mathfrak{a}$ is everywhere defined it must coincide with its closed extension and hence be closed. Thus H is complete with respect to both $\|\cdot\|$ and $\|\cdot\|_{\mathfrak{a}}$. Since we already know that $\|\cdot\| \leq \|\cdot\|_{\mathfrak{a}}$ the closed graph Theorem yields that these two norms are equivalent. But $\mathfrak{a}$ is bounded with respect to the second norm. Thus it also has to be bounded with respect to the equivalent $\|\cdot\|$-norm. □

Before we go on investigation whether a given form is closable, we state a perturbation result for sectorial forms which is very useful.

Definition. Let $\mathfrak{a}$ be a sectorial form. A form $\mathfrak{b}$ which need not be sectorial is $\mathfrak{a}$-*bounded* if $D(\mathfrak{a}) \subset D(\mathfrak{b})$ and

$$|\mathfrak{b}[x]| \leq \alpha\|x\|^2 + \beta|\mathfrak{a}[x]| \quad \forall x \in D(\mathfrak{a})$$

where α, β are nonnegative constants. The infimum of all possible β in this equality is called the $\mathfrak{a}$-*bound* of $\mathfrak{b}$.

Theorem 1.8. *Let $\mathfrak{a}$ be a sectorial form and $\mathfrak{b}$ be $\mathfrak{a}$-bounded with $\mathfrak{a}$-bound less than 1. Then $\mathfrak{a}+\mathfrak{b}$ is sectorial. Furthermore the inner product $(\cdot\,|\,\cdot)_{\mathfrak{a}+\mathfrak{b}}$ is equivalent to $(\cdot\,|\,\cdot)_{\mathfrak{a}}$. In particular, $\mathfrak{a}+\mathfrak{b}$ is closed if and only if $\mathfrak{a}$ is. In this case, a subset D of $D(\mathfrak{a})$ is a core for $\mathfrak{a}$ if and only if it is a core for $\mathfrak{a}+\mathfrak{b}$. The sum $\mathfrak{a}+\mathfrak{b}$ is closable if and only if $\mathfrak{a}$ is. In this case $D(\overline{\mathfrak{a}+\mathfrak{b}}) = D(\bar{\mathfrak{a}})$.*

Proof. If we replace $\mathfrak{a}$ by $\mathfrak{a}-\gamma$ then the β in the above inequality is not changed. Thus we may assume that 0 is a vertex for $\mathfrak{a}$. Let θ be the corresponding semiangle. Then

$$\mathrm{Re}\,(\mathfrak{a}+\mathfrak{b})\,[x] \geq \mathrm{Re}\,\mathfrak{a}\,[x] - |\mathrm{Re}\,\mathfrak{b}\,[x]| \geq (1-\beta)\mathrm{Re}\,\mathfrak{a}\,[x] - \alpha\|x\|^2 \geq -\alpha\|x\|^2$$

which proves that $-\alpha$ is a vertex for $\mathfrak{a}+\mathfrak{b}$. Now we obtain

$$\begin{aligned} |\mathrm{Im}\,(\mathfrak{a}+\mathfrak{b})\,[x]| &\leq |\mathrm{Im}\,\mathfrak{a}\,[x]| + |\mathrm{Im}\,\mathfrak{b}\,[x]| \\ &\leq \tan\theta \mathrm{Re}\,\mathfrak{a}\,[x] + \alpha\|x\|^2 + \beta\mathrm{Re}\,\mathfrak{a}\,[x] \\ &\leq \frac{1}{1-\beta}(\tan\theta+\beta)\,(\mathrm{Re}\,(\mathfrak{a}+\mathfrak{b})\,[x]+\alpha)+\alpha \end{aligned}$$

for $\|x\|=1$, which proves that $\mathfrak{a}+\mathfrak{b}$ is sectorial.

Now observe that

$$\|x\|_{\mathfrak{a}+\mathfrak{b}} \geq (1-\beta)\|x\|_{\mathfrak{a}}$$

and on the other hand

$$\begin{aligned} \|x\|^2_{\mathfrak{a}+\mathfrak{b}} &= (1+\alpha)\|x\|^2 + \mathrm{Re}\,\mathfrak{a}\,[x] + \mathrm{Re}\,\mathfrak{b}\,[x] \\ &\leq (1+\alpha)\|x\|^2 + \mathrm{Re}\,\mathfrak{a}\,[x] + |\mathfrak{b}\,[x]| \\ &\leq (1+\alpha)\|x\|^2 + \mathrm{Re}\,\mathfrak{a}\,[x] + \alpha\|x\|^2 + \beta|\mathfrak{a}\,[x]| \\ &\leq (1+2\alpha)\|x\|^2 + (1+\beta(1+\tan\theta))\mathrm{Re}\,\mathfrak{a}\,[x] \\ &\leq \text{const.}\cdot\|x\|^2_{\mathfrak{a}} \end{aligned}$$

so that the norms $\|\cdot\|_{\mathfrak{a}+\mathfrak{b}}$ and $\|\cdot\|_{\mathfrak{a}}$ are equivalent. This proves the rest of the assertions, considering that $D(\mathfrak{a}+\mathfrak{b}) = D(\mathfrak{a})$. □

Example:
Let $H = L^2(\mathbb{R})$ and consider the forms

$$\begin{aligned} \mathfrak{a}\,[f,g] &:= \textstyle\int_{\mathbb{R}} f'\overline{g'}\,dx \quad D(\mathfrak{a}) = H^1(\mathbb{R}) \\ \mathfrak{b}\,[f,g] &:= f(0)\overline{g(0)} \quad D(\mathfrak{b}) = H^1(\mathbb{R}) \end{aligned}$$

We have that $\mathfrak{a}\,[f,g] = \left(\frac{d}{dx}f \,\middle|\, \frac{d}{dx}g\right)$, so it follows from proposition 1.4, that $\mathfrak{a}$ is a closed, symmetric form, considering, that $\frac{d}{dx}$ is a closed operator on L^2. On the other hand, the form $\mathfrak{b}$ was already seen to be not closed. It will follow from Theorem 1.9, that $\mathfrak{b}$ is not even closable. However, $\mathfrak{b}$ is $\mathfrak{a}$-bounded with bound 0:

$$\begin{aligned}
|\mathfrak{b}\,[f]| &\leq |f(0)^2 - f(T)^2| + |f(T)|^2 \\
&= \left| 2\int_0^T f'(t)f(t)\,dt \right| + |f(T)|^2 \\
&\leq |f(T)|^2 + 2\left| \int_{\mathbb{R}} f'(t)f(t)\,dt \right| \\
&\leq |f(T)|^2 + 2\varepsilon \int_{\mathbb{R}} |f'(t)|^2\,dt + \frac{2}{\varepsilon}\int_{\mathbb{R}} |f(t)|^2\,dt \\
&\to 2\varepsilon \cdot \int_{\mathbb{R}} |f'(t)|^2\,dt + \frac{2}{\varepsilon}\|f\|^2 \quad \text{as} \quad T \to \infty\,.
\end{aligned}$$

Since here $\varepsilon > 0$ is arbitrary, we see that $\mathfrak{b}$ is indeed formbounded with respect to $\mathfrak{a}$ with bound 0. Thus by Theorem 1.8, $\mathfrak{a} + \mathfrak{b}$ is closed.

A tempting thing to do would be the following:
The pre-Hilbert space $(D(\mathfrak{a}), (\,\cdot\,|\,\cdot\,)_{\mathfrak{a}})$ has a completion $(\overline{H_{\mathfrak{a}}}, (\,\cdot\,|\,\cdot\,)_0)$. On this space $\mathfrak{a}$ is a bounded form and thus has an extension $\bar{\mathfrak{a}}$ defined on the whole of $\overline{H_{\mathfrak{a}}}$ by proposition 1.5. But we have that $\|\cdot\|_{\bar{\mathfrak{a}}} = \|\cdot\|_0$[1] which proves that $\bar{\mathfrak{a}}$ is closed.

The problem is of course that we obtained a closed extension of $\mathfrak{a}$ in $\overline{H_{\mathfrak{a}}}$ rather than in H. So the question is whether we can realize all this *within* H. The answer to this question lies in the following observation:
We know that the natural embedding $D(\mathfrak{a}) \hookrightarrow H$ is continuous, since $\|x\| \leq \|x\|_{\mathfrak{a}}$. Thus it extends to a mapping $\iota_{\mathfrak{a}} : \overline{H_{\mathfrak{a}}} \to H$.

$$\begin{tikzcd}
\overline{H_{\mathfrak{a}}} \arrow[dr, "\iota_{\mathfrak{a}}"] & \\
D(\mathfrak{a}) \arrow[u, hook] \arrow[r, hook] & H
\end{tikzcd}$$

So if $\ker \iota_{\mathfrak{a}} = 0$ we can identify $\overline{H_{\mathfrak{a}}}$ with $\iota_{\mathfrak{a}}(\overline{H_{\mathfrak{a}}})$ and see that $\tilde{\mathfrak{a}}\,[x,y] := \bar{\mathfrak{a}}\,[\iota_{\mathfrak{a}}{}^{-1}(x), \iota_{\mathfrak{a}}{}^{-1}(y)]$ with $D(\tilde{\mathfrak{a}}) = \iota(\overline{H_{\mathfrak{a}}})$ is a closed extension of $\mathfrak{a}$.

So we have found a criterion for closability of a form. Unfortunately it is not a very practical one. A better one uses the following

[1] This is seen as follows:
For $x \in \overline{H_{\mathfrak{a}}}$ there exits a sequence x_n in $D(\mathfrak{a})$ converging to x. Hence:

$$\|x\|_0^2 = \lim \|x_n\|_0^2 = \lim(\|x_n\|^2 + (\operatorname{Re}\mathfrak{a} - \gamma)\,[x_n]) = (1-\gamma)\|x\|^2 + \operatorname{Re}\bar{\mathfrak{a}}\,[x] = \|x\|_{\bar{\mathfrak{a}}}^2$$

DEFINITION. Let $\mathfrak{a}$ be a sectorial form. A sequence (x_n) in H is said to be $\mathfrak{a}$ - *convergent* to x , we write $x_n \xrightarrow{\mathfrak{a}} x$, if $(x_n) \subset D(\mathfrak{a})$, $x_n \to x$ in H and (x_n) is a $\|\cdot\|_{\mathfrak{a}}$-Cauchy sequence.

REMARKS. a) Note that x may *not* belong to $D(\mathfrak{a})$.

b) Obviously $\mathfrak{a}$ - convergence is equivalent to both $\operatorname{Re}\mathfrak{a}$ - convergence and $(\mathfrak{a}+\alpha)$ - convergence for any scalar α.

c) It follows from 1.2 that if $x_n \xrightarrow{\mathfrak{a}} x$ then $\mathfrak{a}[x_n - x_m] \to 0$ as $n, m \to \infty$.

THEOREM 1.9. *Let $\mathfrak{a}$ be a sectorial form and $\overline{H_{\mathfrak{a}}}$ and $\iota_{\mathfrak{a}}$ be defined as above. Then the following are equivalent:*

a) *$\mathfrak{a}$ is closable.*

b) *For every sequence $x_n \xrightarrow{\mathfrak{a}} 0$ it follows that $\|x_n\|_{\mathfrak{a}} \to 0$.*

c) *$\iota_{\mathfrak{a}}$ is injective.*

PROOF. a) $\Rightarrow$ b) Let $\mathfrak{b}$ be a closed extension of $\mathfrak{a}$ and suppose that $x_n \xrightarrow{\mathfrak{a}} 0$. Since $(x_n) \subset D(\mathfrak{a})$ we have that $\|x_n\|_{\mathfrak{a}} = \|x_n\|_{\mathfrak{b}}$ for any n so that (x_n) is a $\|\cdot\|_{\mathfrak{b}}$ - Cauchy sequence. Since $H_{\mathfrak{b}}$ is complete x_n converges to some x_0 in $H_{\mathfrak{b}}$. But $x_n \to 0$ in H implies $x_0 = 0$, so $\|x_n\|_{\mathfrak{a}} = \|x_n\|_{\mathfrak{b}} \to 0$.

b) $\Rightarrow$ c) Let $x \in \ker \iota_{\mathfrak{a}}$. Since $D(\mathfrak{a})$ is dense in $\overline{H_{\mathfrak{a}}}$ there exits a sequence (x_n) in $D(\mathfrak{a})$ which converges to x in the $\|\cdot\|_0$ -norm. Since $\iota_{\mathfrak{a}}$ is continuous it follows that $x_n = \iota_{\mathfrak{a}}(x_n) \to \iota_{\mathfrak{a}}(x) = 0$ so that $x_n \xrightarrow{\mathfrak{a}} 0$. From our assumption it follows that $\|x_n\|_{\mathfrak{a}} = \|x_n\|_0 \to 0$ and we can conclude that $x = 0$.

c) $\Rightarrow$ a) This was proved above. $\square$

Closed extensions are usually not unique. This shows the

Example

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and $H = L^2(\Omega, dx)$. Define $\mathfrak{a}$ by

$$\mathfrak{a}[f,g] = \int_\Omega \nabla f \cdot \nabla g \, dx \quad , \quad D(\mathfrak{a}) = C_c^\infty(\Omega) \, .$$

Then $\mathfrak{a}$ is a positive, symmetric form and the corresponding norm is given by

$$\|f\|_{\mathfrak{a}}^2 = \|f\|^2 + \mathfrak{a}[f] = \|f\|^2 + \|\nabla f\|^2 = \|f\|_{H^1}^2 \, .$$

Thus, the same expression is a closed form with domain $H^1(\Omega)$ but also with domain $H_0^1(\Omega)$. $\square$

But we see easily that the closed extension given in Theorem 1.9 has the smallest form domain among all closed extensions. (That is because of the density of $D(\mathfrak{a})$ in $\overline{H_{\mathfrak{a}}}$.)

Definition. If $\mathfrak{a}$ is a closable form the the closed extension with the smallest domain is called the *closure* of $\mathfrak{a}$ and denoted by $\bar{\mathfrak{a}}$.

It is useful to characterize those domains which determine a closed form uniquely. We define:

Definition. Let $\mathfrak{a}$ be a sectorial form. A subset V of $D(\mathfrak{a})$ is called a *core* for $\mathfrak{a}$ if V is $\|\cdot\|_{\mathfrak{a}}$-dense in $D(\mathfrak{a})$.

Proposition 1.10. *Let $\mathfrak{a}$ be a closed, sectorial form and $V \subset D(\mathfrak{a})$. Then*

a) *V is a core for $\mathfrak{a}$ if and only if the closure of the form $\mathfrak{a}_0 = \mathfrak{a}|_V$ is $\mathfrak{a}$.*
b) *If $\mathfrak{b}$ is a closed sectorial form that coincides with $\mathfrak{a}$ on some core for $\mathfrak{a}$ then $\mathfrak{a}$ is an extension of $\mathfrak{b}$. In particular, if two closed sectorial forms coincide on a common core, they are equal.*

Proof. This is an immediate consequence of proposition 1.5. □

Finally we want to introduce the notion of the *regular part* of a symmetric form which is due to B. Simon and was first presented in [**17**]. The idea is to split up any symmetric semibounded form $\mathfrak{a}$ into a closable part $\mathfrak{a}_r$ and some rest $\mathfrak{a}_s = \mathfrak{a} - \mathfrak{a}_r$. For simplicity we may assume the form to be positive.
Recall the definition of $\overline{H_{\mathfrak{a}}}$ and $(\cdot\,|\,\cdot)_0$ from above. Since the form is symmetric (and thus $\operatorname{Re}\mathfrak{a} = \mathfrak{a}$) we have not only $(\cdot\,|\,\cdot)_{\bar{\mathfrak{a}}} = (\cdot\,|\,\cdot)_0$ but also $\bar{\mathfrak{a}}\,[x,y] = (x\,|\,y)_0 - (x\,|\,y)$
We may decompose $\overline{H_{\mathfrak{a}}}$ as $(\ker \iota_{\mathfrak{a}}) \oplus (\ker \iota_{\mathfrak{a}})^{\perp}$ where we of course use the inner product $(\cdot\,|\,\cdot)_{\bar{\mathfrak{a}}}$.
Let P be the orthogonal projection on $(\ker \iota_{\mathfrak{a}})^{\perp}$ and $Q = id - P$ and define the forms $\mathfrak{a}_r$ and $\mathfrak{a}_s$ on $D(\mathfrak{a})$ as follows:

$$\begin{aligned}\mathfrak{a}_r\,[x,y] &= (Px\,|\,y)_0 - (x\,|\,y)\\ \mathfrak{a}_s\,[x,y] &= (Qx\,|\,y)_0\end{aligned}$$

We will prove now that $\mathfrak{a}_r$ has the properties we want and furthermore characterize $\mathfrak{a}_r$ independently of our construction above. For this we will need the

Definition. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two symmetric semibounded forms. We say that $\mathfrak{a}$ *is smaller* then $\mathfrak{b}$ and write $\mathfrak{a} \leq \mathfrak{b}$ if $D(\mathfrak{b}) \subset D(\mathfrak{a})$ and $\mathfrak{a}\,[x] \leq \mathfrak{b}\,[x]$ for all $x \in D(\mathfrak{b})$ [2]

Theorem 1.11. *Let $\mathfrak{a}$ be a positive symmetric, semibounded form and $\mathfrak{a}_r$, $\mathfrak{a}_s$ as above. Then:*

a) $\mathfrak{a}_r + \mathfrak{a}_s = \mathfrak{a}$.
b) *$\mathfrak{a}_r$ is a positive closable form.*
c) *If $\mathfrak{b}$ is a positive closable form smaller than $\mathfrak{a}$ then $\mathfrak{b} \leq \mathfrak{a}_r$.*

[2]Note that this is consistent with the notion of $\mathfrak{a} \geq \gamma$ introduced before.

Proof. We have that

$$(\mathfrak{a}_r + \mathfrak{a}_s)\,[x, y] = ((P+Q)x \,|\, y)_0 - (x \,|\, y) = \mathfrak{a}\,[x, y]$$

for all $x, y \in D(\mathfrak{a})$ which proves a).

To see that $\mathfrak{a}_r$ is positive let $x \in D(\mathfrak{a})$ and observe that
$x = \iota_\mathfrak{a}(x) = \iota_\mathfrak{a}(Px) + \iota_\mathfrak{a}(Qx) = \iota_\mathfrak{a}(Px)$ since $Qx \in \ker \iota_\mathfrak{a}$. Now we obtain:

$$\|x\|^2 = \|\iota_\mathfrak{a}(Px)\|^2 \leq \|Px\|^2_{\bar{\mathfrak{a}}} = (Px \,|\, x)_0 = \mathfrak{a}_r\,[x] + (x \,|\, x)$$

which proves that $\mathfrak{a}_r$ is positive.

We have that $\mathrm{Rg}P$ is a closed subspace of $\overline{H_\mathfrak{a}}$ and $(\,\cdot\,|\,\cdot\,)_{\mathfrak{a}_r} = (\,\cdot\,|\,\cdot\,)_0$ on $\mathrm{Rg}P$. Since $PD(\mathfrak{a})$ is $\|\cdot\|_0$-dense in $\mathrm{Rg}P$ we obtain that $\overline{H_{\mathfrak{a}_r}}$ is isometrically isomorphic to $(\mathrm{Rg}P, (\,\cdot\,|\,\cdot\,)_{\mathfrak{a}_r})$. We also can identify $\iota_{\mathfrak{a}_r}$ with $\iota_\mathfrak{a}|_{\mathrm{Rg}P}$ which is injective. Now 1.9 implies that $\mathfrak{a}_r$ is closable.

To prove the last part observe that $D(\mathfrak{a}_r) = D(\mathfrak{a}) \subset D(\mathfrak{b})$ so that we only need to show that for $x \in D(\mathfrak{a})$, $\mathfrak{b}\,[x] \leq \mathfrak{a}_r\,[x]$ which is equivalent to show that $\|x\|_{\bar{\mathfrak{b}}} \leq \|Px\|_{\bar{\mathfrak{a}}}$. By hypothesis we have that $\|x\|_\mathfrak{b} \leq \|x\|_\mathfrak{a}$. Thus we can extend the inclusion $D(\mathfrak{a}) \hookrightarrow D(\mathfrak{b}) \subset \overline{H_\mathfrak{b}}$ to a contraction $j : \overline{H_\mathfrak{a}} \to \overline{H_\mathfrak{b}}$.
We have the following situation:

$$\begin{tikzcd}
\overline{H_\mathfrak{a}} \arrow[rr, "j"] \arrow[dr, "\iota_\mathfrak{a}"'] & & \overline{H_\mathfrak{b}} \arrow[dl, "\iota_\mathfrak{b}"] \\
& H & \\
D(\mathfrak{a}) \arrow[uu, hook] \arrow[rr, hook] & & D(\mathfrak{b}) \arrow[uu, hook]
\end{tikzcd}$$

We claim that $\ker \iota_\mathfrak{a} = \ker j$.
On $D(\mathfrak{a})$ we have that $\iota_\mathfrak{b} \circ j = \iota_\mathfrak{a}$ but by density this is true everywhere. Since $\mathfrak{b}$ closable, 1.9 implies that $\iota_\mathfrak{b}$ is an injection from which $\ker \iota_\mathfrak{a} = \ker j$ now follows. Thus we have that $j(x) = j(Px)$ for all $x \in D(\mathfrak{a})$ and can compute:

$$\|x\|_\mathfrak{b} = \|j(x)\|_\mathfrak{b} = \|j(Px)\|_{\bar{\mathfrak{b}}} \leq \|Px\|_{\bar{\mathfrak{a}}}$$

□

Definition. Let $\mathfrak{a}$ be a symmetric semibounded form. The largest closable form smaller than $\mathfrak{a}$ (which exists by the preceeding theorem) is called *the regular part* of $\mathfrak{a}$ and is denoted by $\mathfrak{a}_r$.

Corollary 1.12. *Let $\mathfrak{a}$ be a semibounded symmetric form, $\mathfrak{b}$ a bounded form on $\overline{D(\mathfrak{a})}$ as defined in 1.4 and $\mathfrak{c}$ any semibounded symmetric form smaller than $\mathfrak{a}$.*

a) $(\mathfrak{a} + \mathfrak{b})_r = \mathfrak{a}_r + \mathfrak{b}$

b) $\mathfrak{c}_r \leq \mathfrak{a}_r$

PROOF. a) follows from the fact that addition of a bounded form changes neither closedness nor closability. b) is proved by $\mathfrak{c}_r \leq \mathfrak{c} \leq \mathfrak{a}$. □

3. The Relationship Between Forms and Operators

We now come back to the question whether a given form $\mathfrak{a}$ is of the type $\mathfrak{a}[x,y] = (Ax \,|\, y)$ for some operator A. When we define $\mathfrak{a}$ like this on $D(\mathfrak{a}) = D(A)$ we call $\mathfrak{a}$ the form *associated* with A. Of course the operator A should lead to a form which has the right numerical range.

DEFINITION. If A is an operator on a Hilbert space H then the *numerical range* of A is given by

$$\Theta(A) := \{(Ax \,|\, x) \,:\, \|x\| = 1 \,,\, x \in D(A)\}$$

A is *sectorial* if the numerical range of A is contained in some sector $\Sigma_\gamma(\theta)$. We will use the same terminology for operators as for forms and talk of vertices etc.
A is called *m-sectorial* if A is sectorial and for some λ outside of the sector that contains the numerical range of A, $\lambda - A$ is surjective.

We have already seen for positive selfadjoint operators that the operator domain is too small for the associated form to be closed. However the associated form turned out to be closable. Here we have a similar situation.

PROPOSITION 1.13. *If A is a sectorial operator and $\mathfrak{a}[x,y] = (Ax \,|\, y)$, $D(\mathfrak{a}) = D(A)$ then $\mathfrak{a}$ is closable. Furthermore, if the numerical range of $\mathfrak{a}$ is contained in $\Sigma_\gamma(\theta)$, then so is the numerical range of the closure.*

PROOF. We may assume that A has a vertex 0. According to Theorem 1.9 we have to show that for any $x_n \xrightarrow{\mathfrak{a}} 0$ we have $\|x_n\|_{\mathfrak{a}} \to 0$. So let a sequence x_n with $x_n \xrightarrow{\mathfrak{a}} 0$ be given. This implies that $x_n \to 0$ in H and that $\operatorname{Re}\mathfrak{a}[x_n]$ is bounded, say by M^2. We have that

$$\begin{aligned} |\operatorname{Re}\mathfrak{a}[x_n]| &\leq |\operatorname{Re}\mathfrak{a}[x_n, x_n - x_m]| + |\operatorname{Re}\mathfrak{a}[x_n, x_m]| \\ &\leq \operatorname{Re}\mathfrak{a}[x_n]^{\frac{1}{2}} \operatorname{Re}\mathfrak{a}[x_n - x_m]^{\frac{1}{2}} + |\operatorname{Re}(Ax_n \,|\, x_m)| \end{aligned}$$

where we used 1.2.
Given $\varepsilon > 0$ we have for large m, n that $\operatorname{Re}\mathfrak{a}[x_n - x_m] < \varepsilon^2$ Thus for such m, n we have

$$\operatorname{Re}\mathfrak{a}[x_n] \leq M\varepsilon + |\operatorname{Re}(Ax_n \,|\, x_m)| \quad \longrightarrow \quad M\varepsilon \quad \text{as} \quad m \to \infty$$

So for large n we have $|\operatorname{Re}\mathfrak{a}[x_n]| \leq M\varepsilon$ which proves that $\|x_n\|_{\mathfrak{a}} \to 0$. □

M-sectorial operators have nice spectral properties. This is the reason, why they are so interesting in the study of evolution problems. We have

Theorem 1.14. *Let A be an m- sectorial operator on a Hilbert space H. And assume that $\Theta(A) \subset \Sigma_\gamma(\theta)$ and let $\Omega = \Sigma_\gamma(\theta)^C$. Then $\Omega \subset \rho(A)$ and*

$$\|R(\lambda, A)\| \leq \frac{1}{\operatorname{dist}(\lambda, \Theta(A))} \quad \forall \lambda \in \Omega\,.$$

Proof. Let $\lambda \in \Omega$ so that $d = \operatorname{dist}(\lambda, \Theta(A)) > 0$. For $x \in D(A)$ with $\|x\| = 1$ we have

$$d \leq |\lambda - (Ax\,|\,x)| = |((\lambda - A)x\,|\,x)| \leq \|(\lambda - A)x\| \cdot \|x\| = \|(\lambda - A)x\|$$

which proves that $\lambda - A$ is injective. Thus if $\lambda \in \Omega \cap \rho(A) =: M$ then $\|R(\lambda, A)\| \leq \frac{1}{d}$. Our assumption that $\mu - A$ is surjective for some μ shows that M is not empty. Since both Ω and $\rho(A)$ are open M is open in Ω. We show that M is relatively closed in Ω. Then $M = \Omega$ follows since Ω is connected and we have that $\Omega \subset \rho(A)$ as claimed.
So let $(\lambda_n) \subset M$ with $\lambda_n \to \lambda \in \Omega$. From the above it follows that

$$\sup_n \|R(\lambda_n, A)\| \leq \sup_n \frac{1}{\operatorname{dist}(\lambda_n, \Theta(A))} \leq C \,<\, \infty$$

Thus there exists some $n_0 \in \mathbb{N}$ such that $|\lambda - \lambda_{n_0}| < \frac{1}{C} \leq \|R(\lambda_{n_0}, A)\|^{-1}$. Now the Neumann series shows that $\lambda \in \rho(A)$ and we are done. □

From this theorem, the following properties of m-sectorial operators follow:

Corollary 1.15. *M-sectorial operators are closed and densely defined.*

Proof. Let A be an m-sectorial operator and suppose that $x_n \to x$ and $Ax_n \to y$ and let $\lambda \in \rho(A)$. Using the identity $R(\lambda, A)A = \lambda R(\lambda, A) - id$ we see that

$$R(\lambda, A)y = \lim R(\lambda, A)Ax_n = \lim(\lambda R(\lambda, A)x_n - x_n) = \lambda R(\lambda, A)x - x$$

proving that $x \in D(A)$ and (by multiplying with $(\lambda - A)$) that $y = Ax$. Hence A is closed.

To see that A is densely defined suppose that $y \in D(A)^\perp$. We have to show, that $y = 0$. By the above theorem if γ is a vertex for A then $\lambda := \gamma - 1 \in \rho(A)$. So since $D(A) = \operatorname{Rg} R(\lambda, A)$ we have that, $(R(\lambda, A)x\,|\,y) = 0$ for all $x \in H$. Since $\lambda \in \rho(A)$ we find some $z \in D(A)$ such that $y = (\lambda - A)z$. Hence:

$$\begin{aligned} 0 = \operatorname{Re}(R(\lambda, A)y\,|\,y) &= \operatorname{Re}(z\,|\,(\lambda - A)z) \\ &= \operatorname{Re}\lambda\|z\|^2 - \operatorname{Re}(z\,|\,Az) \\ &\leq (\gamma - 1)\|z\|^2 - \gamma\|z\|^2 = -\|z\|^2 \end{aligned}$$

which shows that z and hence y must be equal to 0. □

The name m-sectorial stands for *maximal* sectorial meaning that an m-sectorial operator has no sectorial extensions. This is a consequence of the following

LEMMA 1.16. *If A and B are densely defined m-sectorial operators and $A \subset B$ then $A = B$.*

PROOF. Since A and B are both m-sectorial according to Theorem 1.14 there exists a $\lambda \in \rho(A) \cap \rho(B)$. But then $\varphi = R(\lambda, A)(\lambda - B)$ is a bijection from $D(B)$ to $D(A)$. Since $A \subset B$ we have $\varphi\big|_{D(A)} = id_{D(A)}$ we must have $D(A) = D(B)$ and hence $A = B$. □

Now we come to the main theorem of this section, which links closed sectorial forms and m-sectorial operators.

THEOREM 1.17. *(Representation Theorem)*
Let $\mathfrak{a}$ be a densely defined, closed sectorial form. Then there exists an m-sectorial operator $(A, D(A))$ such that $D(A)$ is a core for $\mathfrak{a}$ and

$$(*) \qquad \mathfrak{a}\,[x, y] = (Ax\,|\,y) \quad \forall x \in D(A)\,,\, y \in D(\mathfrak{a})$$

Furthermore if $x \in D(\mathfrak{a})$, $z \in H$ and $\mathfrak{a}\,[x, y] = (z\,|\,y)$ holds for all y belonging to some core for $\mathfrak{a}$ then $x \in D(A)$ and $Ax = z$. In particular A is uniquely determined by $(*)$.

PROOF. Without loss of generality $\gamma = 0$.
We observe that $y \mapsto \mathfrak{a}\,[x, y]$ is a bounded ($|\mathfrak{a}\,[x, y]| \leq (1+\tan\theta)\|x\|_{\mathfrak{a}}\|y\|_{\mathfrak{a}}$) antilinear functional on $H_{\mathfrak{a}}$. But so is $y \mapsto (z\,|\,y)$. So the question we have to answer is which of the first ones is actually of the second type?

Step 1: LAX-MILGRAM THEOREM
By the Riesz representation Theorem all bounded antilinear functionals on $H_{\mathfrak{a}}$ are of the form $y \mapsto (x\,|\,y)_{\mathfrak{a}}$. We claim that there exists a bounded bijective function $\phi : D(\mathfrak{a}) \to D(\mathfrak{a})$ such that

$$(\mathfrak{a} + 1)\,[x, y] = (\phi x\,|\,y)_{\mathfrak{a}}$$

for all $x, y \in D(\mathfrak{a})$. Furthermore ϕ has a bounded inverse.

The existence of such a function ϕ follows directly from the Riesz representation Theorem. ϕ is bounded since

$$\|\phi x\|_{\mathfrak{a}} = \sup_{\|y\|_{\mathfrak{a}} \leq 1} |(\phi x\,|\,y)_{\mathfrak{a}}| = \sup_{\|y\|_{\mathfrak{a}} \leq 1} |(\mathfrak{a} + 1)\,[x, y]| \leq M\|x\|_{\mathfrak{a}}$$

Since $\|x\|_{\mathfrak{a}}^2 = \operatorname{Re}(\mathfrak{a} + 1)\,[x, x] = \operatorname{Re}(\phi x\,|\,x)_{\mathfrak{a}} \leq \|\phi x\|_{\mathfrak{a}}\|x\|_{\mathfrak{a}}$ we have $\|x\|_{\mathfrak{a}} \leq \|\phi x\|_{\mathfrak{a}}$ and thus ϕ is injective and has closed range. But actually $\mathrm{Rg}\phi = H$ for if $x \in (\mathrm{Rg}\phi)^{\perp}$ then $0 = \operatorname{Re}(\phi x\,|\,x)_{\mathfrak{a}} = \|x\|_{\mathfrak{a}}^2$ thus $x = 0$.
We have shown that ϕ is bijective and has a bounded inverse.

Step 2: $y \mapsto (x\,|\,y)$ is also a bounded antilinear functional on $H_{\mathfrak{a}}$. Again by the Riesz representation Theorem there exists a function $\psi : H \to H_{\mathfrak{a}}$ such that

$$(x\,|\,y) = (\psi x\,|\,y)_{\mathfrak{a}} \qquad \forall x \in H, y \in D(\mathfrak{a})$$

We have that ψ is injective (that is because $\psi x_1 = \psi x_2$ implies $(x_1 - x_2 \,|\, y) = 0$ for all $y \in D(\mathfrak{a})$ and now the density of $D(\mathfrak{a})$ in H implies $x_1 = x_2$) and has $\|\cdot\|_{\mathfrak{a}}$-dense range (because $y \in (\psi H)^\perp$ implies $0 = (\psi x \,|\, y)_{\mathfrak{a}} = (x \,|\, y)$ for all $x \in H$ which shows $y = 0$).

Step 3: So far we have that for any $x \in H$, $y \in D(\mathfrak{a})$

$$(x \,|\, y) = (\psi x \,|\, y)_{\mathfrak{a}} = (\mathfrak{a}+1)\,[\phi^{-1}\psi x, y] = \mathfrak{a}\,[\phi^{-1}\psi x, y] + (\phi^{-1}\psi x \,|\, y)$$

Define $A = \psi^{-1}\phi(id - \phi^{-1}\psi) = \psi^{-1}\phi - id$ on the domain $D(A) = \{x \in D(\mathfrak{a}) : \phi x \in \mathrm{Rg}\psi\}$.

Then $D(A)$ is a core for $\mathfrak{a}$ since $\mathrm{Rg}\psi$ is and ϕ is bicontinuous. We have for $x \in D(A)$ and $y \in D(\mathfrak{a})$

$$\begin{aligned}
(Ax \,|\, y) &= (\psi^{-1}\phi(id - \phi^{-1}\psi)x \,|\, y)) \\
&= (\phi(id - \phi^{-1}\psi)x \,|\, y)_{\mathfrak{a}} \quad \text{by the definition of}\, \psi \\
&= (\mathfrak{a}+1)\,[x - \phi^{-1}\psi x, y] \quad \text{by the definition of}\, \phi \\
&= (\mathfrak{a}+1)\,[x, y] - \underbrace{(\mathfrak{a}+1)\,[\phi^{-1}\psi x, y]}_{=(x \,|\, y)} \\
&= \mathfrak{a}\,[x, y]
\end{aligned}$$

In particular this shows that A is sectorial. That A is m-sectorial follows since $-1 \in \rho(A)$.

We are done except for the uniqueness assertion. If $\mathfrak{a}\,[x, y] = (z \,|\, y)$ holds for all y in some core for $D(\mathfrak{a})$ then it holds for all $y \in D(\mathfrak{a})$ by density. Then of course $(\mathfrak{a}+1)\,[x, y] = (z + x \,|\, y)$ which shows that $\phi x \in \mathrm{Rg}\psi$ and hence $x \in D(A)$. Now $(Ax \,|\, y) = (z \,|\, y)$ for all $y \in D(\mathfrak{a})$ implies $Ax = z$ by the density of $D(\mathfrak{a})$ in H. $\square$

Definition. The operator A constructed in the last theorem is called the operator *associated with* $\mathfrak{a}$.

We now note some consequences of Theorem 1.17

Corollary 1.18. *Let $\mathfrak{a}$ be a closed,densely defined, sectorial form and A be the operator associated with $\mathfrak{a}$. Then*

a) *If $\mathfrak{a}_0\,[x, y] = (Ax \,|\, y)$ on $D(\mathfrak{a}_0) = D(A)$, then $\mathfrak{a}$ is the closure of $\mathfrak{a}_0$.*
b) *If B is an operator with $D(B) \subset D(A)$ and $\mathfrak{a}\,[x, y] = (Bx \,|\, y)$ for all $x \in D(B)$ and y in some core for $\mathfrak{a}$ then $B \subset A$.*
c) *A^* is the operator associated with $\mathfrak{a}^*$.*
d) *If $\mathfrak{a}$ is symmetric, then A is selfadjoint.*
e) *There is a one-to-one correspondence between the set of all densely defined, closed sectorial forms and the set of all m-sectorial operators.*

Proof. a) $D(\mathfrak{a}_0)$ is a core for $\mathfrak{a}$. Hence the domain of any closed extension of $\mathfrak{a}_0$ must contian $D(\mathfrak{a})$, thus $\bar{\mathfrak{a}} = \mathfrak{a}$ is the smallest closed extension of $\mathfrak{a}_0$.

b) This is just a reformulation of the last part of the theorem.

c) Let B be the operator associated with $\mathfrak{a}^*$. Then for any $x \in D(A) \subset D(\mathfrak{a}^*) = D(\mathfrak{a}), y \in D(B)$ we have

$$\begin{aligned} \mathfrak{a}^*[y,x] &= (By \,|\, x) = \overline{(x \,|\, By)} \quad \text{and} \\ \mathfrak{a}^*[y,x] &= \overline{\mathfrak{a}[x,y]} = \overline{(Ax \,|\, y)} \end{aligned}$$

Thus $y \in D(A^*)$ and $By = A^*y$ by uniqueness so that $B \subset A^*$. On the other hand b) implies $A^* \subset B$ thus $A^* = B$.

d) follows directly from c).

e) The mapping $\mathfrak{a} \to A$ is injective since by a) $\mathfrak{a}$ is uniquely determined by A. To see that it is surjective, let B be an m-sectorial operator. By proposition 1.13 the associated form is closable and densely defined. But B must be the associated operator of that form by uniqueness.

□

With help of the representation Theorem we now can prove the following:

Proposition 1.19. *Let $\mathfrak{a}$ be a closed sectorial form and suppose that x_n is a sequence in $D(\mathfrak{a})$ such that $x_n \to x$ (in H!) and $\mathfrak{a}[x_n]$ is bounded. Then $x \in D(\mathfrak{a})$ and* $\mathrm{Re}\,\mathfrak{a}[x] \leq \underline{\lim}\mathrm{Re}\,\mathfrak{a}[x_n]$.

Proof. Since x_n is convergent and thus norm bounded, we have that if $\mathrm{Re}\,\mathfrak{a}[x_n]$ is bounded, then so is $(\mathrm{Re}\,\mathfrak{a} - \gamma)[x_n]$, so we may assume that $\mathfrak{a}$ has a vertex 0. Furthermore we have that $x \in \overline{D(\mathfrak{a})}$, so that we may assume that $\mathfrak{a}$ be densely defined.

By hypothesis, x_n is $\|\cdot\|_{\mathfrak{a}}$-bounded, and thus there exists some weakly convergent subsequence, say

$$(x_{n_k} \,|\, y)_{\mathfrak{a}} \to (z \,|\, y) \quad \forall\, y \in H_{\mathfrak{a}}$$

for some $z \in H_{\mathfrak{a}}$. Since $\mathrm{Re}\,\mathfrak{a}$ is a symmetric, positve, densely defined closed form and thus associated with some positive selfadjoint operator R by 1.18, we have

$$(x_{n_k} \,|\, y)_{\mathfrak{a}} = (x_{n_k} \,|\, (1+R)y) \to (z \,|\, (1+R)y)_{\mathfrak{a}} = (z \,|\, y)_{\mathfrak{a}},$$

for all $y \in H_{\mathfrak{a}}$. But since $x_{n_k} \to x$ in H and $\mathrm{Rg}(1+R) = \overline{D(\mathfrak{a})}$ we must have $x = z \in D(\mathfrak{a})$.

Furthermore, we have that

$$\|x\|_{\mathfrak{a}} = \|\lim x_n\|_{\mathfrak{a}} \leq \underline{\lim}\|x_n\|_{\mathfrak{a}}$$

from which $\mathrm{Re}\,\mathfrak{a}[x] \leq \underline{\lim}\mathrm{Re}\,\mathfrak{a}[x_n]$ follows.

□

The theorems of this section show that forms are excellent means of defining m-sectorial or selfadjoint operators. Usually one is interested in those operators but properties such as selfadjointness are hard to establish. On the other hand it is comparatively easy to construct symmetric or sectorial forms.

However there are some differences between forms and operators. First of all, we have seen that m-sectorial operators have no proper m-sectorial extensions. This is not true for forms. There is no such thing as a "maximal form" (except for everywhere defined forms).

Recall that a symmetric operator is an operator A satisfying $(Ax \mid y) = (x \mid Ay)$ for all $x, y \in D(A)$. Symmetric operators always have closed extensions (the double adjoint), but they may not be selfadjoint. Symmetric forms may have no closed extensions at all (see the example in the preceding section) but if they have, then the extension is necessary associated with an selfadjoint operator.

The same holds for sectorial operators:
If A is a sectorial opererator, then the associated form is closable by proposition 1.13 and the representation Theorem shows that the closure is associated with some m-sectorial operator B extending A. Even if A is closable, then B may differ from the closure of A.

DEFINITION. Let A be a sectorial operator and $\mathfrak{a}$ be the form associated with A. The m-sectorial operator associated with the closure of the form $\mathfrak{a}$ is called the *Friedrichs extension* of A.

PROPOSITION 1.20. a) *The Friedrichs extension of an m-sectorial operator A is A itself.*

b) *Among all m-sectorial extensions of A the Friedrichs extension has the smallest form domain.*

c) *The Friedrichs extension is the only m-sectorial extension of A whose domain is contained in $D(\bar{\mathfrak{a}})$.*

PROOF. Let A be a sectorial operator and denote the form associated with A by $\mathfrak{a}$ and the closure of this form $\bar{\mathfrak{a}}$.
Part a) follows directly from part a) of corollary 1.18.
b) holds since by construction D(A) is a form core for $\bar{\mathfrak{a}}$.
For the last part observe, that if A has another m-sectorial extension B with domain in $D(\bar{\mathfrak{a}})$ then it follows part b) of corollary 1.18 that the Friedrichs extension of A is also an extension of B and hence they must be equal since they are both m-sectorial. □

The representation Theorem has one major weakness:
The domain of the operator associated with a form is not the domain of the form.

Usually the operator domain is smaller. For symmetric positive forms, this problem can be handled and we can connect the domain of the operator with the form domain:

Proposition 1.21. *Let $\mathfrak{a}$ be a densely defined, closed, symmetric, positive form and A be the operator associated with $\mathfrak{a}$. Then $D(A^{\frac{1}{2}}) = D(\mathfrak{a})$ and for all $x, y \in D(\mathfrak{a})$ we have:*

$$\mathfrak{a}\,[x, y] = \left(A^{\frac{1}{2}}x \,\middle|\, A^{\frac{1}{2}}y\right)$$

Proof. Define $\mathfrak{b}$ on $D(\mathfrak{b}) = D(A^{\frac{1}{2}})$ by $\mathfrak{b}\,[x, y] = \left(A^{\frac{1}{2}}x \,\middle|\, A^{\frac{1}{2}}y\right)$. Then $\mathfrak{b}$ is a densely defined, closed form by proposition 1.4. Furthermore $D(A)$ is a core for $\mathfrak{b}$ since it is an operator core for $A^{\frac{1}{2}}$ and the norm $\|\cdot\|_{\mathfrak{b}}$ is equivalent to the graph norm of $A^{\frac{1}{2}}$. So since $\mathfrak{a}$ and $\mathfrak{b}$ coincide on a common core, they have to be equal. $\square$

For sectorial forms $\mathfrak{a}$ it is possible to associate an operator on a larger domain. Let $H_{\mathfrak{a}}^{'}$ be the space of antilinear bounded functionals on $H_{\mathfrak{a}}$ and recall from the proof of the representation Theorem that we have the following situation:

$$\begin{tikzcd} H_{\mathfrak{a}}^{'} & \\ H_{\mathfrak{a}} \arrow[u, "\phi"] \arrow[r, "A"'] & H \arrow[ul, hook, "\psi"'] \end{tikzcd}$$

We used ϕ and ψ to define A in H. Another possibility would have been not to work in H but rather in $H_{\mathfrak{a}}^{'}$ and just say $y \mapsto \mathfrak{a}\,[x, y]$ is an element of $H_{\mathfrak{a}}^{'}$, denote it by $\mathcal{A}x$ acting on $H_{\mathfrak{a}}$ by

$$\langle \mathcal{A}x \ , \ y\rangle = \mathfrak{a}\,[x, y]$$

Then $\mathcal{A}$ is a continuous mapping from $D(\mathfrak{a})$ to $H_{\mathfrak{a}}^{'}$. A is then precisely the part of $\mathcal{A}$ in $\mathrm{Rg}\psi$ transformed to H with the help of ψ:

$$D(A) = \{x \in D(\mathcal{A}) = D(\mathfrak{a}) \ : \ \mathcal{A}x \in \psi H\} \quad Ax = \psi^{-1}\mathcal{A}x$$

4. Boundedness of Universally Sectorial Operators

Now we ask the following question:
The statement that A is a sectorial operator depends on the inner product which is defined on H. What happens if we take an equivalent inner product instead?
We will prove that if A is dissipative with respect to *any* equivalent inner product on H (in particular if A happens to be sectorial for any equivalent inner product on H) then A is necessarily bounded.

Theorem 1.22. *(*Matolcsi *2003) Let A be an operator on H with nonempty resolvent set. If for any equivalent inner product $(\cdot|\cdot)_0$ on H the numerical range of A is contained in some left halfplane (i.e. there exists a constant γ_0 depending on the inner* *product such that* $\operatorname{Re}(Ax\,|\,x)_0 \leq \gamma_0$ *for all* x *with* $\|x\|_0 = 1$ *) then* A *is a bounded operator.*

Before proving the theorem, we state some lemmata that will be used in the proof.

LEMMA 1.23. *Let* A *be an unbounded operator on a Hilbert space* H *and* $0 \in \rho(A)$. *Then there is an orthonormal sequence* x_n *such that* $A^{-1}x_n \to 0$.

PROOF. Without loss of generality, we may assume that A (and thus A^{-1}) is a positive selfadjoint operator. Otherwise, we take the polar decomposition $A = UR$ where U is unitary and R is positive selfadjoint. Using the result for R we find an orthonormal sequence y_n with $\|R^{-1}y_n\| \to 0$.
If we put $x_n := Uy_n$ then x_n is orthonormal since U is unitary and furthermore $\|A^{-1}x_n\| = \|R^{-1}y_n\| \to 0$.

Courtesy of the spectral Theorem, we may assume that $H = L^2(\Omega, \Sigma, \mu)$ for some measure space (Ω, Σ, μ) and A is a multiplication operator, say $Af = hf$.
If we define $M_n = \{\omega : n \leq h(\omega) < n+1\}$ then for any n we have $\mu(M_n) > 0$ (since A is unbounded and so h has to be, thus this is true at least for a subsequence, but for simpler notation, we assume that it is true for any n.)
Now define $f_n = \mu(M_n)^{-\frac{1}{2}}\mathbb{1}_{M_n}$. Then each f_n has norm 1 and since all the sets M_n are disjoint, they are orthogonal as well. Now

$$\|A^{-1}f_n\|_2^2 = \mu(M_n)^{-1}\int_{M_n} |h(\omega)|^{-2}\,d\mu(\omega) \leq \mu(M_n)^{-1}\mu(M_n)n^{-2} \to 0$$

□

LEMMA 1.24. *Let* H *be an inner product space ,* U, V *be subspaces with* $\dim U < \dim V < \infty$. *Then there is a vector* $v \in V$ *such that* $\|V\| = 1$ *and* $v \perp U$.

PROOF. Let $X = \operatorname{span}(U \cup V)$. We extend an orthonormal basis $u_1, \dots u_k$ of U to an orthonormal basis $u_1, \dots, u_k, u_{k+1}, \dots u_{k+l}$ of X. Note that since $V \subset X$ we must have $\dim X > \dim U$ and hence $l \geq 1$. Now take $v = u_{k+1} \in V$. □

PROOF OF 1.22. Assume that A is not a bounded operator, and by shifting if necessary, that $0 \in \rho(A)$. Let $T = A^{-1}$.
We claim that there is a sequence x_n with the following properties:

i) $\|x_n\|$=1 for any n.
ii) $Tx_n \to 0$ with respect to the $\|\cdot\|$-norm
iii) $\operatorname{span}\{x_i, Tx_i\} \perp \operatorname{span}\{x_j, Tx_j\}$ for any $i \neq j$
iv) There is some $1 > \varepsilon > 0$ such that for any $n \in \mathbb{N}$ we have

$$\frac{|(x_n\,|\,Tx_n)|}{\|Tx_n\|} < \varepsilon$$

By lemma 1.23 there is an orthonormal sequence u_n with $Tu_n \to 0$. We construct a sequence x_n with the above properties from this sequence u_n.

Step 1 We construct starting from u_n a new orthonormal sequence $\tilde{u}_n$ such that in addition to the property $T\tilde{u}_n \to 0$ we have that
$\tilde{u}_n \perp \operatorname{span}\{\tilde{u}_1, T\tilde{u}_1, \dots, \tilde{u}_{n-1}, T\tilde{u}_{n-1}\}$.

Take $\tilde{u}_1 = u_{i_1}$ where the index i_1 is chosen such that $\|Tu_{i_1}\| \le 1$.
Now assume that $\tilde{u}_1, \dots, \tilde{u}_{n-1} \in \operatorname{span}\{e_1, \dots e_{l_n}\}$ have already been constructed such that for any $1 \le k \le n-1$ we have $\|\tilde{u}_k\| = 1$ and $\|T\tilde{u}_k\| \le k^{-\frac{1}{2}}$ and for $m < k$ we have $\tilde{u}_k \perp \operatorname{span}\{\tilde{u}_m, T\tilde{u}_m\}$. Pick $i_1, \dots i_n > l_n$ such that $\|Tu_{i_j}\| \le \frac{1}{n}$ for any $1 \le j \le n$. Take $U = \operatorname{span}\{T\tilde{u}_1, \dots, T\tilde{u}_{n-1}\}$ and $V = \operatorname{span}\{u_{i_1}, \dots, u_{i_n}\}$. Note that $\dim U \le n-1$ and $\dim V = n$, so that by lemma 1.24 there exists a vector $\tilde{u}_n \in V$ with $\|\tilde{u}_n\| = 1$ and $\tilde{u}_n \perp U$. But since u_k is an orthogonal sequence, by construction $\tilde{u}_n$ is also perpendicular to $\tilde{u}_1, \dots, \tilde{u}_{n-1}$.
It is only left to show that $\|T\tilde{u}_n\| \le n^{-\frac{1}{2}}$. But if $\tilde{u_n} \in V$ we have $\tilde{u}_n = \sum \lambda_j u_{i_j}$ and thus

$$\|T\tilde{u}_n\|^2 = \sum |\lambda_j|^2 \|Tu_{i_j}\|^2 \le \frac{1}{n} \sum |\lambda_j|^2 \|u_{i_j}\|^2 = \frac{1}{n}\|\tilde{u_n}\|^2 = \frac{1}{n}\,.$$

Step 2 We refine the sequence $\tilde{u}_k$ to an orthonormal sequence v_k such that
$\operatorname{span}\{v_k, Tv_k\} \perp \operatorname{span}\{v_m, Tv_m\}$ for any $k \neq m$
and such that $\|Tu_k\| \le (2k-1)^{-\frac{1}{2}}$.

Let $v_1 = \tilde{u}_1$. For $n \in \mathbb{N}$ let $v_1, \dots, v_{n-1} \in \operatorname{span}\{\tilde{u}_1, \dots \tilde{u}_{l_n}\}$ already be chosen with the above properties.
Pick $i_1, \dots, i_{2n-1} > l_n$ such that $\|T\tilde{u}_{i_j}\| \le (2n-1)^{-1}$ for any $1 \le j \le 2n-1$. Take $U = \operatorname{span}\{v_1, Tv_1, \dots, v_{n-1}, Tv_{n-1}\}$ and $V = T(\operatorname{span}\{\tilde{u}_{i_1}, \dots \tilde{u}_{1_{2n-1}}\})$. Here again, we have $\dim U \le 2n-2$, whereas $\dim V = 2n-1$ since $T = A^{-1}$ is injective. So by applying lemma 1.24 we obtain a vector $T\tilde{v}_n \in V$ satisfying $T\tilde{v}_n \perp U$. If we put $v_n := \|\tilde{v}_n\|^{-1}\tilde{v}_n$ then v_n has norm 1, and $\operatorname{span}\{v_n, Tv_n\} \perp \operatorname{span}\{v_k, Tv_k\}$[3] for any $k \le n$ and a computation analog to step 1 yields $\|Tv_n\| \le (2n-1)^{-\frac{1}{2}}$.

Step 3 In this last step, we finally establish the property iv)
Pick an index k_1 such that $\|Tv_{k_1}\| \le \frac{\varepsilon^2}{4}\|Tv_1\|$ and put

$$x_1 := \frac{\varepsilon}{2}v_1 + (1 - \frac{\varepsilon^2}{4})^{\frac{1}{2}} v_{k_1}$$

[3] $Tv_n \perp \operatorname{span}\{v_k, Tv_k\}$ is clear by construction. But we also have $v_n \perp \operatorname{span}\{v_k, T_k\}$ since v_n is a linear combination of $(\tilde{u_j})$ where these j's are greater than those used for v_k. So this follows from the construction in Step 1.

We then have that

$$\begin{aligned}
\|Tx_1\| &\geq \frac{\varepsilon}{2}\|Tv_1\| - (1-\frac{\varepsilon^2}{4})^{\frac{1}{2}}\|Tv_{k_1}\| \\
&\geq \left(\frac{\varepsilon}{2} - \frac{\varepsilon^2}{4}(1-\frac{\varepsilon^2}{4})^{\frac{1}{2}}\right)\|Tv_1\| \\
|(x_1\,|\,Tx_1)| &= \left|\frac{\varepsilon^2}{4}(v_1\,|\,Tv_1) + (1-\frac{\varepsilon^2}{4})(v_{k_1}\,|\,Tv_{k_1})\right| \\
&\leq \left(\frac{\varepsilon^2}{4} + (1-\frac{\varepsilon^2}{4})\frac{\varepsilon^2}{4}\right)\|Tv_1\|
\end{aligned}$$

So we obtain that

$$\frac{|(x_1\,|\,Tx_1)|}{\|Tx_1\|} \leq \frac{\frac{\varepsilon^2}{4} + (1-\frac{\varepsilon^2}{4})\frac{\varepsilon^2}{4}}{\frac{\varepsilon}{2} - \frac{\varepsilon^2}{4}(1-\frac{\varepsilon^2}{4})^{\frac{1}{2}}} \leq \frac{2}{\varepsilon}\left(\frac{\varepsilon^2}{4} + (1-\frac{\varepsilon^2}{4})\frac{\varepsilon^2}{4}\right) = \varepsilon - \frac{\varepsilon^3}{8} < \varepsilon\,,$$

which is property iv). Furthermore, we have that $\|x_1\| = 1$ and $\|Tx_1\| \leq 1$.

Now suppose that vectors $x_1, \dots x_{n-1} \subset \operatorname{span}\{v_1, \dots, v_{l_n}\}$ with the properties i), iii) and iv) have already been constructed such that $\|Tx_k\| \leq k^{-\frac{1}{2}}$.
Choose indices $m_1, m_2 > l_n$ such that $\|Tv_{m_1}\| \leq n^{-\frac{1}{2}}$ and $\|A^{-1}v_{m_2}\| \leq \frac{\varepsilon}{4}\|Tv_{m_1}\|$, and define

$$x_n := \frac{\varepsilon}{2}v_{m_1} + (1-\frac{\varepsilon^2}{4})^{\frac{1}{2}}v_{m_2}$$

Then one showes as above, that x_n also has the property iv) is satisfied. The properties i) and iii) follow from the construction of x_n and the properties of the sequence v_k. One checks that $\|Tx_n\| \leq n^{-\frac{1}{2}}$.

With the help of the sequence x_n we will now construct an equivalent inner product on H, so that the numerical range of A with respect to this inner product is *not* contained in any left halfplane.

Take a closer look at $H_n := \operatorname{span}\{x_n, Tx_n\}$. x_n and Tx_n are linearly independent, since otherwise, by the Cauchy Schwarz inequality we would have $|(x_n\,|\,Tx_n)| = \|x_n\|\|Tx_n\|$, in violation of the property iv). Thus H_n is 2-dimensional.
Extend x_n to an orthonormal Basis (x_n, y_n) of H_n. Then Tx_n has a unique repersentation $Tx_n = \alpha_n x_n + \beta_n y_n$. Here $\beta_n \neq 0$.

From the property iv) we obtain that

$$\frac{|(x_n\,|\,Tx_n)|^2}{\|Tx_n\|^2} = \frac{|\alpha_n|^2}{|\alpha_n|^2 + |\beta_n|^2} < \varepsilon^2 \quad \text{and thus} \quad \frac{|\beta_n|^2}{|\alpha_n|^2 + |\beta_n|^2} > 1 - \varepsilon^2$$

this implies on the other hand that

$$\frac{|\alpha_n|}{|\beta_n|} < \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}$$

Define an operator Q_n on H_n by having the following matrix representation with respect to the basis (x_n, y_n)

$$Q_n = \begin{pmatrix} 1 & a_n \\ \overline{a_n} & 1+|a_n|^2 \end{pmatrix} \quad Q_n^{-1} = \begin{pmatrix} 1+|a_n|^2 & -a_n \\ -\overline{a_n} & 1 \end{pmatrix}$$

Here $a_n = 2\varepsilon(1-\varepsilon^2)^{-\frac{1}{2}} \frac{\overline{\beta_n}}{|\beta_n|}$

Note, that the entry with the maximal modulus in Q_n is $1+|a_n|^2$. We have $|1+|a_n|^2| < 1+4\varepsilon^2(1-\varepsilon^2)^{-1} =: C$. So as an operator from $(H_n, \|\cdot\|_1)$ to $(H_n, \|\cdot\|_\infty)$, Q_n has norm $\leq C$. But since on a finite dimensional space all norms are equivalent, Q_n has norm $\leq K$ as an operator on $(H_n, \|\cdot\|_2)$. In particular, we have that $\|Q_n\|$ and (similarly) $\|Q_n^{-1}\|$ are bounded independently of n.

Now define Q by

$$Q = P_0 + \sum_{n\in\mathbb{N}} Q_n P_n$$

where P_n is the orthogonal projection onto H_n and $H_0 = (\bigoplus_{n\in\mathbb{N}} H_n)^\perp$. Then Q is a positive, bounded, selfadjoint operator with bounded inverse. So $(x\,|\,y)_0 := (x\,|\,Qy)$ is an equivalent scalar product on H.

Then for $e_n = \|Tx_n\| Tx_n$ we have

$$\begin{aligned}
\mathrm{Re}\,(Ae_n\,|\,e_n)_0 &= \|Tx_n\|^{-2}\mathrm{Re}\,(x_n\,|\,QTx_n) \\
&= \|Tx_n\|^{-2}\mathrm{Re}\,(x_n\,|\,\alpha_n(x_n+\overline{a_n}y_n)+\beta_n(a_nx_n+(1+|a_n|^2y_n)) \\
&= \|Tx_n\|^{-2}\mathrm{Re}\,(\alpha_n+\beta_na_n) \\
&\geq \|Tx_n\|^{-2}(-|\alpha_n|+2\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}|\beta_n|) \\
&\geq \|Tx_n\|^{-2}(\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}|\beta_n|) \\
&> \frac{\varepsilon}{\|A^{-1}x_n\|} \to \infty
\end{aligned}$$

which proves, that $\Theta\,(A)$ is not contained in any left halfplane. $\square$

5. Pseudoresolvents and Degenerate Semigroups

Recall from Theorem 1.14 that for an m-sectorial operator A whose numerical range is contained in the sector $\Sigma_\gamma(\theta)$ we have

$$\Sigma_\gamma(\theta)^C \subset \rho(A) \quad \text{and} \quad \|R(\lambda, A)\| \leq \frac{1}{\mathrm{dist}(\lambda, \Theta\,(A))}.$$

We may reformulate this as follows:

Any λ outside the sector $\Sigma_\gamma(\theta+\varepsilon)$ where $\varepsilon \in (0, \pi-\theta)$ is contained in the resolvent set of A and we have the following estimate for the resolvent:

$$\|R(\lambda, A)\| \leq \frac{M_\varepsilon}{|\lambda - \gamma|}$$

From this it follows that $-A$ generates a quasi bounded analytic semigroup $T(z) = e^{-zA}$ which is defined for $z \in \Sigma(\frac{\pi}{2} - \theta)$. We refer to [**4**] or [**9**] for more details.

For nondensely defined sectorial forms $\mathfrak{a}$ we cannot associate an m-sectorial operator. However, we can think of $\mathfrak{a}$ as a form on $\overline{D(\mathfrak{a})}$, where it *is* densely defined. Let $P_\mathfrak{a}$ be the orthogonal projection onto $\overline{D(\mathfrak{a})}$ and A be the m-sectorial operator associated with $\mathfrak{a}$ as a form on $\overline{D(\mathfrak{a})}$. We define:

$$e^{-z\mathfrak{a}} := e^{-zA} P_\mathfrak{a} \qquad \text{for} \quad z \in \Sigma\left(\frac{\pi}{2} - \theta\right) .$$

Then $e^{-t\mathfrak{a}}$ is a continuous, quasibounded, degenerate, analytic semigroup on H. We clarify the meaning:

Definition. Let X be a Banach space. A *degenerate semigroup* is a strongly continuous mapping $T : (0, \infty) \to \mathcal{L}(X)$ satisfying

$$T(t+s) = T(t)T(s) \quad \text{for all} \quad t, s > 0 .$$

If the strong limit $s-\lim_{t\to 0} T(t)$ exists, we say that T is *continuous.* A degenerate semigroup is *bounded* if there exists a constant $M > 0$ such that $\|T(t)\| \leq M$ for all $t > 0$, and it is *quasi bounded* if $e^{-\omega t}T(t)$ is a bounded degenerate semigroup for some $\omega > 0$. A degenerate semigoup is called *analytic* if it has a holomorphic extension to some sector containing the positive real axis.

For $e^{-t\mathfrak{a}}$ all these properties follow directly form the properties of e^{-tA} on $\overline{D(\mathfrak{a})}$. We may chose the ω in the quasi boundedness condition to be a vertex γ for $\mathfrak{a}$. For simplicity we will frequently call $e^{-t\mathfrak{a}}$ the *degenerate semigroup* (or sometimes just the semigroup) associated with $\mathfrak{a}$ and understand that all the other properties hold.

When dealing with semigroups interesting objects are not only the semigroups and their generators but also the resolvent of the generator. It turns out to be the Laplace transform of the semigroup.
For degenerate semigroups the Laplace transform is a pseudoresolvent.

Definition. Let X be a Banach space and Ω a subset of $\mathbb{C}$. A function $R : \Omega \to \mathcal{L}(X)$ is called a *pseudoresolvent* if for all $\lambda, \mu \in \Omega$ we have that

$$\frac{R(\lambda) - R(\mu)}{\mu - \lambda} = R(\lambda)R(\mu) .$$

This equation is called the *resolvent identity.*

In our case, the Laplace transform is of rather special type. We have

$$\hat{T}(\lambda) = \int_0^\infty e^{-\lambda t} e^{-t\mathfrak{a}}\, dt = \int_0^\infty e^{-\lambda t} e^{-tA}\, P_{\mathfrak{a}}\, dt = R(\lambda, -A) P_{\mathfrak{a}}\,.$$

DEFINITION. Let $\mathfrak{a}$ be a closed sectorial form and A and $P_{\mathfrak{a}}$ as above. $R(\lambda, \mathfrak{a}) := R(\lambda, -A)P_{\mathfrak{a}}$ is called the *pseudoresolvent associated with* $\mathfrak{a}$. We will also write $(\lambda + \mathfrak{a})^{-1}$ for $R(\lambda, \mathfrak{a})$.

We remark that we can compute $e^{-t\mathfrak{a}}$ directly from the pseudoresolvent with the help of a contour integral. We have:

$$\begin{aligned} e^{-z\mathfrak{a}} &= e^{-zA} P_{\mathfrak{a}} \\ &= \frac{1}{2\pi i} \int_\Gamma e^{z\xi} R(\xi, -A) d\xi\, P_{\mathfrak{a}} \\ &= \frac{1}{2\pi i} \int_\Gamma e^{z\xi} R(\xi, \mathfrak{a}) d\xi \end{aligned}$$

Where Γ is a path around $-\Theta\,(\mathfrak{a})$ (and thus around the spectrum of $-A$), which is positively oriented and avoids the vertex γ of $\mathfrak{a}$, such that z lies outside Γ.

Finally we want to somehow describe the form $\mathfrak{a}$ by means of the associated semigroup $e^{-t\mathfrak{a}}$. For this we return to the operator $\mathcal{A} : D(\mathfrak{a}) \to H_{\mathfrak{a}}'$ as defined at the end of section 3. We have the following result. For the proof we refer to OUHABAZ [**14**, Theorem 1.55].

PROPOSITION 1.25. *Let $\mathfrak{a}$ be a densely defined closed sectorial form and A be the operator associated with $\mathfrak{a}$, $\mathcal{A}$ as above. Then $-\mathcal{A}$ generates a strongly continuous semigroup $e^{-t\mathcal{A}}$ on $H_{\mathfrak{a}}'$. Furthermore*

$$e^{-t\mathcal{A}}x = e^{-tA}x = e^{-t\mathfrak{a}}x$$

for all $x \in H$ and $t \geq 0$ (Here, we think of H as a subset of $H_{\mathfrak{a}}'$ where the embedding is as at the end of the last section) .

With the help of this proposition it is now possible to characterize a closed sectorial form entirely by the associated semigroup.

THEOREM 1.26. *Let $\mathfrak{a}$ be a closed sectorial form. Then*

$$D(\mathfrak{a}) = \{x \in H \,:\, \sup_{t>0} t^{-1} \mathrm{Re}\,(x - e^{-t\mathfrak{a}}x \,|\, x) < \infty\}\,.$$

Furthermore, for all $x, y \in D(\mathfrak{a})$ we have

$$\mathfrak{a}\,[x, y] = \lim_{t\to 0} t^{-1}(x - e^{-t\mathfrak{a}}x \,|\, x)\,.$$

Proof. If $0 \neq x \in D(\mathfrak{a})^{\perp}$ then $e^{-t\mathfrak{a}}x = 0$ so that $t^{-1}(x - e^{-t\mathfrak{a}}x \,|\, x) = t^{-1}\|x\|^2$, which tends to ∞ as $t \to 0$. Hence we can assume that $\mathfrak{a}$ is densely defined.

For $x, y \in D(\mathfrak{a}) = D(\mathcal{A})$ we have

$$t^{-1}(x - e^{-t\mathfrak{a}}x \,|\, y) = t^{-1}\langle x - e^{-t\mathcal{A}}x \ , \ y\rangle \longrightarrow \langle \mathcal{A}x \ , \ y\rangle = \mathfrak{a}\,[x, y]$$

as $t \to 0$. Here we used the above proposition and the definition of $\mathcal{A}$. So the only thing left to show is that $\sup_{t>0} t^{-1}(x - e^{-t\mathfrak{a}}x \,|\, x) < \infty$ implies $x \in D(\mathfrak{a})$.
So suppose that $\sup_{t>0} t^{-1}\mathrm{Re}\,(x - e^{-t\mathfrak{a}}x \,|\, x) < \infty$. For $\lambda > 0$ we have $\lambda \in \rho(-A)$ and

$$\begin{aligned}
& \mathrm{Re}\,\mathfrak{a}\,[\lambda(\lambda + A)^{-1}x, \lambda(\lambda + A)^{-1}x] \\
= \ & \mathrm{Re}\,(\lambda A(\lambda + A)^{-1}x \,|\, \lambda(\lambda + A)^{-1}x) \\
= \ & \mathrm{Re}\,\lambda(x - \lambda(\lambda + A)^{-1}x \,|\, \lambda(\lambda + A)^{-1}x) \\
\leq \ & \mathrm{Re}\,\lambda(x - \lambda(\lambda + A)^{-1}x \,|\, x) \\
= \ & \mathrm{Re} \int_0^{\infty} \lambda^2 e^{-\lambda t}(x - e^{-t\mathfrak{a}}x \,|\, x)\, dt \\
\leq \ & \sup_{t>0} t^{-1}\mathrm{Re}\,(x - e^{-t\mathfrak{a}}x \,|\, x) \int_0^{\infty} \lambda^2 t e^{-\lambda t}\, dt \\
= \ & \sup_{t>0} t^{-1}\mathrm{Re}\,(x - e^{-t\mathfrak{a}}x \,|\, x) \int_0^{\infty} r e^{-r}\, dr < \infty
\end{aligned}$$

This shows that $\lambda(\lambda + A)^{-1}x$ is uniformly bounded with respect to the $\|\cdot\|_{\mathfrak{a}}$-norm (recall that $\|\lambda(\lambda + A)^{-1}x\| \leq M\|x\|$ by 1.14). Since $\lambda(\lambda + A)^{-1}x \to x$ as $\lambda \to \infty$ proposition 1.19 implies $x \in D(\mathfrak{a})$. □

Notes and References for Chapter 1: The material presented in this chapter is pretty standard. Our presentation is very close to Kato [**10**, chapter 6]. Another possible reference is Ouhabaz [**14**, chapter 1]. Here, the terminology is somewhat closer to the following approach to forms:

A form is a triplet $(\mathfrak{a}, V, H)$, where H and V are Hilbert spaces and V is densely embedded into H and $\mathfrak{a} : V \times V \to \mathbb{C}$ is a sectorial form. The notion of $\mathfrak{a}$ being sectorial is equivalent to say that $\mathfrak{a}$ is bounded on V, the closedness of $\mathfrak{a}$ is replaced by H-ellipticity of $\mathfrak{a}$, which basically means, that the inner product associated with $\mathfrak{a}$ is equivalent to the inner product on V. This approach reflects the use of form methods for elliptic problems (see chapter 4). Here $H = L^2$, V is a Sobolev space and $\mathfrak{a}$ is a form associated with a differential operator. The closedness of such a form is then proved by showing, that the associated inner product is equivalent to the inner product in the Sobolev space. For a treatment in this spirit, see Arendt et al [**4**, chapter 7].

Yet another different approach for symmetric forms is to interpret the quadratic form as lower semicontinuous convex functionals on H, by defining $\mathfrak{a}\,[x] = \infty$ for $x \notin D(\mathfrak{a})$. This is illustrated in Reed, Simon [**16**, supplement to VIII.7]. This approach is closer to the nonlinear case, which may be found in Brezis [**6**].

The part of section 2 concerning closability and the regular part is inspired by Simon [**17**]. The description of the spectrum of m-sectorial operators may also be found in [**10**], but our presentation is closer to [**18**, lemma VII.1.1], which deals with bounded operators. Finally the representation Theorem is taken from [**10**, chapter 6.2], but the notation was changed so that it may also be used for results concerning the semigroup generated by the operator $\mathcal{A}$ associated with $\mathfrak{a}$ on the antidual of the domain. This presentation is close to Ouhabaz [**14**, chapter 1.5]. Section 4 is taken from Matolcsi [**13**], without many changes.

All these references are mostly concerned with densely defined forms. This caused some minor modification to the theorems and/or the proofs in the first sections of this chapter insofar as they concern nondensely defined forms. The real difference between the densely defined case and the nondensely defined case lies in section 5. The terminology here is mostly inspired by Arendt [**2**]. The characterisation of the degenerate semigroup in terms of the generating forms is taken from [**14**].

CHAPTER 2

Families of Forms

One of the standard techniques in semigroup theory is the approximation of a (possibly complicated) semigroup by other (possibly easier) semigroups. If one has established the "right" convergence of the operators, then the generated semigroups converge. Here, the right notion of convergence is convergence in the strong resolvent sense, that is pointwise convergence of the resolvents.

In the first section of this chapter, we define strong resolvent convergence for forms and prove that if a sequence of forms converges in strong resolvent sense, then the associated degenerate semigroups converge strongly.
Our next goal is to establish strong resolvent convergence from convergence of the forms. This is a big advantage of the form methods, since in many cases this is easy to establish.

Let $\mathfrak{a}_n$ be a uniformly sectorial sequence of closed forms. Here we mean by uniformly sectorial that the numerical range of all forms $\mathfrak{a}_n$ is contained in one common sector $\Sigma_\gamma(\theta)$. We define the limit by

$$D = \{x \in \underline{\lim} D(\mathfrak{a}_n) \,:\, \lim \mathfrak{a}_n\,[x]\,\text{exists}\} \quad \mathfrak{a}\,[x] := \lim \mathfrak{a}_n\,[x]\,.$$

If D is a vector space, we obtain by polarisation a form $\mathfrak{a}$ on $D(\mathfrak{a}) = D$.

So convergence of forms is a really mild assumption on the associated operators. For example, even if all the forms $\mathfrak{a}_n$ and $\mathfrak{a}$ are bounded (so that the associated operators are bounded as well) form convergence means nothing more than weak convergence of the associated operators. Often this is not enough to draw interesting conclusions from. Therefore we have to assure that $\mathfrak{a}_n$ converges to $\mathfrak{a}$ in a certain way to conclude strong resolvent convergence of the associated operators.

For unbounded forms, it is not even clear what form convergence means for the associated operators. This is mainly due to the fact that even if all forms $\mathfrak{a}_n$ have the same domain, the associated operators may have different, even disjoint domains.
When working with forms alone, different questions occur:

- When is the limit defined on a vector space ?
- What conditions assure that the limit form is closed as well?

- If the forms $\mathfrak{a}_n$ are densely defined when is $\mathfrak{a}$?

In section 3, we will present conditions on the sequence $\mathfrak{a}_n$, that will assure a positive answer to these questions and establish strong resolvent convergence, so that the associated semigroup will converge as well.

Section 2 deals not with sequences of forms, but with families that are holomorphic in a certain sense. This will provide a powerful tool to extend results for symmetric forms to sectorial forms. This will be applied in the proof of Trotter's formula in the next chapter and also at the end of section 3 to obtain a convergence result for sectorial forms from one for symmetric forms.

1. Convergence in the Strong Resolvent Sense

Definition. Let $\mathfrak{a}_n$ and $\mathfrak{a}$ be uniformly sectorial closed forms, say $\Theta\,(\mathfrak{a}_n), \Theta\,(\mathfrak{a}) \subset \Sigma$. We say that $\mathfrak{a}_n$ *converges* to $\mathfrak{a}$ *in strong resolvent sense* (and write $\mathfrak{a}_n \xrightarrow{R} \mathfrak{a}$) if for all $x \in H$ and $\lambda \in (-\Sigma)^C$ we have

$$R(\lambda, \mathfrak{a}_n)x \to R(\lambda, \mathfrak{a})x$$

As with resolvents it is enough to check this convergence for *one* λ and we always obtain uniform convergence on compact subsets of $(-\Sigma)^C$:

Theorem 2.1. *If $\mathfrak{a}_n$ and $\mathfrak{a}$ are closed sectorial forms and $\Theta\,(\mathfrak{a}_n), \Theta\,(\mathfrak{a}) \subset \Sigma$ and if for one $\mu \in \Omega := (-\Sigma)^C$ and all $x \in H$ we have $R(\mu, \mathfrak{a}_n)x \to R(\mu, \mathfrak{a})x$ then $\mathfrak{a}_n \xrightarrow{R} \mathfrak{a}$ and the convergence is uniform on compact subsets of Ω.*

Proof. Let $\lambda \in \Omega$ and $x \in H$ and define

$$y = x + (\mu - \lambda)R(\lambda, \mathfrak{a})x \quad , \quad y_n = y + (\lambda - \mu)R(\mu, \mathfrak{a}_n)y\,.$$

By the resolvent identity we have $R(\lambda, \mathfrak{a})x = R(\mu, \mathfrak{a})y$ and $R(\lambda, \mathfrak{a}_n)y_n = R(\mu, \mathfrak{a}_n)y$. By assumption we have that $y_n \to y + (\lambda - \mu)R(\mu, \mathfrak{a})y = y + (\lambda - \mu)R(\lambda, \mathfrak{a})x = x$. We further know that $\|R(\lambda, \mathfrak{a}_n)\| \le (\operatorname{dist}(\lambda, \Sigma))^{-1} < \infty$ and we can conclude that $\lim R(\lambda, \mathfrak{a}_n)(y_n - x) = 0$. Now it follows that

$$\begin{aligned}
\lim R(\lambda, \mathfrak{a}_n)x &= \lim R(\lambda, \mathfrak{a}_n)y_n \\
&= \lim R(\mu, \mathfrak{a}_n)y \\
&= R(\mu, \mathfrak{a})y = R(\lambda, \mathfrak{a})x
\end{aligned}$$

Which proves the required convergence for λ. Since pseudoresolvents are holomorphic and locally bounded, the uniform convergence on compact subsets of Ω follows form Vitali's Theorem. ☐

For a symmetric form $\mathfrak{a}$, the associated operator is selfadjoint and for *real* λ we also have that the pseudoresolvents $R(\lambda, \mathfrak{a})$ are selfadjoint. In this situation even weak convergence of the resolvents is enough:

THEOREM 2.2. *Let* $\mathfrak{a}_n$, $\mathfrak{a}$ *be symmetric closed forms and suppose that* $\mathfrak{a}_n$, $\mathfrak{a} \geq \gamma$. *Assume that*

$$(R(\lambda, \mathfrak{a}_n)x \,|\, x) \to (R(\lambda, \mathfrak{a})x \,|\, x)$$

for all $x \in H$ *and all* $\lambda \in \Omega_0$ *where* Ω_0 *is a subset of the real line which has an accumulation point.*
Then $\mathfrak{a}_n \xrightarrow{R} \mathfrak{a}$.

PROOF. According to the preceding theorem it suffices to show strong convergence of the pseudoresolvent in one point λ_0.
We can think of $(R(\lambda, \mathfrak{a}_n)x \,|\, x)$ as a semibounded symmetric form in x and thus by polarization obtain the convergence

$$(R(\lambda, \mathfrak{a}_n)x \,|\, y) \to (R(\lambda, \mathfrak{a})x \,|\, y)$$

for all $x, y \in H$ and all $\lambda \in \Omega_0$. That is we have $R(\lambda, \mathfrak{a}_n)x \rightharpoonup R(\lambda, \mathfrak{a})x$.
It is well known that in Hilbert spaces norm convergence is equivalent to weak convergence and convergence of norms. Thus we are done if we can show for one λ_0 that $\|R(\lambda_0, \mathfrak{a}_n)x\| \to \|R(\lambda_0, \mathfrak{a})x\|$ for all $x \in H$.

Fix $\lambda_0 \in \Omega_0 \subset \mathbb{R}$ and $x \in H$ and define $f_n(\lambda) = (R(\lambda_0, \mathfrak{a}_n)x \,|\, R(\lambda, \mathfrak{a}_n)x)$. We use the resolvent equation and the hypothesis and obtain:

$$\begin{aligned} f_n(\lambda) &= (R(\lambda, \mathfrak{a}_n)R(\lambda_0, \mathfrak{a}_n)x \,|\, x) \quad \text{by selfadjointness} \\ &= \frac{1}{\lambda_0 - \lambda}((R(\lambda, \mathfrak{a}_n) - R(\lambda_0, \mathfrak{a}_n))x \,|\, x) \\ &\to \frac{1}{\lambda_0 - \lambda}((R(\lambda, \mathfrak{a}) - R(\lambda_0, \mathfrak{a}))x \,|\, x) \\ &= (R(\lambda_0, \mathfrak{a})x \,|\, R(\lambda, \mathfrak{a})x) =: f(\lambda) \end{aligned}$$

for all $\lambda \in \Omega_0$. Since furthermore f_n is holomorphic and locally bounded we obtain from Vitali's Theorem and the uniqueness of holomorphic functions that

$$f_n(\lambda) \to f(\lambda) \qquad \forall \lambda \in \mathbb{C} \setminus (-\infty, \gamma]$$

In particular we have

$$f_n(\lambda_0) = \|R(\lambda_0, \mathfrak{a}_n)x\|^2 \to f(\lambda_0) = \|R(\lambda_0, \mathfrak{a})x\|^2$$

This finishes the proof. □

We now prove, that (as for semigroups) convergence in the strong resolvent sense and convergence of the degenerate semigroup are equivalent. We have:

THEOREM 2.3. *Let $\mathfrak{a}_n$ be a sequence of closed sectorial forms with $\Theta(\mathfrak{a}_n) \subset \Sigma_\gamma(\theta)$ for all n. Then $\mathfrak{a}_n \xrightarrow{R} \mathfrak{a}$ if and only if $e^{-z\mathfrak{a}_n}x \to e^{-z\mathfrak{a}}x$ for all z with $|\arg z| \le \frac{\pi}{2} - \theta$ and all $x \in H$.*

PROOF. Assume without loss that $\gamma = 0$.

First assume that $\mathfrak{a}_n \xrightarrow{R} \mathfrak{a}$.
Fix z and x and let Γ be the positively oriented boundary of $B(0, \delta_1) \cup (-\Sigma(\theta + \delta_2))$ where δ_1 and δ_2 are chosen so that z is on the outside of Γ. Define $C = \int_\Gamma |e^{-z\xi}|\,|d\xi| < \infty$. Let $\varepsilon > 0$ be given. By 1.14 there exists some $M > 0$ such that $\|R(\lambda, \mathfrak{a})\| \le \frac{M}{|\lambda|}$ for all $\lambda \in \Gamma$. Pick some $R > \frac{4 \cdot C \cdot M \|x\|}{\varepsilon}$. By Theorem 2.1 the pseudoresolvents converge *uniformly* on $\Gamma \cap \overline{B(0,R)}$. So we find some $n_0 \in \mathbb{R}$ such that $\|R(\lambda, \mathfrak{a}_n)x - R(\lambda, \mathfrak{a})x\| \le \frac{\varepsilon}{2 \cdot C}$ for all $n \ge n_0$. Denote $\Gamma_1 = \Gamma \cap \overline{B(0,R)}$ and $\Gamma_2 = \Gamma \cap \overline{B(0,R)^C}$
Now we have:

$$
\begin{aligned}
\|e^{-z\mathfrak{a}_n}x - e^{-z\mathfrak{a}}x\| &= \| \textstyle\int_\Gamma e^{z\xi} \left(R(\xi, \mathfrak{a}_n) - R(\xi, \mathfrak{a}) \right) x \, d\xi \| \\
&\le \int_{\Gamma_1} |e^{z\xi}| \cdot \|R(\xi, \mathfrak{a}_n)x - R(\xi, \mathfrak{a})x)\| \, |d\xi| \\
&\quad + \int_{\Gamma_2} |e^{z\xi}| \cdot \|R(\xi, \mathfrak{a}_n)x\| \, |d\xi| \\
&\quad + \int_{\Gamma_2} |e^{z\xi}| \cdot \|R(\xi, \mathfrak{a})x\| \, |d\xi| \\
&< \varepsilon
\end{aligned}
$$

for all $n \ge n_0$ which proves that $e^{-z\mathfrak{a}_n}x \to e^{-z\mathfrak{a}}x$.

The other implication follows since the pseudoresolvents are the Laplace transforms of the degenerate semigroups. ☐

2. Holomorphic Families of Forms

The goal of this section is to show that if a family of forms depends holomorphically on some complex parameter then the associated pseudoresolvents are also analytic with respect to that parameter. We start with bounded sesquilinear forms. The results are then easily generalized to unbounded forms.

DEFINITION. Let $\Omega \subset \mathbb{C}$ be an open set. A bounded family of sesquilinear forms $(\mathfrak{a}(z))_{z \in \Omega}$ is called *holomorphic*, if for every fixed $x \in H$ $\mathfrak{a}(z)[x]$ is an analytic function of z.

By polarization we obtain that if $\mathfrak{a}(z)$ is a holomorphic family of bounded forms, then also $\mathfrak{a}(z)[x, y]$ is a holomorphic function of z for fixed x and y.

PROPOSITION 2.4. *Let* $\mathfrak{a}(z)$, $z \in \Omega \subset \mathbb{C}$ *be a holomorphic family of bounded sesquilinear forms. And let* $A(z) \in \mathcal{L}(H)$ *be the the operator associated with* $\mathfrak{a}(z)$.
Then $A(z)$ *is a holomorphic family of operators. Furthermore if* $\lambda \in \rho(A(z_0))$ *then for* $|z - z_0|$ *small enough* $\lambda \in \rho(A(z))$ *and* $R(\lambda, A(z))$ *is holomorphic in* z.

PROOF. To see that $A(z)$ is holomorphic it suffices to show that $(A(z)x \,|\, y)$ is holomorphic for all $x, y \in H$. This is a consequence of [**4**, prop. A.3]. But we have

$$\frac{1}{z-w}((A(z) - A(w))x \,|\, y) = \tfrac{\mathfrak{a}(z)-\mathfrak{a}(w)}{z-w}\,[x,y]$$

which converges as $z \to w$ by hypothesis.

Now suppose that $\lambda \in \rho(A(z_0))$. We have

$$\begin{aligned}\lambda - A(z) &= \lambda - A(z_0) - (A(z) - A(z_0)) \\ &= \Big[id - (A(z) - A(z_0))R(\lambda, A(z_0))\Big](\lambda - A(z_0))\end{aligned}$$

If we choose $|z - z_0|$ small enough so that

$$\|A(z) - A(z_0)\| < \|R(\lambda, A(z_0))\|$$

then the first factor in the above equation is invertible and we have

$$\Big[id - (A(z) - A(z_0))R(\lambda, A(z_0))\Big]^{-1} = \sum_{k=0}^{\infty} \Big[(A(z) - A(z_0))R(\lambda, A(z_0))\Big]^k$$

Now it follows that $\lambda - A(z)$ is invertible for such z and

$$(\lambda - A(z))^{-1} = R(\lambda, A(z_0)) \sum_{k=0}^{\infty} \Big[(A(z) - A(z_0))R(\lambda, A(z_0))\Big]^k$$

This also shows that $R(\lambda, A(z))$ is a holomorphic function of z. □

Now we talk about unbounded forms. KATO develops in [**10**] a whole theory of holomorpic families of forms and their associated operators. We will only need the analyticity of the pseudoresolvents of the forms and thus only present this result:

THEOREM 2.5. *Let* $\mathfrak{a}(z)$ *for* z *in some open subset* Ω *of* $\mathbb{C}$ *be a family of closed, sectorial forms such that* $D(\mathfrak{a}(z)) \equiv D$ *and* $\mathfrak{a}(z)\,[x]$ *is holomorphic for fixed* $x \in D$[1]. *Then* $R(\lambda, \mathfrak{a}(z))$ *is holomorphic in* z.

PROOF. We may assume that D is dense in H since otherwise everything is just multiplied by the orthogonal projection onto $\overline{D}$, which does not change the analyticity.

[1]Kato calls this a holomorphic family of type (a) and the family of the associated operators is called a holomorphic family of type (B)

Let $z_0 \in \Omega$. Assume without loss that $\operatorname{Re} \mathfrak{a}(z_0) \geq 1$ and let S^2 be the selfadjoint operator associated with $\operatorname{Re} \mathfrak{a}(z_0)$. Let $\lambda \in \rho(A(z_0))$ and define

$$\tilde{\mathfrak{a}}(z)\,[x,y] = \mathfrak{a}(z)\,[S^{-1}x, S^{-1}y] - \lambda(x - S^{-1}x \,|\, y + S^{-1}y)$$

Then $\tilde{\mathfrak{a}}(z)$ is everywhere defined and closable. By proposition 1.7, $\tilde{\mathfrak{a}}(z)$ is bounded and hence by 2.4 associated with a holomorphic family of bounded operators $\tilde{A}(z)$.
But we have that

$$(\mathfrak{a}(z) + \lambda)\,[x,y] = (\tilde{\mathfrak{a}}(z) + \lambda)\,[Sx, Sy]$$

for all $x, y \in D(S) = D(\operatorname{Re} \mathfrak{a}(z_0)) = D$. This implies that $\lambda + A(z) = S(\lambda + \tilde{A}(z))S$ and hence we have that $\lambda + A(z)$ is invertible and

$$(\lambda + A(z))^{-1} = S^{-1}(\lambda + \tilde{A}(z))^{-1}S^{-1}$$

which is holomorphic in z.

□

Now we apply this theorem to a rather special family of forms. But this is a really powerful application of this theory, since it will allow us to generalize results for symmetric forms to sectorial forms.

PROPOSITION 2.6. *Let $\mathfrak{a}$ be a closed sectorial form , say $\Theta(\mathfrak{a}) \subset \Sigma_\gamma(\theta)$ and let*

$$\mathfrak{a}(z) = \operatorname{Re} \mathfrak{a} + z \operatorname{Im} \mathfrak{a} \quad \textit{with} \quad D(\mathfrak{a}(z)) \equiv D(\mathfrak{a})$$

Then for $z \in \Omega = (-\varepsilon, \varepsilon) \times \mathbb{R}$ where $\varepsilon = \frac{1}{1+\tan\theta}$ we have that $\mathfrak{a}(z)$ is a closed, sectorial form. Moreover, for $z \in [-\alpha_0, \alpha_0] \times [-\beta_0, \beta_0]$ where $\alpha_0 < \varepsilon$ and β_0 are fixed positive numbers the family $\mathfrak{a}(z)$ is uniformly sectorial. In particular for z in such a rectangle and suitable λ the pseudoresolvent $R(\lambda, \mathfrak{a}(z))$ is analytic in z.

PROOF. We may assume that $\gamma = 0$. Fix $z = \alpha + i\beta \in [-\alpha_0, \alpha_0] \times [-\beta_0, \beta_0] \subset \Omega$. Then

$$\operatorname{Re} \mathfrak{a}(z) = \operatorname{Re} \mathfrak{a} + \alpha \operatorname{Im} \mathfrak{a} \geq (1 - |\alpha| \cdot \tan\theta) \operatorname{Re} \mathfrak{a} \geq 0$$

so that 0 is a vertex for $\mathfrak{a}(z)$.
Now let $x \in D(\mathfrak{a})$ with $\|x\| = 1$. Then

$$\begin{aligned} \operatorname{Re} \mathfrak{a}(z)\,[x] &= \operatorname{Re} \mathfrak{a}\,[x] + \alpha \operatorname{Im} \mathfrak{a}\,[x] =: u + \alpha v \\ \operatorname{Im} \mathfrak{a}(z)\,[x] &= \beta \operatorname{Im} \mathfrak{a}\,[x] = \beta v \end{aligned}$$

Hence we obtain that

$$\begin{aligned} \left| \frac{\operatorname{Im} \mathfrak{a}(z)\,[x]}{\operatorname{Re} \mathfrak{a}(z)\,[x]} \right| &= \left| \frac{\beta v}{u + \alpha v} \right| \\ &= \left| \frac{v}{u} \right| \cdot \left| \frac{\beta}{1 + \alpha \frac{v}{u}} \right| \\ &\leq \tan\theta \cdot \frac{|\beta_0|}{1 - |\alpha_0| \tan\theta} \end{aligned}$$

This shows that $\mathfrak{a}(z)$ is uniformly sectorial in this rectangle with vertex 0 and corresponding semingle

$$\tilde{\theta} = \arctan\left(\tan\theta\frac{|\beta_0|}{1-|\alpha_0|\tan\theta}\right)$$

Since the family is clearly analytic in z the rest of the assertions follow from Theorem 2.5 $\square$

3. Convergence Theorems for Forms

We begin this section with theorems about monotone convergence of symmetric forms. Such theorems are are easier obtained than theorems about sectorial forms. This is mainly due to the following fact:

PROPOSITION 2.7. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be closed symmetric forms. Then:*

$$\mathfrak{a} \le \mathfrak{b} \quad \Longleftrightarrow \quad (1+\mathfrak{b})^{-1} \le (1+\mathfrak{a})^{-1}\,.$$

PROOF. Without loss of generality, we may assume that $\mathfrak{a}, \mathfrak{b} \ge 0$. First, suppose that $\mathfrak{a} \le \mathfrak{b}$.
For $x \in H$ we have $(1+\mathfrak{b})^{-1} \in D(B) \subset D(\mathfrak{b}) \subset D(\mathfrak{a})$. Hence

$$\begin{aligned}((1+\mathfrak{b})^{-1}x\,|\,x) &= \left((1+\mathfrak{a})^{\frac{1}{2}}(1+\mathfrak{b})^{-1}x\,\middle|\,(1+\mathfrak{a})^{-\frac{1}{2}}x\right)\\ &\le \|(1+\mathfrak{a})^{-\frac{1}{2}}x\|\cdot\|(1+\mathfrak{a})^{\frac{1}{2}}(1+\mathfrak{b})^{-1}x\|\\ &= ((1+\mathfrak{a})^{-1}x\,|\,x)^{\frac{1}{2}}((1+\mathfrak{a})(1+\mathfrak{b})^{-1}x\,|\,(1+\mathfrak{b})^{-1}x)^{\frac{1}{2}}\\ &\le ((1+\mathfrak{a})^{-1}x\,|\,x)^{\frac{1}{2}}((1+\mathfrak{b})(1+\mathfrak{b})^{-1}x\,|\,(1+\mathfrak{b})^{-1}x)^{\frac{1}{2}}\\ &= ((1+\mathfrak{a})^{-1}x\,|\,x)^{\frac{1}{2}}((1+\mathfrak{b})^{-1}x\,|\,x)^{\frac{1}{2}}\end{aligned}$$

This proves $((1+\mathfrak{b})^{-1}x\,|\,x)^{\frac{1}{2}} \le ((1+\mathfrak{a})^{-1}x\,|\,x)^{\frac{1}{2}}$ which implies $(1+\mathfrak{b})^{-1} \le (1+\mathfrak{a})^{-1}$.

Now suppose $(1+\mathfrak{b})^{-1} \le (1+\mathfrak{a})^{-1}$.
If $x \in \overline{D(\mathfrak{a})}^{\perp}$, then $0 = ((1+\mathfrak{a})^{-1}x\,|\,x) \ge ((1+\mathfrak{b})^{-1}x\,|\,x)$, so we obtain that $x \in \overline{D(\mathfrak{b})}^{\perp}$.
This implies that $\overline{D(\mathfrak{b})} \subset \overline{D(\mathfrak{a})}$.
Thus, for $x \in D(\mathfrak{a})$, we have that

$$\begin{aligned}\|(1+\mathfrak{b})^{-\frac{1}{2}}(1+\mathfrak{a})^{\frac{1}{2}}x\|^2 &= \left((1+\mathfrak{b})^{-1}(1+A)^{\frac{1}{2}}x\,\middle|\,(1+A)^{\frac{1}{2}}x\right)\\ &\le \left((1+\mathfrak{a})^{-1}(1+A)^{\frac{1}{2}}x\,\middle|\,(1+A)^{\frac{1}{2}}x\right) = \|x\|^2\,.\end{aligned}$$

Hence $(1+\mathfrak{b})^{-\frac{1}{2}}(1+A)^{\frac{1}{2}} : D(\mathfrak{a}) \to \overline{D(\mathfrak{b})} \subset \overline{D(\mathfrak{a})}$ is a bounded operator with respect to the $\|\cdot\|$ - norm. By density it may be extended to $\overline{D(\mathfrak{a})}$. Denote this extension by T.
Now let $x \in D(\mathfrak{b})$. Then $x = (1+\mathfrak{b})^{-\frac{1}{2}}y$ for some $y \in \overline{D(\mathfrak{b})} \subset \overline{D(\mathfrak{a})}$. Now

$$\left((1+A)^{\frac{1}{2}}z\,\middle|\,x\right) = (Tz\,|\,y) = (z\,|\,T^*y) \quad \forall\, z \in D(\mathfrak{a})$$

which shows that $x \in D((1+A)^{\frac{1}{2}^*}) = D(\mathfrak{a})$ and thus $D(\mathfrak{b}) \subset D(\mathfrak{a})$.
Furthermore for $x \in D(\mathfrak{b})$ we have $x = (1+\mathfrak{b})^{-\frac{1}{2}}y$ for some y and

$$(1+\mathfrak{a})\,[x] = \left\|(1+A)^{\frac{1}{2}}(1+\mathfrak{b})^{-\frac{1}{2}}y\right\|^2 \leq \|y\|^2 = (1+\mathfrak{b})\,[x]$$

□

Now we come to the first convergence theorem for forms:

THEOREM 2.8. *Let $\mathfrak{a}_n$ be an increasing sequence of closed, symmetric forms that are uniformly bounded from below, say by γ. Define*

$$D(\mathfrak{a}) = \{x \in \bigcap D(\mathfrak{a}_n) \,:\, \sup \mathfrak{a}_n\,[x] < \infty\} \quad \mathfrak{a}\,[x,y] := \lim \mathfrak{a}_n\,[x,y]$$

where the last expression exists by polarization.
Then $\mathfrak{a}$ is closed and $\mathfrak{a}_n \xrightarrow{R} \mathfrak{a}$.
Moreover, if $\mathfrak{a}_n \leq \mathfrak{a}_0$ for all n then the domain of $\mathfrak{a}$ contains $D(\mathfrak{a}_0)$.

PROOF. We may assume for simplicity that $\gamma = 0$.
As a concequence of the Cauchy Schwarz inequality, we obtain that $\mathfrak{a}_n\,[x+y]$ is bounded if $\mathfrak{a}_n\,[x]$ and $\mathfrak{a}_n\,[y]$ are. Thus, in particular, $D(\mathfrak{a})$ is a vectorspace. Since all forms $\mathfrak{a}_n$ are closed, we have that $\mathfrak{a}_n \leq \mathfrak{a}_r$ and thus also $\mathfrak{a} \leq \mathfrak{a}_r$ from which we conclude that $\mathfrak{a} = \mathfrak{a}_r$ is closable. We prove that it is actually closed:
Let x_n be a $\|\cdot\|_{\mathfrak{a}}$- Cauchy sequence. Then x_n converges to some x in H. Since $\|\cdot\|_{\mathfrak{a}_m} \leq \|\cdot\|_{\mathfrak{a}}$ we have that x_n is a $\|\cdot\|_{\mathfrak{a}_m}$-Cauchy sequence for every m so that we conclude that $x \in D(\mathfrak{a}_m)$ for every m since $\mathfrak{a}_m$ is closed.

Now, we have that

$$\begin{aligned} \sup_m \mathfrak{a}_m\,[x] &= \sup_m \lim_n \mathfrak{a}_m\,[x_n] \\ &\leq \sup_m \sup_n \mathfrak{a}_m\,[x_n] \\ &= \sup_n \sup_m \mathfrak{a}_m\,[x_n] \\ &= \sup_n \mathfrak{a}\,[x_n] < \infty\,, \end{aligned}$$

where $\sup \mathfrak{a}\,[x_n] < \infty$ holds since x_n is a $\|\cdot\|_{\mathfrak{a}}$ -Cauchy sequence and hence $\|\cdot\|_{\mathfrak{a}}$-bounded and since $\mathfrak{a}\,[\cdot]^{\frac{1}{2}} \leq \|\cdot\|_{\mathfrak{a}}$. This implies that $x \in D(\mathfrak{a})$. Now look at $y_n = x_n - x \in D(\mathfrak{a})$. We have that $y_n \xrightarrow{\mathfrak{a}} 0$ and now since $\mathfrak{a}$ is closable 1.9 implies $\|y_n\|_{\mathfrak{a}} \to 0$ proving that $\mathfrak{a}$ is closed.

By 2.7 we have that $((1+\mathfrak{a}_n)^{-1}x\,|\,x)$ is decreasing and that $((1+\mathfrak{a}_n)^{-1}x\,|\,x) \geq ((1+\mathfrak{a})^{-1}x\,|\,x)$ for all n. So

$$\mathfrak{b}\,[x] = \lim\,((1+\mathfrak{a}_n)^{-1}x\,|\,x) \geq ((1+\mathfrak{a})^{-1}x\,|\,x)$$

exists. $\mathfrak{b}$ is a bounded symmetric form. Thus it is associated with a bounded selfadjoint operator B. We have the decomposition $H = \ker B \oplus \ker B^{\perp}$. Clearly, $B_{|_{\ker B^{\perp}}}$ is injective. But since B is selfadjoint, we have $\ker B^{\perp} = \overline{\mathrm{Rg} B} = \overline{\mathrm{Rg} B_{|_{\ker B^{\perp}}}}$, so that $B_{|_{\ker B^{\perp}}}$ has dense range in $\ker B^{\perp}$. Hence we may think of B as $(1+\mathfrak{c})^{-1}$ where $\mathfrak{c}$ is the closed form associated with the operator $C = (B_{|_{\ker B^{\perp}}})^{-1} - 1$ on the domain $D(C) = \mathrm{Rg}(B)$.
We obtain $\mathfrak{a}_n \leq \mathfrak{c} \leq \mathfrak{a}$ for all n. If we let $n \to \infty$ we obtain $\mathfrak{a} = \mathfrak{c}$. This shows that $((1+\mathfrak{a}_n)^{-1}x \,|\, x) \to ((1+\mathfrak{a})^{-1}x \,|\, x)$ for any $x \in H$. But a similar computation can be carried out for any $\lambda > 0$ instead of 1, so by Theorem 2.2 we obtain $\mathfrak{a}_n \overset{R}{\longrightarrow} \mathfrak{a}$.

If $\mathfrak{a}_n \leq \mathfrak{a}_0$, then we have that $D(\mathfrak{a}_0) \subset D(\mathfrak{a}_n)$ for all n and the sequence $\mathfrak{a}_n[x]$ is bounded for all $x \in D(\mathfrak{a}_0)$. Thus, $D(\mathfrak{a}_0) \subset D(\mathfrak{a})$.

□

Our next theorem deals with decreasing sequences of forms. Here the situation is not as nice as before. We still obtain strong resolvent convergence of the forms, but we cannot conclude that the limiting form is closed. This shows the following

Example:
Let $H = L^2(\mathbb{R})$ and look at the sequence

$$\mathfrak{a}_n[f,g] = n^{-1} \cdot \int_{\mathbb{R}} f'\overline{g'}dx \,+\, f(0)\overline{g(0)} \quad D(\mathfrak{a}_n) = H^1(\mathbb{R})$$

The first part of this form is clearly closed, since the associated inner product is equivalent to that of the Sobolev space H^1. The form $\mathfrak{a}_n$ arises from this form by perturbing with $\mathfrak{b}$, where $\mathfrak{b}[f] = |f(0)|^2$. Similar as in an earlier example, we see that $\mathfrak{b}$ is form-bounded with respect to the first part with bound 0, so that $\mathfrak{a}_n$ is indeed a closed form. Furthermore, the sequence $\mathfrak{a}_n$ is decreasing to $\mathfrak{a}[f,g] = f(0)\overline{g(0)}$. We already saw, that $\mathfrak{a}$ is not closable and thus in particular not closed. However the following theorem shows that the pseudoresolvents converge strongly to 0.

Theorem 2.9. *Let $\mathfrak{a}_n$ be a decreasing sequence of closed, symmetric forms that are uniformly bounded from below, say by γ. Define*

$$\mathfrak{a}[x,y] = \lim \mathfrak{a}_n[x,y] \quad \text{on} \quad D(\mathfrak{a}) = \bigcup D(\mathfrak{a}_n)$$

Then we have $\mathfrak{a}_n \overset{R}{\longrightarrow} \overline{\mathfrak{a}_r}$.

Proof. We suppose again, that $\gamma = 0$. By 2.7 we have that $(1+\mathfrak{a}_n)^{-1}$ is increasing to a limit which we can identify with some pseudoresolvent $(1+\mathfrak{b})^{-1}$ as in the last theorem.
We have that $\mathfrak{b} \leq \mathfrak{a}_n$ for all n and thus we obtain $\mathfrak{b} \leq \mathfrak{a}$ and since $\mathfrak{b}$ is closed $\mathfrak{b} \leq \overline{\mathfrak{a}_r}$. On the other hand, since $\mathfrak{a}_r \leq \mathfrak{a} \leq \mathfrak{a}_n$, we obtain that $\overline{\mathfrak{a}_r} \leq \mathfrak{a}_n$ and thus $(1+\mathfrak{a}_n)^{-1} \leq$

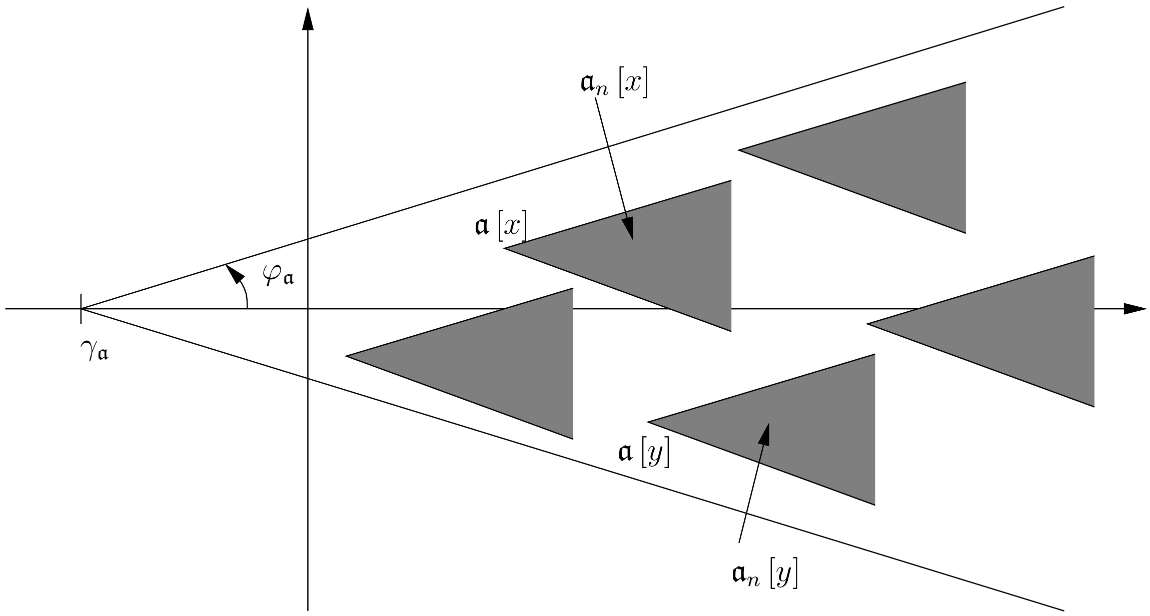

FIGURE 1. Situation in Theorem 2.10

$(1+\overline{\mathfrak{a}_r}\,)^{-1}$ and with $n \to \infty$ we obtain $(1+\mathfrak{b})^{-1} \leq (1+\overline{\mathfrak{a}_r}\,)^{-1}$ and hence $\bar{\mathfrak{a}_r} \leq \mathfrak{b}$. Now again, we can finish the proof by doing this for different λ again and by applying 2.2. □

For sectorial forms we obtain slightly different results. We will generalize the result for increasing sequences with the help of proposition 2.6. For the decreasing case, we present the following result. It does not require the form domains to be increasing, but instead requires the knowledge of a limiting form.

THEOREM 2.10. *Let $\mathfrak{a}_n$ and $\mathfrak{a}$ be closed, sectorial forms satisfying:*

a) *$D(\mathfrak{a}_n) \subset D(\mathfrak{a})$ for all $n \in \mathbb{N}$.*
b) *The forms $\tilde{\mathfrak{a}}_n := \mathfrak{a}_n - \mathfrak{a}$ are uniformly sectorial. That is there exists some $M \geq 1$ such that $|\mathrm{Im}\,\tilde{\mathfrak{a}}_n| \leq M \mathrm{Re}\,\tilde{\mathfrak{a}}_n$*
c) *There is a core D for $\mathfrak{a}$ which is contained in $D(\mathfrak{a}_n)$ from some N on (i.e. $D \subset \underline{\lim} D(\mathfrak{a}_n)$) and $\mathfrak{a}_n\,[x] \to \mathfrak{a}\,[x]$ for all $x \in D$.*

Then $\mathfrak{a}_n \xrightarrow{R} \mathfrak{a}$.

REMARK. Condition b) means that for fixed x all numbers $\mathfrak{a}_n\,[x]$ are contained in the sector $\mathfrak{a}\,[x] + \Sigma(\phi)$. Here ϕ is $\arctan(M-1)$ and thus ϕ is independent of x. In particular the forms $\mathfrak{a}_n$ are uniformly sectorial. See figure 1:
For fixed x $\mathfrak{a}\,[x]$ is the vertex of one of the shaded sectors. The numbers $\mathfrak{a}_n\,[x]$ lie in that sector.

PROOF OF 2.10. We may assume that 0 is a vertex for $\mathfrak{a}$. As a consequence of a) and b) we obtain $0 \leq \mathrm{Re}\,\mathfrak{a} \leq \mathrm{Re}\,\mathfrak{a}_n$. In particular $(1+\mathfrak{a}_n)^{-1} =: R_n$ and $(1+\mathfrak{a})^{-1} =: R$

exist.

For $x \in H$ we have $R_n x \in D(A_n) \subset D(\mathfrak{a}_n) \subset D(\mathfrak{a})$ and

$$\begin{aligned}(1 + \operatorname{Re} \mathfrak{a})\,[R_n x] &\leq (1 + \operatorname{Re} \mathfrak{a}_n)\,[R_n x] \\ &= \operatorname{Re}\,((1 + A_n)R_n x \,|\, R_n x) \\ &= \operatorname{Re}\,(P_{\mathfrak{a}_n} x \,|\, R_n x) \\ &\leq \|x\|^2 .\end{aligned}$$

From this it follows that both $(1+\mathfrak{a})\,[R_n x]$ and $\tilde{\mathfrak{a}}_n x\,[R_n] \leq (1+\mathfrak{a}_n)\,[R_n x]$ are uniformly bounded by $\|x\|^2$.

Now for $x \in H$ and $y \in D$ we have that

$$\begin{aligned}(1+\mathfrak{a})\,[R_n x - Rx, y] &= (1+\mathfrak{a})\,[R_n x, y] - (1+\mathfrak{a})\,[Rx, y] \\ &= (1+\mathfrak{a}_n)\,[R_n x, y] - \tilde{\mathfrak{a}}_n\,[R_n x, y] - (1+\mathfrak{a})\,[Rx, y] \\ &= (P_{\mathfrak{a}_n} x \,|\, y) - \tilde{\mathfrak{a}}_n\,[R_n x, y] - (P_{\mathfrak{a}} x \,|\, y) \\ &= -\tilde{\mathfrak{a}}_n\,[R_n x, y]\end{aligned}$$

The last equality holds for all $n \geq n_0$ where n_0 is chosen such that $y \in D(\mathfrak{a}_n)$ for all $n \geq n_0$ and thus $y = P_{\mathfrak{a}} y = P_{\mathfrak{a}_n} y$.

For such n we have

$$|(1+\mathfrak{a})\,[R_n x - Rx, y]| \leq (1+M)\,\underbrace{\tilde{\mathfrak{a}}_n\,[R_n x]^{\frac{1}{2}}}_{\text{bounded}} \cdot \underbrace{\tilde{\mathfrak{a}}_n\,[y]^{\frac{1}{2}}}_{\to 0} .$$

We have shown that $(1+\mathfrak{a})\,[R_n x - Rx, y] \to 0$ for all $x \in H$ and all $y \in D$. By the density of D in $D(\mathfrak{a})$, this holds for all $y \in D(\mathfrak{a})$.

Now we have

$$\begin{aligned}(1+\mathfrak{a})\,[R_n x - Rx] + \tilde{\mathfrak{a}}_n\,[R_n x] &= (1+\mathfrak{a})\,[R_n x] - (1+\mathfrak{a})\,[R_n x, Rx] \\ &\quad -(1+\mathfrak{a})\,[Rx, R_n x] + (1+\mathfrak{a})\,[Rx] \\ &\quad +(1+\mathfrak{a}_n)\,[R_n x] - (1+\mathfrak{a})\,[R_n x] \\ &= (1+\mathfrak{a})\,[Rx - R_n x, Rx] \\ &\quad - \underbrace{(P_{\mathfrak{a}} x \,|\, R_n x) + (P_{\mathfrak{a}_n} x \,|\, R_n x)}_{=0 \text{ since } R_n x \in D(\mathfrak{a}_n) \subset D(\mathfrak{a})} \\ &\to 0 \quad \text{since}\, Rx \in D(\mathfrak{a})\end{aligned}$$

Taking the real part we see that

$$\underbrace{\operatorname{Re} \mathfrak{a}\,[R_n x - Rx]}_{\geq 0} + \|R_n x - Rx\|^2 + \underbrace{\operatorname{Re} \tilde{\mathfrak{a}}\,[R_n x]}_{\geq 0} \to 0$$

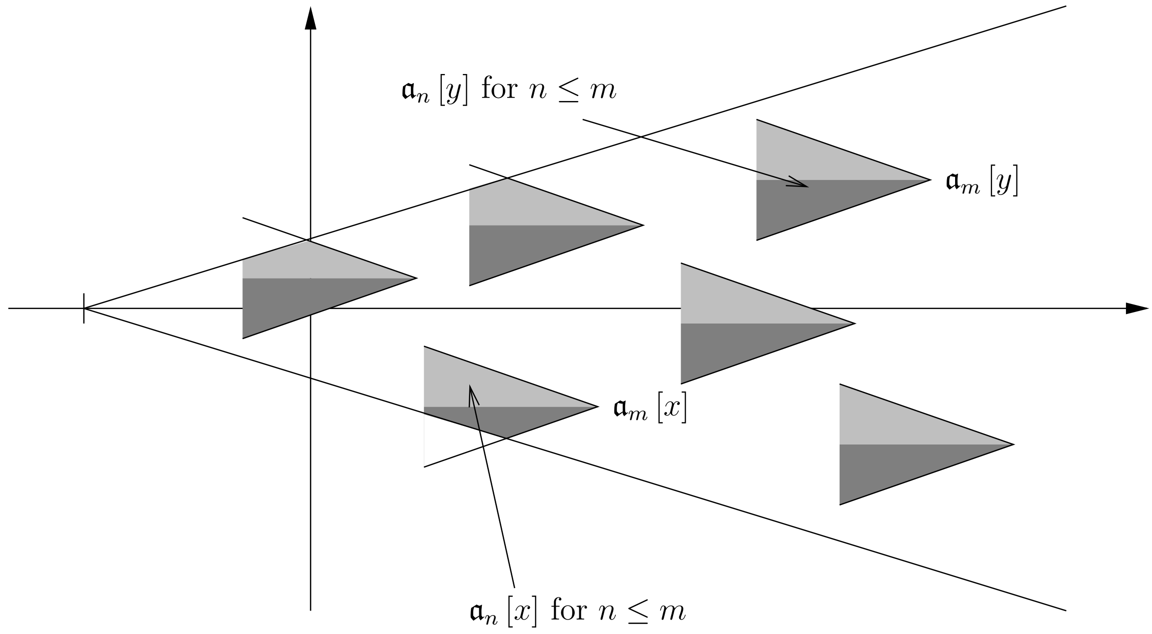

FIGURE 2. Situation in Theorem 2.11

which implies $\|R_n x - Rx\| \to 0$. Now we use Theorem 2.1 to see that $\mathfrak{a}_n \xrightarrow{R} \mathfrak{a}$. $\square$

One often refers to the above convergence as convergence *from above*. Now we show a corresponding theorem for convergence from below.

THEOREM 2.11. *Let $\mathfrak{a}_n$ be a sequence of closed, uniformly sectorial forms satisfying*

a) $D(\mathfrak{a}_{n+1}) \subset D(\mathfrak{a}_n)$

b) *There exists a constant $M > 0$ such that for any $n \in \mathbb{N}, m \geq n$ and any $x \in D(\mathfrak{a}_m)$*

$$0 \leq |\mathrm{Im}\,(\mathfrak{a}_m - \mathfrak{a}_n)\,[x]| \leq M\mathrm{Re}\,(\mathfrak{a}_m - \mathfrak{a}_n)\,[x] \qquad (*)$$

Let $\mathfrak{a} := \lim \mathfrak{a}_n$ with domain $D(\mathfrak{a}) = \{x : \lim \mathfrak{a}_n\,[x]$ exists$\}$. Then $\mathfrak{a}$ is a closed, sectorial form and $D(\mathfrak{a}) = \{x : \lim \mathrm{Re}\,\mathfrak{a}_n\,[x]$ exists$\}$. If one can replace in $(*)$ $|\mathrm{Im}\,(\mathfrak{a}_m - \mathfrak{a}_n)\,[x]|$ by either

$$i) \quad \mathrm{Im}\,(\mathfrak{a}_m - \mathfrak{a}_n)\,[x] \qquad or \qquad ii) \quad \mathrm{Im}\,(\mathfrak{a}_n - \mathfrak{a}_m)\,[x]$$

then in addition $\mathfrak{a}_n \xrightarrow{R} \mathfrak{a}$.

REMARK. Here, condition b) may be interpreted as follows:
For fixed x and m , $\mathfrak{a}_m\,[x]$ is the vertex of one of the shaded sectors. For any $n \leq m$, the number $\mathfrak{a}_n\,[x]$ is then contained in that sector. That condition i) (ii)) is satisfied means that $\mathfrak{a}_n\,[x]$ for such n may only lie in the dark (light) shaded part of this sector.

PROOF. As a consequence of the conditions a) and b) we have that $\mathrm{Re}\,\mathfrak{a}_n$ is an increasing sequence of closed, symmetric forms. By Theorem 2.8 the form $\lim \mathrm{Re}\,\mathfrak{a}_n$ is closed. But as a consequence of condition b) $\mathrm{Im}\,\mathfrak{a}_n$ is a Cauchy sequence if $\mathrm{Re}\,\mathfrak{a}_m$

is. In particular $\lim \mathfrak{a}_n$ exists if and only if $\lim \operatorname{Re} \mathfrak{a}_n$ exists. This proves that $D(\mathfrak{a}) = \{x : \lim \operatorname{Re} \mathfrak{a}_n [x] \text{ exists}\} =: D(\mathfrak{b})$. It follows from Theorem 2.8 , that $\mathfrak{b} := \lim \operatorname{Re} \mathfrak{a}_n$ is a closed, symmetric form. In particular, $D(\mathfrak{a})$ is a vectorspace. On the other hand, it follows from the above, that $\mathfrak{b} = \lim \operatorname{Re} \mathfrak{a}_n = \operatorname{Re} \lim \mathfrak{a}_n = \operatorname{Re} \mathfrak{a}$, which implies that $\mathfrak{a}$ is closed, since a sectorial form is closed if and only if its real part is closed.

Now consider the forms $\mathfrak{a}_n(z) = \operatorname{Re} \mathfrak{a}_n + z \operatorname{Im} \mathfrak{a}_n$ for $n \in \mathbb{N}$ and the form $\mathfrak{a}(z) = \operatorname{Re} \mathfrak{a} + z \operatorname{Im} \mathfrak{a}$. These are for $\operatorname{Re} z < \varepsilon$, holomorphic families of closed, sectorial forms by proposition 2.6. (Note that because of the uniform sectoriality we can always take the same ε .) Furthermore, for fixed z $\mathfrak{a}_n(z)\,[x] \to \mathfrak{a}(z)\,[x]$ for any $x \in D(\mathfrak{a})$.
For $z \in (-\varepsilon, \varepsilon) \times \{0\}$ the forms $\mathfrak{a}_n(z)$ and $\mathfrak{a}(z)$ are symmetric. Now if we assume condition i), then $\mathfrak{a}_n(z)$ is an increasing sequence of forms for $z \in [0, \varepsilon)$, so that by Theorem 2.8 $R(\lambda, \mathfrak{a}_n(z))x \to R(\lambda, \mathfrak{a}(z))x$ for these z, any $x \in H$ and suitable λ. But since the forms $\mathfrak{a}_n(z)$ are uniformly sectorial for any n and any z in a rectangle we obtain from Theorem 1.14 that $R(\lambda, \mathfrak{a}_n(z))$ is normbounded independently of n and z. Now Vitali's Theorem implies that we have this convergence uniformly on any rectangle. By setting $z = i$ we obtain strong convergence of $R(\lambda, \mathfrak{a}_n)$ to $R(\lambda, \mathfrak{a})$, which proves the claimed resolvent convergence.

If we assume condition ii), then for $z \in (-\varepsilon, 0] \times \{0\}$ the sequence $\mathfrak{a}_n(z)$ is increasing, and we obtain the resolvent convergence in the same way.

□

Notes and References for Chapter 2: It is a widely known fact, that the resolvent of a generator of a strongly continuous semigroup is the Laplacetransform of that semigroup. It is thus no surprise, that the convergence of semigroups is equivalent to the convergence of the resolvents. This connection is discussed in ARENDT ET AL [**4**].
The adaption of this to pseudoresolvents and degenerate semigroups causes little trouble. Our presentation is close to ARENDT [**2**].
What makes forms so much easier to handle is the observation, that for symmetric forms weak convergence of the resolvent suffices to imply strong convergence. This fact is mentioned in REED, SIMON [**16**, chapter VIII-7]. We note, that this fact is used explicitly in the proof of both convergence theorems for symmetric forms and thus implicitly in the proof of Theorem 2.11.

The goal of section 2 is to establish proposition 2.6, which goes back to an idea of SIMON which appears in [**11**, addendum]. This is the right tool to generalize results for symmetric forms to sectorial forms. This is illustrated in the proof of Theorem 2.11 and it will be used again in the next chapter in the proof of Trotter's formula. The rest of section 2 is an excerpt of KATO [**10**, chapter 7-4].

The first part of section 3 concerning convergence of symmetric forms is close to SIMON [**17**] and was adapted to the nondensely defined case. A different proof of these theorems using convex, lower semicontinuous functionals may be found in REED, SIMON [**16**, supplement to VIII.7]. The theorem concerning convergence from above may be found in KATO [**10**, chapter 8-3], whereas the theorem concerning convergence from below was presented in OUHABAZ [**15**] under the assumption, that the limiting form $\mathfrak{a}$ is known to be closed and either condition i) or ii) is satisfied.

CHAPTER 3

Trotter's Product Formula

Let us look again at the Cauchy Problem

$$(CP)_A \begin{cases} u'(t) &= Au(t) \\ u(0) &= u_0 \end{cases}$$

When A is a complex number the solution is given by $u(t) = T_A(t)u_0$ where $T_A(t) = e^{tA}$. By the functional equation of the exponential function we have $T_A(t+s) = T_A(t)T_A(s)$. The theory of abstract Cauchy problems and semigroups yields the same results on Banach spaces provided that A is a generator.
In case of complex numbers, we could also interpret the functional equation as follows:

$$T_{A+B}(t) = T_A(t)T_B(t)$$

and thus we may compute the solution of the "sum-problem" $(CP)_{A+B}$ from the solutions of the problems $(CP)_A$ and $(CP)_B$. This is not true in a more general setting, even if A and B are matrices the above equation does not hold in general:

Example: Consider the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Then, the solutions of the associated Cauchy problems are given by

$$T_A(t) = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} T_B(t) = \begin{pmatrix} 0 & 0 \\ -t & 0 \end{pmatrix}$$

We then have

$$T_A(t)T_B(t) = \begin{pmatrix} -t^2 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = T_{A+B}(t)$$

However, if A and B are bounded operators that *commute*, then this formula remains valid. This follows by computing the Cauchy product of the power series expansion for e^{tA} and e^{tB}.
More general, we could ask the following question:
Is it possible to compute the solution of $(CP)_{A+B}$ from those for $(CP)_A$ and $(CP)_B$ for general A and B? A possible approach to answer this question is given by the Lie product formula:

THEOREM 3.1. *(Lie)*
If $A, B \in \mathbb{C}^{d\times d}$, then

$$e^{t(A+B)} = \lim_{n\to\infty} \left(e^{\frac{t}{n}A}e^{\frac{t}{n}B}\right)^n \qquad (*)$$

However for unbounded operators even the meaning of $A + B$ is not clear. It may happen, that even though A and B are generators $D(A)\cap D(B)$ - which is the natural domain for the sum - is not dense, so that $A+B$ cannot be the generator of a semigroup. However the limit in $(*)$ may exist even if $A + B$ is no generator.
In fact CHERNOFF investigates these phenomena in [**7**]. He proves that if the limit $(*)$ exists, it always has the semigroup property (see [**7**, 2.5.3]) and uses this process to define a generalized addition process $+_L$.
But this generalized Lie-addition has several weaknesses:

- It is not applicable to all generators. CHERNOFF even proves that the only universally addable operators are bounded (see [**7**, §6]).
- It is not associative ([**7**, prop. 5.3]).

In the case of sesquilinear forms these problems do not occur. Addition is not a problem but it may happen that $e^{t(\mathfrak{a}+\mathfrak{b})}$ is degenerate even if the operators associated with the two forms $\mathfrak{a}$ and $\mathfrak{b}$ generate semigroups on H. However the Trotter product formula for forms respects this. In fact we are going to prove:

THEOREM 3.2. *(Trotter's product formula)*
Let $\mathfrak{a}$ and $\mathfrak{b}$ be closed sectorial forms. Then

$$s-\lim_{n\to\infty} \left(e^{-\frac{t}{n}\mathfrak{a}}e^{\frac{t}{n}\mathfrak{b}}\right)^n = e^{-t(\mathfrak{a}+\mathfrak{b})}$$

for all $t > 0$, uniformly on compact subsets of $(0,\infty)$.

In the last section, we will see, that in fact Trotter's formula as stated here characterizes operators that are associated with sectorial forms entirely. This characterisation is due to MATOLCSI.

1. The Proof of Trotter's Formula

A proof of this formula was given in KATO [**11**] under the assumption that both $\mathfrak{a}$ and $\mathfrak{b}$ are densely defined. We imitate this proof and work in several steps:

Step 1 We prove an appropriate resolvent convergence under the assumption that both $\mathfrak{a}$ and $\mathfrak{b}$ are symmetric.
Step 2 We use this result to prove the theorem for symmetric forms.
Step 3 We use the theorem for symmetric forms to prove the general result.

PROOF OF THEOREM 3.2, PART 1.
Let $\mathfrak{a}$ and $\mathfrak{b}$ be closed symmetric forms. By adding suitable constants, we may assume

that $\mathfrak{a}$ and $\mathfrak{b}$ be positive. Let $\mathfrak{c} = \mathfrak{a} + \mathfrak{b}$. Define for $x \in H$

$$S(t) = \frac{1}{t}(id - e^{-t\mathfrak{a}}e^{-t\mathfrak{b}})$$

In this first part of the proof, we show that for all $x \in H$

$$(1 + S(t))^{-1}x \to (1 + \mathfrak{c})^{-1}x \quad \text{as} \quad t \to 0$$

which implies by 2.1 that the (bounded) symmetric forms associated with $S(t)$ converge to $\mathfrak{c}$ in the strong resolvent sense.

Let $A(t) = t^{-1}(id - e^{-t\mathfrak{a}})$, $B(t) = t^{-1}(id - e^{-t\mathfrak{b}})$ and $C(t) = A(t) + B(t)$. We now rewrite $1 + S(t)$ as

$$\begin{aligned} 1 + S(t) &= 1 + C(t) - tA(t)B(t) \\ &= (1 + C(t))^{\frac{1}{2}}(1 - Q(t))(1 + C(t))^{\frac{1}{2}} \end{aligned}$$

where $Q(t) = t(1 + C(t))^{-\frac{1}{2}}A(t)B(t)(1 + C(t))^{-\frac{1}{2}}$.
Lemma 3.3 shows that $1 - Q(t)$ is invertible. Thus we obtain

$$\begin{aligned} (1 + S(t))^{-1}x &= (1 + C(t))^{-\frac{1}{2}}(1 - Q(t))^{-1}(1 + C(t))^{-\frac{1}{2}}x \\ &= (1 + C(t))^{-1}x + (1 + C(t))^{-\frac{1}{2}}(1 - Q(t))^{-1}Q(t)(1 + C(t))^{-\frac{1}{2}}x \quad (*) \end{aligned}$$

As a consequence of Theorem 1.26 we have that $(C(t)x \,|\, y)$ converges as $t \to 0$ if and only if $x, y \in D(\mathfrak{c})$ and that the limit is $\mathfrak{c}\,[x, y]$. By the spectral Theorem $(C(t)x \,|\, x)$ is increasing as t decreases, so that by Theorem 2.8 $(1+C(t))^{-1}x \to (1+\mathfrak{c})^{-1}x$ as $t \to 0$.

Thus it remains to show that the second summand in $(*)$ converges to 0. But according to Lemma 3.3 $(1+C(t))^{-\frac{1}{2}}(1-Q(t))^{-1}$ is bounded independently of t, so it suffices to show that

$$Q(t)(1 + C(t))^{-\frac{1}{2}} = t(1 + C(t))^{-\frac{1}{2}}A(t)B(t)(1 + C(t))^{-1}$$

converges to 0.
But $\|(1 + C(t))^{-\frac{1}{2}}A(t)^{\frac{1}{2}}\| = \|A(t)^{\frac{1}{2}}(1 + C(t))^{-\frac{1}{2}}\| \leq 1$ by Lemma 3.3[1] . We also have that $tA(t)^{\frac{1}{2}}B(t)^{\frac{1}{2}}$ converges strongly to $P_{\mathfrak{a}}P_{\mathfrak{b}}$. By the Banach-Steinhaus Theorem, this convergence is uniform on compact subsets of H. Now Lemma 3.4 shows that $B(t)^{\frac{1}{2}}(1 + C(t))^{-1}x \to B^{\frac{1}{2}}(1 + \mathfrak{c})^{-1}x$ which is an element of $\overline{D(\mathfrak{b})}$. These three facts imply the required convergence and finish the proof.

□

LEMMA 3.3. *$(1 - Q(t))$ is invertible and* $\|(1 - Q(t))^{-1}\| \leq 2$.
Furthermore $\|A(t)^{\frac{1}{2}}(1 + C(t))^{-\frac{1}{2}}\| \leq \frac{1}{2}$ *and* $\|B(t)^{\frac{1}{2}}(1 + C(t))^{-\frac{1}{2}}\| \leq \frac{1}{2}$.

[1]Note that the first operator is the adjoint of the second one

PROOF. It suffices to show that $\Theta\left(Q(t)\right) \subset \overline{B(0,\frac{1}{2})}$, since then $\operatorname{dist}\Big(1, \Theta\left((Q(t)\right)\Big) \geq \frac{1}{2}$ and 1.14 shows that $1 \in \rho(Q(t))$ and $\|(1-Q(t))^{-1}\| \leq 2$.

For $x \in H$ we have:

$$\begin{aligned}
|(Q(t)x \,|\, x)| &= \left|\left(t(1+C(t))^{-\frac{1}{2}}A(t)B(t)(1+C(t))^{-\frac{1}{2}}x \,\middle|\, x\right)\right| \\
&= \left|\left(tA(t)^{\frac{1}{2}}B(t)^{\frac{1}{2}}B(t)^{\frac{1}{2}}(1+C(t))^{-\frac{1}{2}}x \,\middle|\, A(t)^{\frac{1}{2}}(1+C(t))^{-\frac{1}{2}}x\right)\right| \\
&\leq \underbrace{\|tA(t)^{\frac{1}{2}}B(t)^{\frac{1}{2}}\|}_{\leq 1} \cdot \|B(t)^{\frac{1}{2}}(1+C(t))^{-\frac{1}{2}}x\| \cdot \|A(t)^{\frac{1}{2}}(1+C(t))^{-\frac{1}{2}}x\| \\
&\leq \frac{1}{2}\left(\|B(t)^{\frac{1}{2}}(1+C(t))^{-\frac{1}{2}}x\|^2 + \|A(t)^{\frac{1}{2}}(1+C(t))^{-\frac{1}{2}}x\|^2\right) \\
&= \frac{1}{2}\left(\left(B(t)(1+C(t))^{-\frac{1}{2}}x \,\middle|\, (1+C(t))^{-\frac{1}{2}}x\right)\right. \\
&\quad \left. + \left(A(t)(1+C(t))^{-\frac{1}{2}}x \,\middle|\, (1+C(t))^{-\frac{1}{2}}x\right)\right) \\
&= \frac{1}{2}(C(t)(1+C(t))^{-1}x \,|\, x) \\
&= \frac{1}{2}\Big(\|x\|^2 - ((1+C(t))^{-1}x \,|\, x)\Big) \leq \frac{\|x\|^2}{2}
\end{aligned}$$

Here we used the selfadjointness of $A(t)$ and $(1+C(t))^{-\frac{1}{2}}$ in the second and the Cauchy-Schwarz inequality in the third step. The fourth inequality uses the fact that $2\alpha\beta \leq \alpha^2+\beta^2$. In the last equations we use again selfadjointness and the fact that $C(t)$ and $(1+C(t))^{-\frac{1}{2}}$ commute. $\square$

LEMMA 3.4. *For every $x \in H$*

$$\begin{aligned}
A(t)^{\frac{1}{2}}(1+C(t))^{-1}x &\to A^{\frac{1}{2}}(1+\mathfrak{c})^{-1}x \\
\textit{and} \quad B(t)^{\frac{1}{2}}(1+C(t))^{-1}x &\to B^{\frac{1}{2}}(1+\mathfrak{c})^{-1}x\,.
\end{aligned}$$

PROOF.

a) By a similar manipulation as in the proof of the last lemma, we obtain that

$$\begin{aligned}
&\|(1+C(t))^{-1}x\|^2 + \|A(t)^{\frac{1}{2}}(1+C(t))^{-1}x\|^2 + \|B(t)^{\frac{1}{2}}(1+C(t)^{-1}x\|^2 \\
=\ &((1+C(t))^{-1}x \,|\, x) \leq \|x\|^2\,.
\end{aligned}$$

Thus since H is reflexive for any given sequence $t_n \to 0$ we may extract a subsequence t_k such that $y_a(t_k) := A(t_k)^{\frac{1}{2}}(1+C(t_k))^{-1}x$ and $y_b(t_k) := B(t_k)^{\frac{1}{2}}(1+C(t_k))^{-1}$ are weakly convergent, say $y_a(t_k) \rightharpoonup y_a$ and $y_b(t_k) \rightharpoonup y_b$.

b) We clearly have that $P_{\mathfrak{a}} y_a(t_k) \rightharpoonup P_{\mathfrak{a}} y_a$ and similar for indices b. But on the other hand for $u \in D(\mathfrak{a}) = D(A^{\frac{1}{2}})$

$$\begin{aligned} (y_a(t_k) \,|\, u) &= \left((1+C(t_k))^{-1} x \,\middle|\, A(t_k)^{\frac{1}{2}} u \right) \\ &\to \left((1+\mathfrak{c})^{-1} x \,\middle|\, A^{\frac{1}{2}} u \right) = \left(A^{\frac{1}{2}} (1+\mathfrak{c})^{-1} x \,\middle|\, u \right), \end{aligned}$$

since $(1+C(t))^{-1}x \to (1+\mathfrak{c})^{-1}x \in D(\mathfrak{c}) \subset D(\mathfrak{a})$ and since for $u \in D(A^{\frac{1}{2}})$ we have $A(t)^{\frac{1}{2}} u \to A^{\frac{1}{2}} u$.
Now by density of $D(A^{\frac{1}{2}})$ in $\overline{D(\mathfrak{a})} = \mathrm{Rg} P_{\mathfrak{a}}$ we conclude that
$P_{\mathfrak{a}} y_a = A^{\frac{1}{2}} (1+\mathfrak{c})^{-1} x$ and similar $P_{\mathfrak{b}} y_b = B^{\frac{1}{2}} (1+\mathfrak{c})^{-1} x$.

c) We know that $((1+C(t))^{-1}x \,|\, x) \to ((1+\mathfrak{c})^{-1}x \,|\, x)$ and since $\mathrm{Rg}(1+\mathfrak{c})^{-1} \subset \mathrm{Rg} P_{\mathfrak{c}}$ we can split up the limit as:

$$\begin{aligned} ((1+\mathfrak{c})^{-1}x \,|\, x) &= ((1+\mathfrak{c})^{-1}x \,|\, P_{\mathfrak{c}} x) \\ &= ((1+\mathfrak{c})^{-1}x \,|\, (1+C)(1+\mathfrak{c})^{-1}x) \\ &= \|(1+\mathfrak{c})^{-1}x\|^2 + \|A^{\frac{1}{2}}(1+\mathfrak{c})^{-1}x\|^2 + \|B^{\frac{1}{2}}(1+\mathfrak{c})^{-1}x\|^2 \\ &= \|(1+\mathfrak{c})^{-1}x\|^2 + \|P_{\mathfrak{a}} y_a\|^2 + \|P_{\mathfrak{b}} y_b\|^2 \end{aligned}$$

Now we use the first equality from a) and $(1+C(t))^{-1}x \to (1+\mathfrak{c})^{-1}x$ and obtain

$$\|y_a(t_k)\|^2 + \|y_b(t_k)\|^2 \to \|P_{\mathfrak{a}} y_a\|^2 + \|P_{\mathfrak{b}} y_b\|^2 \le \|y_a\|^2 + \|y_b\|^2$$

d) Now we exploit what we have done so far:
From the weak convergence it follows that

$$\|y_a\| \le \underline{\lim} \|y_a(t_k)\| \quad \text{and} \quad \|y_b\| \le \underline{\lim} \|y_b(t_k)\|$$

Thus given $\varepsilon > 0$ for all but finitely many k we have

$$\|y_a(t_k)\|^2 \ge \|y_a\|^2 - \varepsilon \quad \text{and} \quad \|y_b(t_k)\|^2 \ge \|y_b\|^2 - \varepsilon$$

thus for these k we have

$$\|y_a(t_k)\|^2 + \|y_b(t_k)\|^2 \ge \|y_a\|^2 + \|y_b\|^2 - 2\varepsilon \ge \|P_{\mathfrak{a}} y_a\|^2 + \|y_b\|^2 - 2\varepsilon$$

If we now let $k \to \infty$ we obtain using c):

$$\|P_{\mathfrak{a}} y_a\|^2 + \|y_b\|^2 - 2\varepsilon \le \|P_{\mathfrak{a}} y_a\|^2 + \|P_{\mathfrak{b}} y_b\|^2 \le \|P_{\mathfrak{a}} y_a\|^2 + \|y_b\|^2$$

Since $\varepsilon > 0$ was arbitrary, the last inequality is actually an equality and $\|P_{\mathfrak{b}} y_b\|^2 = \|y_b\|^2$ and hence $P_{\mathfrak{b}} y_b = y_b$. Similar $P_{\mathfrak{a}} y_a = y_a$.
Altogether we have showed that

$$\|y_a(t_k)\|^2 + \|y_b(t_k)\|^2 \to \|y_a\|^2 + \|y_b\|^2$$

e) First we show strong convergence as claimed along the sequence t_k. In fact we have:

$$\begin{aligned} & \|y_a(t_k) - y_a\|^2 + \|y_b(t_k) - y_b\|^2 \\ = \quad & \|y_a(t_k)\|^2 + \|y_a\|^2 - (y_a(t_k) \mid y_a) - (y_a \mid y_a(t_k)) \\ & + \|y_b(t_k)\|^2 + \|y_b\|^2 - (y_b(t_k) \mid y_b) - (y_b \mid y_b(t_k)) \\ \longrightarrow \quad & 0 \quad \text{by d) and a)} \end{aligned}$$

But now we see that this convergence takes place as $t \to 0$ without any restriction to a sequence, for otherwise, we would find $\varepsilon > 0$ and a sequence $t_n \to 0$ such that $\|A^{\frac{1}{2}}(1 + C(t_n))^{-1}x - A^{\frac{1}{2}}(1 + \mathfrak{c})^{-1}x\| \geq \varepsilon$ for all n. But then by what we have done, we could extract a subsequence of t_n such that the norm goes to 0 – a contradiction.

□

This resolvent convergence would finish the proof if $\mathfrak{c}$ was densely defined, by a theorem of Chernoff [**7**]. We will now generalize this theorem to symmetric forms which need *not* be densely defined.

Theorem 3.5. *(Chernoff product formula)*
Let $F(t)$ for $t \in (0, \infty)$ be a family of selfadjoint, positive contractions and define $A(t) = t^{-1}(id - F(t))$. Assume that there exits a closed symmetric form $\mathfrak{a}$ such that $(1+A(t))^{-1}$ converges strongly to $(1+\mathfrak{a})^{-1}$. Then $F(\frac{t}{n})^n \to e^{-t\mathfrak{a}}$ strongly and uniformly on intervals of the form $[t_0, t_1]$.

The proof uses the following lemma. For the proof of the lemma we refer to Engel-Nagel [**9**, III, Lemma 5.1]

Lemma 3.6. *Let X be a Banach space, $T \in \mathcal{L}(X)$ satisfying $\|T^m\| \leq M$ for some $M \geq 1$ and all $m \in \mathbb{N}$. Then for every $n \in \mathbb{N}$ and $x \in X$*

$$\|e^{n(T-id)}x - T^n x\| \leq \sqrt{n} M \|Tx - x\|$$

Proof of 3.5. As a consequence of 2.1 we obtain that $A(t) \xrightarrow{R} \mathfrak{a}$. Since all forms are positive, and thus uniformly sectorial we have $e^{-sA(t)}x \to e^{-s\mathfrak{a}}x$ for all x by 2.3.

Decompose $H = \overline{D(\mathfrak{a})} \oplus D(\mathfrak{a})^\perp$. First suppose that $x \in D(\mathfrak{a})$. By the lemma, we obtain

$$\begin{aligned} \|e^{-tA(\frac{t}{n})}x - F(\tfrac{t}{n})^n x\| &= \|e^{n(F(\frac{t}{n})-id)}x - F(\tfrac{t}{n})^n x\| \\ &\leq \sqrt{n}\|F(\tfrac{t}{n})x - x\| \\ &= \frac{t}{\sqrt{n}}\|A(\tfrac{t}{n})x\|\,. \end{aligned}$$

For $x \in D(A)$, where A is the operator associated with $\mathfrak{a}$ as usual, we find $x = (1+\mathfrak{a})^{-1}y$ for some $y \in \overline{D(\mathfrak{a})}$. Now we use that both F and $e^{-tA(\frac{t}{n})}$ are uniformly bounded (say by C) and obtain:

$$\begin{aligned}
&\|e^{-tA(\frac{t}{n})}x - F(\tfrac{t}{n})^n x\| = \|e^{-tA(\frac{t}{n})}(1+\mathfrak{a})^{-1}y - F(\tfrac{t}{n})^n(1+\mathfrak{a})^{-1}y\| \\
\leq\ & 2C \cdot \|(1+\mathfrak{a})^{-1}y - (1+A(\tfrac{t}{n}))^{-1}y\| \\
&+\|e^{-tA(\frac{t}{n})}(1+A(\tfrac{t}{n}))^{-1}y - F(\tfrac{t}{n})^n(1+A(\tfrac{t}{n}))^{-1}y\| \\
\leq\ & 2C \cdot \|(1+\mathfrak{a})^{-1}y - (1+A(\tfrac{t}{n}))^{-1}y\| + \frac{t}{\sqrt{n}}\|A(\tfrac{t}{n})(1+A(\tfrac{t}{n}))^{-1}y\| \\
\leq\ & 2C \cdot \|(1+\mathfrak{a})^{-1}y - (1+A(\tfrac{t}{n}))^{-1}y\| + \frac{t}{\sqrt{n}}\|y - (1+A(\tfrac{t}{n}))^{-1}y\|
\end{aligned}$$

Here, since $A(t) \xrightarrow{R} \mathfrak{a}$, the first term tends to zero uniformly for t in compact subsets of $\mathbb{R}_+$ whereas the term in the second norm is convergent, and hence bounded. Thus we obtain the desired convergence for $x \in D(A)$ and by density also for all $x \in \overline{D(\mathfrak{a})}$.

Now let $x \in D(\mathfrak{a})^\perp$. We have to show that $F(\frac{t}{n})^n x \to 0$.
By the definition of $A(t)$ we obtain that

$$0 \leq F(t) = 1 - tA(t) \leq (1 + tA(t))^{-1}$$

where the last inequality is proved this way:

$$\begin{aligned}
((1-tA(t))x \,|\, x) &= ((1+tA(t))^{-1}(1-t^2A(t)^2)x \,|\, x) \\
&= ((1+tA(t))^{-1}x \,|\, x) - \underbrace{((1+tA(t))^{-1}tA(t)x \,|\, tA(t)x)}_{\geq 0} \\
&\leq ((1+tA(t))^{-1}x \,|\, x)
\end{aligned}$$

Using the fact that $F(t)$ and $A(t)$ are selfadjoint and commute we obtain inductively that $F(t)^{2n} \leq (1+tA(t))^{-2n}$ for all $n \in \mathbb{N}$. But we have that

$$\begin{aligned}
(1+tA(t))^{2n} &= \sum_{k=0}^{2n} \binom{2n}{k} t^k A(t)^k \\
&= 1 + 2ntA(t) + \underbrace{\cdots + t^{2n}A(t)^{2n}}_{\geq 0} \\
&\geq 1 + 2ntA(t)
\end{aligned}$$

and hence $0 \leq F(t)^{2n} \leq (1+tA(t))^{-2n} \leq (1+2ntA(t))^{-1}$ by 2.7. Hence

$$\begin{aligned}
\|F(\tfrac{t}{n})^n x\|^2 &= \big(F(\tfrac{t}{n})^{2n}x \,\big|\, x\big) \\
&\leq \big((1+2tA(\tfrac{t}{n}))^{-1}x \,\big|\, x\big) \\
&= (2t)^{-1}\big((2t + A(\tfrac{t}{n}))^{-1}x \,\big|\, x\big) \to 0
\end{aligned}$$

uniformly on intervals of the form $[t_0, t_1]$.

□

Now we are ready to finish the proof of Trotter's product formlua for positive, symmetric forms. We want to apply Chernoffs formula, and we are done. The only thing to do, is to "symmetrize" everything[2]

Proof of Theorem 3.2, Part 2.
Define $\tilde{F}(t) = e^{-t\frac{\mathfrak{a}}{2}} e^{-t\mathfrak{b}} e^{-t\frac{\mathfrak{a}}{2}}$ and $F(t) = e^{-t\mathfrak{a}} e^{-t\mathfrak{b}}$, $\tilde{S}(t) = t^{-1}(id - \tilde{F}(t))$ and $S(t)$ as in Part I.

We claim that also $(1 + \tilde{S}(t))^{-1} \to (1 + \mathfrak{c})^{-1}$ strongly. Note that

$$\begin{aligned} 1 &= (1 + \tilde{S}(t))(1 + \tilde{S}(t))^{-1} \\ &= t^{-1}(t + 1 - \tilde{F}(t))(1 + \tilde{S}(t))^{-1} \end{aligned}$$

so that we may conclude that

$$(1 + t)(1 + \tilde{S}(t))^{-1} = t + \tilde{F}(t)(1 + \tilde{S}(t))^{-1}$$

We claim that $e^{-t\frac{\mathfrak{a}}{2}}(1 + \tilde{S}(t))^{-1} = (1 + S(t))^{-1} e^{-t\frac{\mathfrak{a}}{2}}$ from which it then follows that

$$(1 + \tilde{S}(t))^{-1} = (t + 1)^{-1} \left(t + e^{-t\frac{\mathfrak{a}}{2}} e^{-t\mathfrak{b}} (1 + S(t))^{-1} e^{-t\frac{\mathfrak{a}}{2}} \right) \quad (*)$$

To prove the last claim observe that

$$\begin{aligned} (1 + S(t))^{-1} e^{-t\frac{\mathfrak{a}}{2}} (1 + \tilde{S}(t)) &= (1 + S(t))^{-1} e^{-t\frac{\mathfrak{a}}{2}} t^{-1} (1 + ti - \tilde{F}(t)) \\ &= (1 + S(t))^{-1} t^{-1} (1 + tF(t)) e^{-t\frac{\mathfrak{a}}{2}} \\ &= e^{-t-\frac{\mathfrak{a}}{2}} \end{aligned}$$

where we used that $e^{-t\frac{\mathfrak{a}}{2}} \tilde{F}(t) = F(t) e^{-t\frac{\mathfrak{a}}{2}}$. From this the desired equality follows immediately.

Now observe that this equation for the resolvent of $\tilde{S}$ implies that

$$(1 + \tilde{S}(t))^{-1} x - (1 + S(t))^{-1} x \to 0 + P_{\mathfrak{a}} P_{\mathfrak{b}} (1 + \mathfrak{c})^{-1} x - (1 + \mathfrak{c})^{-1} x$$

which is equal to zero, since $D(\mathfrak{c}) \subset D(\mathfrak{a}) \cap D(\mathfrak{b})$ and we thus may omit the projections.
Now Chernoffs formula implies that $\tilde{F}(\frac{t}{n})^n x \to e^{-t\mathfrak{c}} x$
But we have that

$$F(\frac{t}{n})^{n+1} = e^{-t\frac{\mathfrak{a}}{2}} \tilde{F}(\frac{t}{n})^n e^{-t\frac{\mathfrak{a}}{2}} e^{-t\mathfrak{b}} \to P_{\mathfrak{a}} e^{-t\mathfrak{c}} P_{\mathfrak{a}} P_{\mathfrak{b}} = e^{-t\mathfrak{c}}$$

which completes the proof. □

[2]The reason why we didn't start with the "right" approximants is to have a more readable proof. For densely defined $\mathfrak{c}$ a direct proof of the resolvent convergence we will need in order to apply Chernoffs formula can be found in Reed-Simon [**16**, Supplement VIII.8].

PROOF OF THEOREM 3.2, PART 3.
We now generalize our result to sectorial forms by an argument about holomorphic functions. We let $\mathfrak{a}, \mathfrak{b}$ be closed sectorial forms and we assume that 0 is a vertex and θ a corresponding semiangle for both of them We define $\mathfrak{a}(z) = \operatorname{Re}\mathfrak{a} + z\operatorname{Im}\mathfrak{a}$ and $\mathfrak{b}(z) = \operatorname{Re}\mathfrak{b} + z\operatorname{Im}\mathfrak{b}$.

For $z \in (-\varepsilon, \varepsilon) \times \mathbb{R}$ where $\varepsilon = \frac{1}{1+\tan\theta}$ $\mathfrak{a}(z)$ is a sectorial form. Moreover, for $z \in [-\alpha_0, \alpha_0] \times [-\beta_0, \beta_0]$ where α_0 and β_0 are fixed positive numbers the family $\mathfrak{a}(z)$ is uniformly sectorial by Proposition 2.6. Thus, we may locally use the same integration path for $e^{-t\mathfrak{a}(z)}$. This together with the analyticity of the resolvents (which also follows from 2.6) implies, that the semigroups $e^{-t\mathfrak{a}(z)}$ and $e^{-t\mathfrak{b}(z)}$ depend analytically on z. The first two parts of the proof show that

$$s - \lim \left(e^{-\frac{t}{n}\mathfrak{a}(z)} e^{-\frac{t}{n}\mathfrak{b}(z)} \right)^n = e^{-t\mathfrak{c}}$$

for all $z \in (-\varepsilon, \varepsilon)$ since these forms are symmetric. But since in this equation all occuring functions depend holomorphically on z. Vitali's Theorem implies the convergence for any z in the strip $(-\varepsilon, \varepsilon) \times \mathbb{R}$ and the desired result follows by setting $z = i$. Furthermore, since the left hand side is analytic in t too, Vitali's Theorem implies uniform convergence on compact t-subsets of $(0, \infty)$. □

2. Projections

Now we consider some rather special closed sectorial forms. If U is a closed subspace of H, then $\mathfrak{o}_U$ defined by

$$\mathfrak{o}_U\,[x, y] := 0 \qquad D(\mathfrak{o}) = U$$

is a closed sectorial form[3] . Then we have that the operator associated with $\mathfrak{o}_U$ is the zero-operator on U, so that we obtain $e^{-t\mathfrak{o}} \equiv P_U$ where P_U is the orthogonal projection onto U.

Thus we obtain as a consequence of 3.2 the following corollary, which is a special case of a theorem of VON NEUMANN:

COROLLARY 3.7. *Let U, V be closed subspaces of a Hilbert space H and P_U, P_V be the orthogonal projections onto U resp. V. Then for any $x \in H$ we have*

$$\lim (P_U P_V)^n x = P_{U\cap V} x$$

where $P_{U\cap V}$ is the orthogonal projection onto the intersection of U and V.

PROOF. This follows immediately form 3.2 by choosing $\mathfrak{a} = \mathfrak{o}_U, \mathfrak{b} = \mathfrak{o}_V$ and observing that $\mathfrak{a} + \mathfrak{b} = \mathfrak{o}_{U\cap V}$. □

[3]Clearly $\mathfrak{o}_U$ is sectorial. It is closed, since $(\cdot\,|\,\cdot)_{\mathfrak{o}_U} = (\cdot\,|\,\cdot)$.

As symmetric forms may also be viewed as subdifferentials of convex functionals it is also interesting, that we have a Trotter formula for those as well:
If φ is a lower semicontinuous convex function on H then the subdifferential $\partial\varphi$ generates a semigroup $(S(t))$ of nonlinear contractions on $\Omega_\varphi = \overline{D(\partial\varphi)}$ which is always a closed, convex set. Thus we may view this as a degenerate semigroup of nonlinear contractions on H by projecting onto Ω_φ first. We then write $e^{-t\partial\varphi} = S(t)P_{\Omega_\varphi}$ where P_{Ω_φ} is the (nonlinear) projection onto Ω_φ. We now have the following theorem of Kato and Masuda which was proved in [**12**]:

Theorem 3.8. *Let $\varphi_1, \dots, \varphi_k$ be lower semicontinuous, convex functionals on H and $\varphi := \sum \varphi_j$. Then for any $x \in \overline{\Omega_\varphi}$ we have that*

$$\lim \left(e^{-\frac{t}{n}\partial\varphi_k} \cdot \dots \cdot e^{-\frac{t}{n}\partial\varphi_1}\right)^n x = e^{-t\partial\varphi} x$$

However, one cannot omit the requirement that $x \in \overline{\Omega_\varphi}$ in this theorem (and thus in particular obtain Theorem 3.2 from a more general theorem concerning nonlinear semigroups). This is, because a nonlinear generalisation of corollary 3.7 does not hold. Look at the following

Example:
Let $H = \mathbb{R}^2$ and let $V = \{x : \|x\| \le 1\}$ and $U = \{x : x_2 = 0\}$ then U and V are closed, convex subsets of H and the orthogonal projections onto them are given by

$$P_U x = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \quad \text{and} \quad P_V x = \begin{cases} x & , \text{ if } \|x\| \le 1 \\ \frac{x}{\|x\|} & , \text{ otherwise} \end{cases}$$

Figure 1 shows, that a convergence analoguous to 3.7 is not valid.

If we are given a closed sectorial form $\mathfrak{a}$ it is also a consequence of Trotters formula, that

$$s - \lim_{n\to\infty} \left(e^{-\frac{t}{n}\mathfrak{a}} P\right)^n$$

exists for any orthogonal projection P. We will see, that this characterizes (operators associated with) closed sectorial forms entirely.

But first we explore a much simpler case, namely that where P is an orthogonal projection onto a 1-dimensional subspace.

Proposition 3.9. *Let A be the generator of a strongly continuous semigroup and define f_n for $x \neq 0$ by*

$$f_n(x) = \left(e^{\frac{1}{n}A} P_x\right)^n x$$

where P_x is the orthogonal projection onto $\operatorname{span}\{x\}$. *Then we have:*

a) *For any $x \in D(A)$ with $\|x\| = 1$ we have* $\lim f_n(x) = e^{(Ax\,|\,x)}x$.
b) *Any f_n is a continuous function of x.*

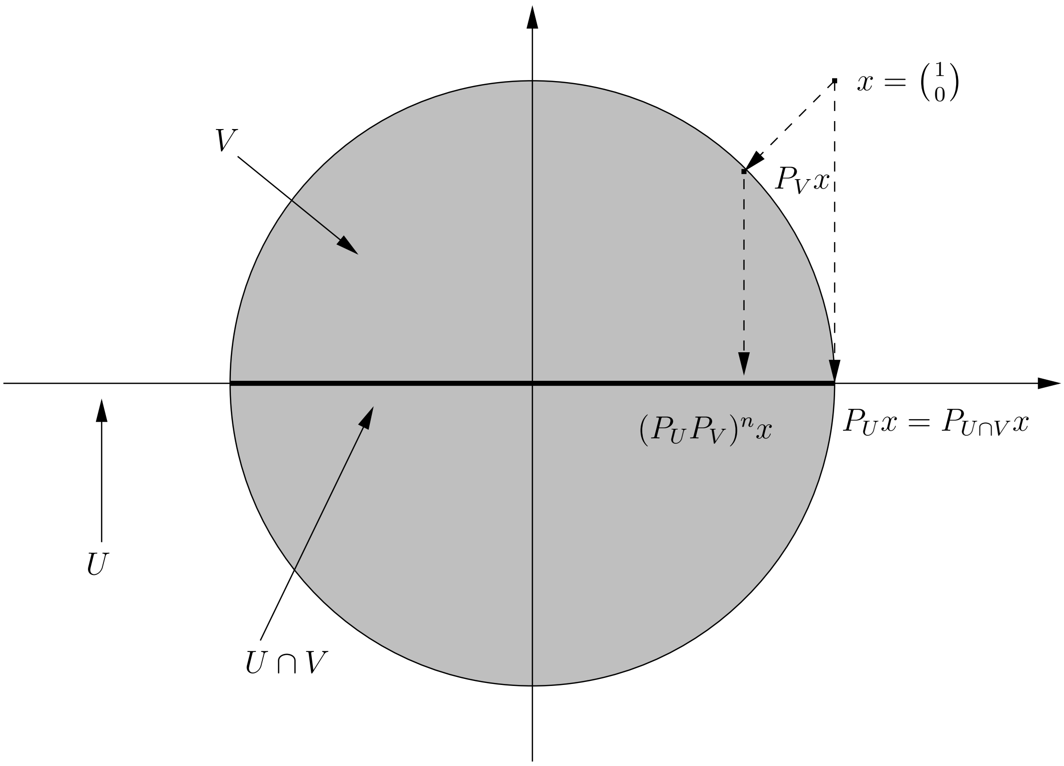

FIGURE 1. Acting of the Projections in this example

PROOF. a) We have that

$$f_n(x) = e^{\frac{1}{n}A} \left(P_x e^{\frac{1}{n}A} P_x \right)^{n-1} x = e^{\frac{1}{n}A} \lambda_n x$$

To find λ_n observe that for $n = 2$ we have

$$\lambda_2 = (\lambda_2 x \,|\, x) = \left(P_x e^{\frac{1}{n}A} P_x x \,\middle|\, x \right)$$

from which one obtains by induction that $\lambda_n = \left(P_x e^{\frac{1}{n}A} P_x x \,\middle|\, x \right)^{n-1}$. Thus we have

$$\begin{aligned} \lim f_n(x) &= \lim e^{\frac{1}{n}A} \left(P_x e^{\frac{1}{n}A} P_x x \,\middle|\, x \right)^{n-1} x = id \cdot \lim \left(P_x e^{\frac{1}{n}A} P_x x \,\middle|\, x \right)^{n-1} x \\ &= \lim \left(1 + \frac{z_n}{n} \right)^{n-1} x = e^{\lim z_n} x \end{aligned}$$

where

$$z_n = \frac{\left(P_x e^{\frac{1}{n}A} P_x x \,\middle|\, x \right) - 1}{\frac{1}{n}} = \left(n(e^{\frac{1}{n}A} - id)x \,\middle|\, x \right) \to (Ax \,|\, x)$$

b) It suffices to show, that the orthogonal projection onto span$\{x\}$ is a continuous function of x, since then f_n may be represented as the composition of continuous functions evaluated at a point.

But we have $P_x u = \left(u \,\middle|\, \frac{x}{\|x\|}\right)\frac{x}{\|x\|}$ and thus

$$\begin{aligned}\|P_x u - P_y u\| &\leq \left\|\left(u \,\middle|\, \frac{x}{\|x\|}\right)\frac{x}{\|x\|} - \left(u \,\middle|\, \frac{x}{\|x\|}\right)\frac{y}{\|y\|}\right\| + \left\|\left(u \,\middle|\, \frac{x}{\|x\|}\right)\frac{y}{\|y\|} - \left(u \,\middle|\, \frac{y}{\|y\|}\right)\frac{y}{\|y\|}\right\| \\ &\leq 2\|u\| \cdot \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|\end{aligned}$$

which implies that $\|P_x - P_y\| \leq 2\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|$. This shows the continuous dependence of P_x on x. Moreover, we can estimate this difference by

$$\begin{aligned}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| &\leq \frac{1}{\|x\|\cdot\|y\|}\Big\| \|y\|x - \|x\|y \Big\| \\ &= \frac{1}{\|x\|\cdot\|y\|}\left(\Big\| \|y\|x - \|y\|y \Big\| + \Big\| \|y\|y - \|x\|y \Big\|\right) \\ &\leq \frac{2}{\|x\|}\|x-y\|\end{aligned}$$

So that for x, y outside a ball $B(0,r)$ the projection P_x is Lipschitz continuous with Lipschitz constant $2r^{-1}$. □

THEOREM 3.10. *Let A be the generator of a C_0-semigroup e^{tA} on a Hilbert space H. Then the following are equivalent:*

a) *$-A$ is associated with a closed, densely defined sectorial form.*
b) *$(e^{\frac{t}{n}A}P)^n x$ converges for any orthogonal projection P and any $t > 0, x \in H$.*

PROOF. a) $\Rightarrow$ b) follows directly from Theorem 3.2

b) $\Rightarrow$ a) We adopt the notation $f_n(x) = (e^{\frac{1}{n}A}P_x)^n x$ from proposition 3.9 and put $f_\infty(x) = \lim f_n(x) = e^{(Ax\,|\,x)}x$ for $x \in D(A)$ with $\|x\| = 1$. We also use the abbreviation y^o for $\frac{y}{\|y\|}$.
We prove this implication by contradiction and consider two cases:

1st case: $\Theta(A)$ is not bounded from the right.
Let $\varepsilon > 0$ be given. We construct sequences $x_k \subset D(A), n_k \subset \mathbb{N}$ and $\delta_k \subset (0,\infty)$ as follows:

Choose $x_1 \in D(A)$ such that $\|x_1\| = 1$ and $\operatorname{Re}(Ax_1\,|\,x_1) \geq 1$. Now pick $n_1 \in \mathbb{N}$ such that $\|f_{n_1}(x_1) - f_\infty(x_1)\| < \varepsilon$. By continuity, there exists $\delta_1 < \frac{1}{2}$ such that

$$\|f_{n_1}(y^o) - f_\infty(x_1)\| < 2\varepsilon$$

for all $y \in B(x_1, \delta_1)$.

Now let $x_1, \ldots, x_k$, $n_1, \ldots n_k$ and $\delta_1, \ldots \delta_k$ already be chosen such that for any $1 \leq l \leq k$

i) $\|x_{l+1} - x_l\| < \min\{\frac{\delta_j}{2^{l+1-j}} \,:\, 1 \leq j \leq l\}$
ii) $\mathrm{Re}\,(Ax_l \,|\, x_l) \geq l$
iii) $\|f_{n_l}(y^o) - f_\infty(x_l^o)\| < 2\varepsilon$ for all $y \in B(x_l, \delta_l)$.

Since $\Theta\,(A)$ is not bounded from the right, we can find $u \in D(A)$, such that

$$\|u\| < \min\{\|Ax_k\|^{-1}, \frac{\delta_l}{2^{k+1-l}} \,:\, 1 \leq l \leq k\} \quad \text{and} \quad \mathrm{Re}\,(Au \,|\, u) \geq 2$$

Now choose real α such that $\mathrm{Re}\,(Ae^{i\alpha}u \,|\, x_k) \geq 0$ and put $x_{k+1} = x_k + e^{i\alpha}u$. Then by construction condition i) is fulfilled, so we check condition ii):
We have

$$\begin{aligned} \mathrm{Re}\,(Ax_{k+1} \,|\, x_{k+1}) &= \mathrm{Re}\,(Ax_k \,|\, x_k) + \mathrm{Re}\,(Ax_k \,|\, e^{i\alpha}u) \\ &\quad + \mathrm{Re}\,(Ae^{i\alpha}u \,|\, x_k) + \mathrm{Re}\,(Ae^{i\alpha}u \,|\, e^{i\alpha}u) \\ &\geq k - \|Ax_k\| \cdot \|u\| + 0 + 2 \\ &\geq k - 1 + 2 = k + 1 \end{aligned}$$

And we may find n_{k+1} and δ_{k+1} such that iii) is satisfied as above.

Now observe, that for $m > n$ we have

$$\|x_m - x_n\| \leq \sum_{k=n}^{m-1} \|x_{k+1} - x_k\| \leq \sum_{k=n}^{m-1} \frac{\delta_1}{2^k} \to 0 \quad \text{as} \quad m, n \to \infty$$

Thus x_n is a Cauchy sequence and hence convergent, say to x. But we also have for $m > n$ that

$$\|x_m - x_n\| \leq \sum_{k=n}^{m-1} \|x_{k+1} - x_k\| < \sum_{k=n}^{m-1} \frac{\delta_n}{2^{k+1-n}} = \delta_n \sum_{k=1}^{m-n-1} 2^k = \delta_n \frac{1 - 2^{-(m-n)}}{1 - \frac{1}{2}} \leq \delta_n$$

So that $x_m \in B(x_n, \delta_n)$ for all $m \geq n$. This implies in particular, that $x \in B(x_n, \delta_n)$ for all n and thus

$$\|f_{n_k}(x^o) - f_\infty(x_k^o)\| < 2\varepsilon\,.$$

But since $\|x_k\| \in (\frac{1}{2}, \frac{3}{2})$ we have

$$\|f_\infty(x_k^o)\| = e^{\frac{\mathrm{Re}\,(Ax_k \,|\, x_k)}{\|x_k\|^2}} > e^{\frac{4k}{9}} \to \infty\,.$$

This shows that $f_n(x^o) = (e^{\frac{1}{n}A}P_{x^0})^n x^o$ cannot converge, since the subsequence $f_{n_k}(x^o)$ has unbounded norms.

2nd case: We assume that $\Theta\,(A)$ is bounded from the right but $\Theta\,(A)$ is not contained in any sector. By rescaling, we may assume, that $\mathrm{Re}\,(Ax \,|\, x) \leq -1$. We also assume, that the upper half of the numerical range is not contained in any sector.
Again, we construct sequences $x_k \subset D(A), n_k \subset \mathbb{N}$ and $\delta_k \subset (0, \infty)$ as follows:

Let $x_1 \in D(A)$ with $\|x_1\| = 1$ be arbitrary and put $\alpha_1 = (Ax_1 \,|\, x_1) = a_1 + b_1 i$.
Let $0 < \varepsilon < \frac{e^{2a_1}}{5}$ be given. Choose $\rho > 0$ such that $|e^{\varepsilon i} - 1| < \frac{e^{2a_1+1}}{2}$ for all $|\varepsilon| < \rho$ and $\delta_1 < \frac{3e^{2a_1+1}}{16}$.

Now assume, that $x_1, \dots x_k, n_1, \dots, n_k$ and $\delta_1, \dots \delta_k$ already be chosen, such that for any $1 \le l \le k$

i) $\|x_{l+1} - x_l\| < \min\{\frac{\delta_j}{2^{l+1-j}} \ : \ 1 \le j \le l\}$
ii) If $\alpha_l = \|x_l\|^{-2}(Ax_l \,|\, x_l) = a_l + b_l i$ then we have

$$2a_1 \le a_l \le -1 \quad \text{and} \quad |b_l - b_{l-1} - \pi| = |\varepsilon_l| < \rho$$

iii) $\|f_{n_l}(y^o) - f_\infty(x_l^o)\| < 2\varepsilon$ for all $y \in B(x_l, \delta_l)$.

Since A is not sectorial, there is a sequence $z_n \subset D(A)$ such that $\|z_n\| = 1$ and $\mathrm{Im}\,(Az_n \,|\, z_n) \ge n|\mathrm{Re}\,(Az_n \,|\, z_n)| \ge n$.
Define

$$\tilde{u}_n = \frac{\sqrt{\pi}\|x_k\|}{\sqrt{\mathrm{Im}\,(Az_n \,|\, z_n)}} z_n$$

and now pick φ_n such that $(Ae^{i\varphi_n}\tilde{u}_n \,|\, x_k) \ge 0$ (and thus in particular real!). Put $u_n = e^{i\varphi_n}\tilde{u}_n$. Then we have that

i) $\|u_n\| \le \text{const.} \cdot \frac{1}{\sqrt{n}} \to 0$
ii) $\mathrm{Im}\,(Au_n \,|\, u_n) = \pi\|x_k\|^2$
iii) $|\mathrm{Re}\,(Au_n \,|\, u_n)| \le \frac{1}{n}\mathrm{Im}\,(Au_n \,|\, u_n) = \frac{\pi\|x_k\|^2}{n} \to 0$

We then have:

$$\begin{aligned} \frac{(A(x_k + u_n) \,|\, x_k + u_n)}{\|x_k\|^2} &= \|x_k\|^{-2}\left((Ax_k \,|\, x_k) + \underbrace{(Ax_k \,|\, u_n)}_{\to 0} + \underbrace{(Au_n \,|\, x_k)}_{\ge 0} + (Au_n \,|\, u_n) \right) \\ &=: c_n + d_n i \end{aligned}$$

Here we have that

$$c_n \ge a_k - \|x_k\|^{-2} \cdot |(Ax_k \,|\, u_n)| - \|x_k\|^{-2} \cdot |(Au_n \,|\, u_n)| \to a_k \quad \text{as} \quad n \to \infty$$

and

$$d_n = b_k + \|x_k\|^{-2}\mathrm{Im}\,(Ax_k \,|\, u_n) + \pi \to b_k + \pi \quad \text{as} \quad n \to \infty$$

So considering that $\|x_k + u_n\|^{-2}\|x_k\|^2 \to 1$, we may choose n_0 large enough, such that $\|u_{n_0}\| < \min\{\frac{\delta_k}{2^{k+1-j}} \ : \ 1 \le j \le k\}$ and

$$c_{n_0}\frac{\|x_k\|^2}{\|x_k + u_{n_0}\|^2} > a_k \ge 2a_1 \quad \text{and} \quad \left| d_{n_0} \cdot \frac{\|x_k\|^2}{\|x_k + u_{n_0}\|^2} - b_k - \pi \right| < \rho$$

Then if we put $x_{k+1} = x_k + u_{n_0}$ then conditions i) and ii) are fulfilled.
Now as in the first case, we may choose n_{k+1} and δ_{k+1} such that the third condition is satisfied.

Exactly as in the first case, we see that the sequence x_n converges to x (say) and that for any k we have that

$$\|f_{n_k}(x^o) - f_\infty(x^o_k)\| < 2\varepsilon\,.$$

But now observe, that consecutive members of $f_\infty(x_k)$ are far away from each other. More precisely:

$$\begin{aligned}
& \|f_\infty(x^o_{k+1}) - f_\infty(x^o_k)\| = \|e^{\alpha_{k+1}}x^o_{k+1} - e^{\alpha_k}x^o_k\| \\
= \;& \|e^{\alpha_{k+1}}x^o_{k+1} - e^{a_{k+1}+(b_k+\pi)i}x^o_k + e^{a_{k+1}+(b_k+\pi)i}x^o_k - e^{\alpha_k}x^o_k\| \\
\geq \;& \|e^{a_{k+1}+(b_k+\pi)i}x^o_k - e^{\alpha_k}x^o_k\| - \|e^{\alpha_{k+1}}x^o_{k+1} - e^{a_{k+1}+(b_k+\pi)i}x^o_k\| \\
\geq \;& e^{a_k} + e^{a_{k+1}} - e^{a_{k+1}}\|e^{b_{k+1}i}x^o_{k+1} - e^{(b_k+\pi)i}x^o_k\| \\
\geq \;& 2e^{2a_1} - e^{-1}\|e^{\varepsilon_{k+1}i}x^o_{k+1} - x^o_k\| \\
\geq \;& 2e^{2a_1} - e^{-1}\left(|e^{\varepsilon_{k+1}i} - 1| + \|x^o_{k+1} - x^o_k\|\right) \\
\geq \;& 2e^{2a_1} - e^{-1}\left(\frac{e^{2a_1+1}}{2} + \frac{8}{3}\delta_1\right) \geq e^{2a_1}
\end{aligned}$$

Here, we used in the last step the choice of ρ, the fact that the projection on the unitsphere is Lipschitz continuous outside the ball $B(0,\frac{3}{4})$ with Lipschitz constant $\frac{8}{3}$ (see the proof of 3.9) and that $\|x_k\| \in (\frac{3}{4},\frac{5}{4})$.
From this, it follows, that

$$\begin{aligned}
& \|f_{n_k}(x^o) - f_{n_{k+1}}(x^o)\| \\
\geq \;& \|f_\infty(x^o_{k+1}) - f_\infty(x^o_k)\| - \|f_\infty(x^o_k) - f_{n_k}(x^o)\| - \|f_\infty(x^o_{k+1}) - f_{n_{k+1}}(x^o)\| \\
\geq \;& 5\varepsilon - 2\varepsilon - 2\varepsilon = \varepsilon
\end{aligned}$$

□

Notes and References for Chapter 3: The main reference for section 1 is of course KATO [**11**], where the proof was given in the case where $\mathfrak{a}$ and $\mathfrak{b}$ are densely defined. Here, Kato also suggested an adaption of his proof to nondensely defined forms by using a generalized spectral Theorem. The proof presented here however uses only the standard spectral Theorem and the convergence theorems of the previous chapter (which do not appear in Kato's proof). This was inspired by the proof given in REED, SIMON [**16**, chapter VIII.8 and supplement]. The idea for the proof of lemma 3.4 was suggested by CHARLES BATTY during a stay in Ulm. The proof of the Chernoff product formula given here is also not the one suggested in [**11**] but inspired by the proof given in ENGEL, NAGEL [**9**] for operators.

For more information on the nonlinear case mentioned in section 2, we refer to BREZIS [**6**]. The example, that Trotter's formula does not hold for nonlinear projections was found in KATO, MASUDA [**12**]. The rest of this section is taken from MATOLCSI [**13**].

CHAPTER 4

Applications to Elliptic Forms

Elliptic operators belong to the most interesting operators for applications of semigroup theory. Also, they are the prototype of operators associated with forms. Therefore, we want to introduce elliptic forms in this chapter.

We will work in the Hilbert space $H = L^2(\mathbb{R}^d)$ (where we use Lebesgue-measure dx). Let Ω be an open subset of $\mathbb{R}^d$. By an elliptic form, we mean a form looking as follows:

$$\mathfrak{a}\,[f,g] = \underbrace{\int\limits_{\Omega} \sum_{i,j=1}^{d} a_{ij} D_i f \overline{D_j g}\, dx}_{:=\mathfrak{a}_0\,[f,g]} + \underbrace{\int\limits_{\Omega} \sum_{k=1}^{d} \left(b_k D_k f \overline{g}\, dx + c_k f \overline{D_k g} \right)}_{:=\mathfrak{b}\,[f,g]} + \underbrace{\int\limits_{\Omega} a_o f \overline{g}\, dx}_{:=\mathfrak{c}\,[f,g]}$$

defined on a suitable domain $D(\mathfrak{a}) \subset H_0^1(\Omega)$. Here, $D_j = \frac{\partial}{\partial x_j}$. We call $\mathfrak{a}_0$ the *principal part* of $\mathfrak{a}$. $\mathfrak{b}$ is called the *drift term* of $\mathfrak{a}$, $\mathfrak{c}$ the *potential part.* We will always assume the coefficients $A = (a_{ij}) : \Omega \to \mathbb{C}^{d\times d}$ to satisfy the following uniform ellipticity condition:

There exists some constant $\eta > 0$ such that

$$\operatorname{Re} \sum_{i,j=1}^{d} a_{ij}(x) \xi_i \bar{\xi_j} \geq \eta |\xi|^2 \quad \text{for a.e. } x \text{ and every } \xi \in \mathbb{C}^d \qquad (E)$$

In the following, we will impose more conditions on A and the coefficients $b = (b_k), c = (c_k) : \Omega \to \mathbb{C}^d$ and $a_0 : \Omega \to \mathbb{C}$. So far, we will only assume that the coefficients are measurable.
It is then interesting, to investigate, how the forms (and then also the semigroups) depend on the coefficients.

Another interesting aspect is the dependence of the forms on the domain Ω. For this, we require, that $D(\mathfrak{a}) \subset H_0^1(\Omega)$, which means, that we impose Dirichlet boundary conditions. This allows us, to identify the form domains with closed subspaces of $L^2(\mathbb{R}^d)$ by extending functions by 0. For this, we use that

$$H_0^1(\Omega) := \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)} = \overline{\{f \in \mathcal{D}(\mathbb{R}^d) \,:\, \operatorname{supp} f \subset \Omega\}}^{H^1(\mathbb{R}^d)}.$$

1. The Principal Part of an Elliptic Form

What makes the form approach to elliptic operators so easy is the fact, that we can take care of the different parts of the form separately and then use form methods to 'paste' them together. So we start with the principal part.

PROPOSITION 4.1. *Let $A = (a_{ij}) \in L^\infty(\Omega, \mathbb{C}^{d\times d})$ satisfy condition (E). Then $\mathfrak{a}_0$ as defined above with domain $D(\mathfrak{a}_0) = H_0^1(\Omega)$ is a closed, sectorial form.*

PROOF. First observe that

$$\begin{aligned} \operatorname{Re} \mathfrak{a}_0 [f] &= \operatorname{Re} \int_\Omega \sum a_{ij} D_i f \overline{D_j f}\, dx \\ &\geq \int_\Omega \eta |\nabla f|^2\, dx \\ &= \eta \sum_{k=1}^d \|D_k f\|_2^2 \geq 0 \end{aligned}$$

and thus $\operatorname{Re} \mathfrak{a}_0 \geq 0$. Now pick C such that $\|a_{ij}\|_\infty \leq C$ for all $1 \leq i, j \leq d$. Then

$$\begin{aligned} |\operatorname{Im} \mathfrak{a}_0 [f]| &\leq C \cdot \int_\Omega \sum_{i,j=1}^d |D_i f \overline{D_j f}|\, dx \\ &\leq C \left(\sum_{k=1}^d \|D_k f\|_2 \right)^2 \quad \text{by the Hölder inequality} \\ &\leq Cd \sum_{k=1}^d \|D_k f\|_2^2 \quad \text{by convexity of the square function} \\ &\leq \frac{Cd}{\eta} \operatorname{Re} \mathfrak{a}_0 [f] \end{aligned}$$

Furthermore $\mathfrak{a}_0$ is closed since the associated norm $\|\cdot\|_{\mathfrak{a}_0}$ is equivalent to the Sobolev norm $\|\cdot\|_{H^1}$. To see this, observe

$$\begin{aligned} \|f\|_{\mathfrak{a}_0}^2 = \|f\|_2^2 + \operatorname{Re} \mathfrak{a} [f] &\geq \|f\|_2^2 + \eta \sum_{i=1}^d \|D_i f\|_2^2 \\ &\geq \min\{1, \eta\} \cdot \|f\|_{H^1}^2 \\ \|f\|_{\mathfrak{a}_0}^2 = \|f\|_2^2 + \operatorname{Re} \mathfrak{a}_0 [f] &\leq \|f\|_2^2 + Cd \sum_{i=1}^d \|D_i f\|_2^2 \\ &\leq \max\{1, Cd\} \cdot \|f\|_{H^1}^2 . \end{aligned}$$

□

COROLLARY 4.2. *If the conditions of proposition 4.1 are satisfied, then $\mathcal{D}(\Omega)$ is a form core for $\mathfrak{a}_0$.*

PROOF. This follows, since the norm $\|\cdot\|_{\mathfrak{a}_0}$ is equivalent to the Sobolev norm, and the testfunctions are dense in H_0^1. □

Next we consider elliptic forms with unbounded coefficients. Here, we have to assume a little bit more on the coefficients to make the associated form sectorial. We have the following

PROPOSITION 4.3. *Let $A = (a_{ij}) : \Omega \to \mathbb{C}^{d\times d}$ with measurable entries such that the ellipticity condition (E) is satisfied. Furthermore assume that there exists a constant $M > 0$ such that*

$$|\sum_{i,j=1}^{d} \operatorname{Im} a_{ij}(x)\,\xi_i\bar{\xi_j}| \le M \sum_{i,j=1}^{d} \operatorname{Re} a_{ij}(x)\xi_i\bar{\xi_j} \quad \forall \xi \in \mathbb{C}^d \quad , a.e.\ x$$

And define

$$\begin{aligned} \mathfrak{a}_0\,[f,g] &= \int_\Omega \sum_{i,j=1}^{d} a_{ij} D_i f \overline{D_j g}\,dx \\ D(\mathfrak{a}_0) &= \{f \in H_0^1(\Omega) : \int_\Omega |\sum_{i,j=1}^{d} a_{ij} D_i f \overline{D_j f}\,|\,dx < \infty\} \end{aligned}$$

Then $\mathfrak{a}_0$ is a closed, sectorial form.

PROOF. First, we check, that $D(\mathfrak{a}_0)$ is a vector space. Obviously, $0 \in D(\mathfrak{a}_0)$ and also if $f \in D(\mathfrak{a}_0)$ then also $\lambda f \in D(\mathfrak{a}_0)$ for any scalar λ. To see that $f, g \in D(\mathfrak{a}_0)$ implies $f + g \in D(\mathfrak{a}_0)$ note that

$$\sum_{i,j=1}^{d} a_{ij}(x)D_i(f+g)\overline{D_j(f+g)} = \sum_{i,j=1}^{d} a_{ij}(x)\left(D_i f\overline{D_j f} + D_i g\overline{D_j f} + D_i f\overline{D_j g} + D_i g\overline{D_j g}\right)$$

for almost every x. The first and last summand are integrable by hypothesis. For the two in the middle, note, that the mapping

$$(\xi,\eta) \mapsto \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\overline{\eta_i}$$

is a sectorial form on $\mathbb{C}^d$ for almost every x. Thus, by applying proposition 1.2 and the Cauchy inequality, we see, that these are integrable as well.

Let $A_n = (a_{ij}^n) \in L^\infty(\Omega, \mathbb{C}^{d\times d})$ be defined by

$$a_{ij}^n(x) = a_{ij}(x)\mathbb{1}_{\Omega_n} + \frac{\eta}{2}\delta_{ij}\mathbb{1}_{\Omega_n{}^C}$$

where $\Omega_n = \{x : \max_{i,j} \operatorname{Re} a_{i,j}(x) \le n\}$. The sequence Ω_n is increasing. Now observe, that A_n satisfies (E) with ellipticity constant $\frac{\eta}{2}$ independent of n. So we can consider

the elliptic form $\mathfrak{a}_n$ associated with A_n as above, which is a closed, sectorial form. But we also obtain, that the family $\mathfrak{a}_n$ is uniformly sectorial:
By Theorem 4.1 $\operatorname{Re}\mathfrak{a}_n \geq 0$ for any n. Now observe, that

$$\begin{aligned}
|\operatorname{Im}\mathfrak{a}_n\,[f]| &= \left|\int_{\Omega_n} \operatorname{Im} \sum_{i,j=1}^{d} a_{ij} D_i f \overline{D_j f}\, dx\right| \\
&\leq M \int_{\Omega_n} \operatorname{Re} \sum_{i,j=1}^{d} a_{ij} D_i f \overline{D_j f}\, dx \\
&\leq M \int_{\Omega_n} \operatorname{Re} \sum_{i,j=1}^{d} a_{ij} D_i f \overline{D_j f}\, dx + \underbrace{M \int_{{\Omega_n}^C} \eta |\nabla f|^2\, dx}_{\geq 0} \\
&= M \operatorname{Re}\mathfrak{a}_n\,[f]
\end{aligned}$$

Furthermore we have for $m \geq n$ that

$$\begin{aligned}
|\operatorname{Im}(\mathfrak{a}_m - \mathfrak{a}_n)\,[f]| &= \left|\int_{\Omega_m \setminus \Omega_n} \operatorname{Im} \sum_{i,j=1}^{d} a_{ij}\, D_i f \overline{D_j f}\, dx\right| \\
&\leq M \int_{\Omega_m \setminus \Omega_n} \operatorname{Re} \sum_{i,j=1}^{d} a_{ij}\, D_i f \overline{D_j f}\, dx \\
&\leq M \int_{\Omega_m \setminus \Omega_n} \operatorname{Re} \sum_{i,j=1}^{d} a_{ij}\, D_i f \overline{D_j f}\, dx \\
&\quad + M \int_{\Omega_m \setminus \Omega_n} \underbrace{\operatorname{Re} \sum_{i,j=1}^{d} a_{ij}\, D_i f \overline{D_j f} - \eta |\nabla f|^2\, dx}_{\geq 0} \\
&= 2M \operatorname{Re}(\mathfrak{a}_m - \mathfrak{a}_n)\,[f]
\end{aligned}$$

Now Theorem 2.11 shows that the limit form $\lim \mathfrak{a}_n$ is a closed and sectorial form. It remains to show that actually $\lim \mathfrak{a}_n = \mathfrak{a}_0$:

Assume that $\lim \operatorname{Re}\mathfrak{a}_n\,[f]$ exists. We have to show that $f \in D(\mathfrak{a}_0)$ and that $\lim \mathfrak{a}_n\,[f] = \mathfrak{a}_0\,[f]$. Put

$$g_n = \sum_{i,j=1}^{d} a_{ij}^n D_i f \overline{D_j f} \qquad \text{and} \qquad g = \sum_{i,j=1}^{d} a_{ij} D_i f \overline{D_j f}$$

We have that $g_n \to g$ pointwise, furthermore $\operatorname{Re} g_n$ is increasing. It is a consequence of the monotone convergence Theorem, that $\operatorname{Re} g$ is integrable. From this it follows, that g is integrable, since $|g| \leq (1+M)\operatorname{Re} g$ and this implies that $f \in D(\mathfrak{a})$. The fact that $\lim \mathfrak{a}_n\,[f] = \mathfrak{a}_0\,[f]$ follows now from the dominated convergence Theorem.

Conversely, suppose that $f \in D(\mathfrak{a}_0)$. Then we have that $\lim \mathfrak{a}_n[f]$ exists and is equal to $\mathfrak{a}_0[f]$ as a consequence of the dominated convergence Theorem. □

Corollary 4.4. *If in addition to the hypotheses of Proposition 4.3 the coefficients A have an almost everywhere realvalued quadratic form, i.e.*

$$\operatorname{Im} \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\bar{\xi_j} = 0 \quad \forall \xi \in \mathbb{C}^d \quad , a.e.\ x\,.$$

Then the forms $\mathfrak{a}_n$ defined as in the last theorem converge in strong resolvent sense to $\mathfrak{a}$.

Proof. This is an immediate consequence of Theorem 2.11 and the fact that $\operatorname{Im}(\mathfrak{a}_n - \mathfrak{a}_m) = 0$. □

2. Schrödinger Operators

A Schrödinger operator is an operator of the form

$$\mathcal{S} = \Delta - V.$$

Here Δ is the Laplace operator on an open set $\Omega \subset \mathbb{R}^d$ (where we will impose Dirichlet boundary conditions as usual) and V is multiplication with a function (which we will call V as well). We can interpret this formal definition by using form methods:
$-\Delta$ is associated with the closed, symmetric form

$$\mathfrak{a}_0[f] := \int_\Omega |\nabla f|^2\,dx \qquad D(\mathfrak{a}) = H_0^1(\Omega).$$

On the other hand, V is associated with the form

$$\mathfrak{c}[f] := \int_\Omega V|f|^2\,dx \qquad D(\mathfrak{c}) = \{f \in L^2(\Omega)\ :\ \int_\Omega |V|\cdot|f|^2\,dx < \infty\}.$$

If $V(\Omega) \subset \Sigma_\gamma(\theta)$, then obviously $\mathfrak{c}$ is a sectorial form with $\Theta(\mathfrak{c}) \subset \Sigma_\gamma(\theta)$. Furthermore $\mathfrak{c}$ is closed, since $(D(\mathfrak{c}), \|\cdot\|_{\mathfrak{c}}) \simeq L^2(\Omega, (\operatorname{Re} V - \gamma)dx)$. So the operator associated with the form $\mathfrak{s} = \mathfrak{a}_0 + \mathfrak{c}$ may be viewed as a realisation of $\mathcal{S}$. However, the meaning of this operator is in general not clear. If for example V is nowhere locally integrable, then $D(\mathfrak{s}) = \{0\}$, so that also $\mathcal{S}x$ is only defined for $x = 0$. However, if $V \in L^1_{\mathrm{loc}}(\Omega)$, then $\mathcal{S}$ is densely defined, since the test functions belong to $D(\mathfrak{s})$.

The semigroup generated by Δ is the heat semigroup and well known. The semigroup generated by $-V$ is just multiplication by e^{-tV}. So, by using the Trotter formula, one can compute the semigroup generated by $\mathcal{S}$:

$$e^{t\mathcal{S}}f = e^{-t(\mathfrak{a}_0+\mathfrak{c})}f = \lim_{n\to\infty}\left(e^{\frac{t}{n}\Delta}e^{-\frac{t}{n}V}\right)^n f$$

Sometimes, one is also interested in potentials $V : \Omega \to [0,\infty]$. If $\infty \in V(\Omega)$ one calls V an *absorbing potential.* How is $V(\omega) = \infty$ to be interpreted?

Example:
Let $M \subset \Omega$ be measurable and define $V = \infty \cdot \mathbb{1}_M$.
In some sense, $V = \lim n \cdot \mathbb{1}_M$ and the latter function may be associated with a closed, symmetric form:
If we define $\mathfrak{c}_n$ by

$$\mathfrak{c}_n\,[f] = \int_M n \cdot |f|^2\,dx \qquad D(\mathfrak{c}) = L^2(\Omega),$$

then the operator associated with $\mathfrak{c}_n$ is multiplication with $V_n := n \cdot \mathbb{1}_M$. So we could define V to be associated with the form $\mathfrak{c} = \lim \mathfrak{c}_n$. But the latter limit exists only if $f(x) = 0$ for almost every $x \in M$.

This leads to the following

Definition. Let $V : \Omega \to [0,\infty]$ be measurable and $M := V^{-1}(\infty)$. Then we identify V with the operator associated with the closed, symmetric form

$$\mathfrak{c}\,[f] = \int_\Omega V \cdot |f|^2\,dx \qquad D(\mathfrak{c}) = \{f \in L^2(\Omega)\,;\, f_{|M} = 0 \text{ a.e. and } \int_{\Omega\setminus M} V|f|^2\,dx < \infty\}.$$

Now we obtain the following approximation result:

Proposition 4.5. *Let $V : \Omega \to [0,\infty]$ be a measurable function and V_n be a sequence of bounded, positive functions pointwise increasing to V. Then if V and V_n are associated with $\mathfrak{c}$ and $\mathfrak{c}_n$ and $-\Delta$ is associated with $\mathfrak{a}_0$, then $\mathfrak{a}_0 + \mathfrak{c}_n \xrightarrow{R} \mathfrak{a}_0 + \mathfrak{c}$, so in particular*

$$e^{t(\Delta - V_n)} f \to e^{t(\Delta - V)} f$$

for every $f \in L^2(\Omega)$.

Proof. We have that $D(\mathfrak{a}_0 + \mathfrak{c}_n) \equiv H^1_0(\Omega)$ independent of n and also that $(\mathfrak{a}_0 + \mathfrak{c}_n)\,[f]$ is increasing for fixed f. Thus, $\mathfrak{a}_0 + \mathfrak{c}_n$ is an increasing sequence of closed, symmetric forms. By Theorem 2.8, it suffices show, that $\mathfrak{a}_0 + \mathfrak{c}$ really is the limiting form to finish the proof.
For $f \in D(\mathfrak{a}_0 + \mathfrak{c})$ we have $\lim\,(\mathfrak{a}_0 + \mathfrak{c}_n)\,[f] = (\mathfrak{a}_0 + \mathfrak{c})\,[f]$ by the dominated convergence Theorem. This proves that $\mathfrak{a}_0 + \mathfrak{c} \subset \lim(\mathfrak{a}_0 + \mathfrak{c}_n)$.
Conversely, let $f \in H^1_0(\Omega)$ be given, such that $\sup\,(\mathfrak{a}_0 + \mathfrak{c}_n)\,[f] < \infty$. Let $k \in \mathbb{N}$ and define $M_k = M \cap \{|f| > \frac{1}{k}\}$. If $|M_k| > 0$, then

$$(\mathfrak{a}_0 + \mathfrak{c}_n)\,[f] \geq \int_{M_k} V_n \frac{1}{k^2}\,dx \to \infty\,,$$

since otherwise, the monotone convergence Theorem would imply, that $\int_{M_k} V\,dx < \infty$. This contradicts that $\sup\,(\mathfrak{a}_0 + \mathfrak{c}_n)\,[f] < \infty$. Thus $|M_k| = 0$ and hence also $|\{x \in M :$

$f(x) \neq 0\}| = \left|\bigcup_{n\in\mathbb{N}} M_n\right| = 0$. That $f \in D(\mathfrak{a}_0 + \mathfrak{c})$ and that $(\mathfrak{a}_0 + \mathfrak{c}_n)[f] \to (\mathfrak{a}_0 + \mathfrak{c})[f]$ now follows from the monotone convergence Theorem.

$\square$

REMARKS. a) If $V \in L^1_{\text{loc}}$, then $\mathfrak{a}_0 + \mathfrak{c}$ is not only densely defined but also $\mathcal{D}(\Omega)$ is a core for $\mathfrak{a}_0 + \mathfrak{c}$. The proof of this may be found in DAVIES [**8**, Theorem 8.2.1].

b) In this section, it is *not* necessary, that the principal part of our elliptic form is given by the form associated with the Dirichlet-Laplacian. We can do exactly the same with arbitrary principal parts. To prove the equivalent of the preceding proposition in this case, use Theorem 2.11. Here note, that the potential part has no influence on the imaginary part of the forms whatsoever, so that nearly the same proof works.

3. Elliptic Forms with Drift

The drift part of an elliptic form is a more delicate object. Usually, it does *not* define a sectorial form by itself, so that one cannot simply add it to the principal part. Thus, the drift term is usually treated as a perturbation of the other parts of the elliptic form.

First, we formulate a requirement on the coefficients b_k and c_k which will ensure, that $\mathfrak{a}_0 + \mathfrak{b}$ is a closed, sectorial form defined on $D(\mathfrak{a}_0)$ such that the associated inner product $(\,\cdot\,|\,\cdot\,)_{\mathfrak{a}_0+\mathfrak{b}}$ is equivalent to $(\,\cdot\,|\,\cdot\,)_{\mathfrak{a}_0}$. We suppose, that $c_k = 0$ for every k, since this part of the form may be dealt with similary.

THEOREM 4.6. *Let $\Omega \subset \mathbb{R}^d$ be an open set and let the coefficients b_k belong to $L^q(\Omega)$, where $q = q(k) \geq d$ if $d \geq 3$ and $q \geq 2$ if $d \in \{1, 2\}$. Then the form*

$$\mathfrak{b}[f, g] := \int_\Omega \sum_{k=1}^{d} b_k D_k f \overline{g}\, dx \qquad D(\mathfrak{b}) = H^1_0(\Omega)$$

is $\mathfrak{a}_0$-bounded with bound 0 for any principal part $\mathfrak{a}_0$. Hence, $\mathfrak{a}_0 + \mathfrak{b}$ is a closed, sectorial form and the associated inner product is equivalent with $(\,\cdot\,|\,\cdot\,)_{\mathfrak{a}_0}$.

The proof depends on Sobolev embeddings, which we state for completeness:

THEOREM 4.7. *(Sobolev Embedding)*
Let $\Omega \subset \mathbb{R}^d$ be open then the following hold:

a) *If $d \geq 3$, then $H^1_0(\Omega) \hookrightarrow L^p(\Omega)$ for every $2 \leq p \leq \frac{2d}{d-2}$.*
b) *If $d \in \{1, 2\}$, then $H^1_0(\Omega) \hookrightarrow L^p(\Omega)$ for every $2 \leq p < \infty$.*

PROOF. See ADAMS [**1**, Theorem 5.4] $\square$

PROOF OF THEOREM 4.6. First, suppose that $d \geq 3$ and $q > d$.
Suppose, that $b_k \in L^q(\Omega)$. For any $f \in D(\mathfrak{b}) = H^1_0(\Omega)$ we have that $f \in L^p(\Omega)$, where

p is chosen such that $\frac{1}{q}+\frac{1}{2}+\frac{1}{p}=1$. This follows from the Sobolev imbedding 4.7 and the computation

$$\frac{1}{p}=\frac{1}{2}-\frac{1}{q}>\frac{1}{2}-\frac{1}{d}=\frac{d-2}{2d}$$

Thus, we can apply the Hölder inequality and obtain

$$\left|\int_\Omega b_k D_k f\bar{f}\,dx\right| \leq \|b_k\|_q \|f\|_{H^1}\|f\|_p \,.$$

By the Lyapunov inequality (cf. Werner [**18**, Lemma II.4.1]) we have, that if $g \in L^p \cap L^q$, then $g \in L^r$ for every r between p and q and the inequality

$$\|g\|_r \leq \|g\|_p^\theta \cdot \|g\|_q^{1-\theta} \quad \text{,where} \quad \frac{1}{r}=\frac{\theta}{p}+\frac{1-\theta}{q}\,,$$

holds. Applying this in the situation $2 \leq p \leq \frac{d-2}{2d}$ yields

$$\|f\|_p \leq \|f\|_2^\theta \cdot \|f\|_{\frac{d-2}{2d}}^{1-\theta} \leq c\cdot \|f\|_2^\theta \cdot \|f\|_{H_0^1}^{1-\theta} \qquad (*)$$

where the constant c is taken from the Sobolev embedding.
Thus, we have the estimate

$$\begin{aligned}\left|\int_\Omega b_k D_k f\bar{f}\,dx\right| &\leq c\cdot\|b_k\|_p\,\varepsilon\|f\|_{H^1}^{2-\theta}\cdot\frac{1}{\varepsilon}\|f\|_2^\theta \\ &\leq c\cdot\|b_k\|_p\left(\frac{2-\theta}{\theta}\varepsilon^{\frac{2}{2-\theta}}\|f\|_{H^1}^2+\frac{\theta}{2}\frac{1}{\varepsilon^{\frac{2}{\theta}}}\|f\|_2^2\right)\end{aligned}$$

where the last inequality is the Young inequality. Thus summing up and using that $\operatorname{Re}\mathfrak{a}_0[f] \geq \eta\sum\|D_k f\|$ we obtain

$$|\mathfrak{b}[f]| \leq const.\cdot\frac{g(\varepsilon)}{\eta}\operatorname{Re}\mathfrak{a}_0[f]+const.\cdot\|f\|_2^2\,,$$

where g is a continuous function with $g(0+)=0$. Thus $\varepsilon\to 0$ implies that the $\mathfrak{a}_0$-bound of $\mathfrak{b}$ is in fact 0. Now Theorem 1.8 finishes the proof.

If $d \geq 3$ and $q=d$ we can split up b_k as follows:

$$b_k = \underbrace{b_k\mathbb{1}_{\{|b_k|<n\}}}_{=:b_k^n} + \underbrace{b_k\mathbb{1}_{\{|b_k|>n\}}}_{=:\tilde{b}_k^n}\,,$$

where $b_k^n \in L^\infty(\Omega)\cap L^d(\Omega)$ and thus $b_k^n \in L^q(\Omega)$ for every $d \leq q \leq \infty$, whereas $\tilde{b}_k^n \in L^d(\Omega)$ with $\|\tilde{b}_k^n\|_d \to 0$ as $n\to\infty$. By what was done so far, we know, that for every $\varepsilon>0$ there exist a constant $c(\varepsilon)>0$ such that

$$\begin{aligned}\left|\int_\Omega b_k d_k f\overline{f}\,dx\right| &\leq \int_\Omega |b_k^n d_k f\overline{f}|\,dx+\int_\Omega|\tilde{b}_k^n d_k f\overline{f}|\,dx \\ &\leq \varepsilon\operatorname{Re}\mathfrak{a}_0[f]+c(\varepsilon)\|f\|_2^2+\tilde{b}_{k\,d}^n\|f\|_{H1}\|f\|_{\frac{2d}{d-2}} \\ &\to 0 \quad \text{as } n\to\infty\,.\end{aligned}$$

From this it follows as above, that also in this case, $\mathfrak{b}$ is $\mathfrak{a}_0$-bounded with bound 0.

If $d \in \{1, 2\}$ the same proof works: Here, $f \in L^p$ for $\frac{1}{p} = \frac{1}{2} - \frac{1}{q}$ by the Sobolev embedding, so that the first step can be done in this situation as well. Apart from that, we used $d \geq 3$ only in $(*)$. But if $d \in \{1, 2\}$ then we have a Sobolev embedding of H_0^1 into every L^p for $p \geq 2$ so that we have such an estimate as well. $\square$

If Ω is an unbounded domain, one is also interested in allowing unbounded coefficients in the drift terms, i.e. coefficients which tend to infinity as $|x| \to \infty$. In this case, the preceding theorem can not be applied. However, it is sometimes possible to balance the effect of these drifts by the potential part:

THEOREM 4.8. *Let $\mathfrak{a}_0$ be a principal part with ellipticity constant η and $V : \Omega \to [0, \infty)$ be a measurable potential, so that $\mathfrak{a} + \mathfrak{c}$ is a closed, sectorial form on $D(\mathfrak{a}) \cap D(\mathfrak{c}) = \{f \in D(\mathfrak{a}) : \int V|f|^2 \, dx < \infty\}$ by the last section. And let coefficients $b_k : \Omega \to \mathbb{C}$ be given with $|b|^2 = \sum |b_k|^2 \leq \gamma V$, where $\gamma < 4\eta$. Then, the form $\mathfrak{a} = \mathfrak{a}_0 + \mathfrak{b} + \mathfrak{c}$, where $\mathfrak{b}$ is the drift with the coefficients b_k, is closed and sectorial. In addition, the associated inner product is equivalent to $(\cdot \,|\, \cdot)_{\mathfrak{a}_0 + \mathfrak{c}}$.*

PROOF. By Theorem 1.8, it suffices to show that the form

$$\mathfrak{b}\,[f, g] := \int_\Omega \sum_{k=1}^d b_k D_k f \bar{g} \, dx \qquad D(\mathfrak{b}) = D(\mathfrak{a}_0) \cap D(\mathfrak{c})$$

is $\mathfrak{a}_0 + \mathfrak{c}$-bounded with bound < 1.

We have, that

$$\begin{aligned}
|\mathfrak{b}\,[f]| &\leq \sum_{k=1}^d \int_\Omega |b_k| \cdot |D_k f| \cdot |\bar{f}| \, dx \\
&\leq \sum_{k=1}^d \int_\Omega \frac{1}{2\varepsilon} |b_k|^2 \cdot |f|^2 + \frac{\varepsilon}{2} |D_k f|^2 \, dx \quad \text{for any } \varepsilon > 0 \\
&= \frac{1}{2\varepsilon} \int_\Omega |b|^2 \, |f|^2 \, dx + \frac{\varepsilon}{2} \sum_{k=1}^d \|D_k f\|_2^2 \\
&\leq \frac{\gamma}{2\varepsilon} \mathfrak{c}\,[f] + \frac{\varepsilon}{2\eta} \operatorname{Re} \mathfrak{a}_0\,[f] \\
&= \sqrt{\frac{\gamma}{4\eta}} \operatorname{Re}\,(\mathfrak{a}_0 + \mathfrak{c})\,[f] \quad \text{for } \varepsilon = \sqrt{\eta\gamma} > 0
\end{aligned}$$

But by hypothesis, we have, that $\sqrt{\gamma(4\eta)^{-1}} < 1$ which finishes the proof. $\square$

REMARK. It is a consequence of this theorem, that $\mathcal{D}(\Omega)$ is a core for $\mathfrak{a}$ if V is locally integrable and the coefficients in the principal part are bounded. See the remark at the end of section 2.

4. Dependence on the Domain

In this section we study dependence of the semigroups generated by elliptic operators on the domain Ω. If $\mathfrak{a}$ is an elliptic form with coefficients defined on some open set $\Omega \subset \mathbb{R}^d$, then we may obtain different elliptic forms just thinking of the coefficients to be defined on some smaller domain $\Omega_0 \subset \Omega$. Our first result will be a corollary of Trotter's product formula and will allow us to recover a semigroup generated by an elliptic operator on a domain Ω from a semigroup acting on a larger space $\tilde{\Omega}$ (which we assume to be $\mathbb{R}^d$).
The natural Hilbert space to work in is $L^2(\Omega)$. We will consider $L^2(\Omega)$ as a subset of $L^2(\mathbb{R}^d)$ by extending functions $f \in L^2(\Omega)$ by 0 on $\mathbb{R}^d \setminus \Omega$. Then the orthogonal projection onto $L^2(\Omega)$ is given by multiplication with $\mathbb{1}_\Omega$, the indicator function of Ω.

If we want to apply Trotter's formula for this projection we need a mild regularity assumption on Ω. Namely, we require that $H^1(\mathbb{R}^d) \cap L^2(\Omega) =: \tilde{H}_0^1 = H_0^1(\Omega)$. This is satisfied, if the boundary of Ω is sufficiently regular, e.g. if Ω has Lipschitz boundary. However, for $\Omega = (0,1) \cup (1,2)$ one has $\tilde{H}_0^1(\Omega) \neq H_0^1(\Omega)$. Now we have:

Proposition 4.9. *Let $\Omega \subset \mathbb{R}^d$ be given and assume that $\tilde{H}_0^1(\Omega) = H_0^1(\Omega)$. Let $\mathfrak{a}$ be an elliptic form as in the previous section, with coefficients defined on $\mathbb{R}^d$, so we have $D(\mathfrak{a}) \subset H^1(\mathbb{R}^d)$. Denote by $\mathfrak{a}_\Omega$ the elliptic form with the same coefficients as $\mathfrak{a}$ defined on $L^2(\Omega)$. Let P_Ω be multiplication with the indicator function $\mathbb{1}_\Omega$ of Ω. Then for any $f \in L^2(\mathbb{R}^d)$ we have*

$$\lim_{n\to\infty} \left(e^{-\frac{t}{n}\mathfrak{a}} P_\Omega\right)^n f = e^{-t\mathfrak{a}_\Omega} f.$$

Proof. P_Ω is the semigroup associated with the closed sectorial form $\mathfrak{o}[\,\cdot\,,\,\cdot\,] \equiv 0$ defined on $D(\mathfrak{o}) = L^2(\Omega)$. So by Trotter's formula it suffices to prove that $\mathfrak{a} + \mathfrak{o} = \mathfrak{a}_\Omega$. This is only a requirement on the domains, since clearly these forms coincide on $D(\mathfrak{a}+\mathfrak{o}) \cap D(\mathfrak{a}_\Omega)$. But the equality of the domains of these two forms follows directly from the regularity requirement on Ω. $\square$

The next thing we want to do is to approximate Ω by a sequence of sets Ω_n. Here we consider two cases:

- The sequence Ω_n is increasing to Ω, that is for any n we have $\Omega_n \subset \Omega_{n+1}$ and $\Omega = \bigcup \Omega_n$.
- The sequence Ω_n is shrinking to Ω. Hereby we mean that $\Omega_{n+1} \subset \Omega_n$ for any n and that $\Omega = \bigcap \Omega_n$.

This is useful to approximate Ω by a sequence Ω_n with a more regular boundary (think of Ω_n as having smooth boundary). The procedure is illustrated in figure 1.
The first theorem deals with approximation from within. Here we want to apply Theorem 2.10 to obtain convergence of the associated semigroups.

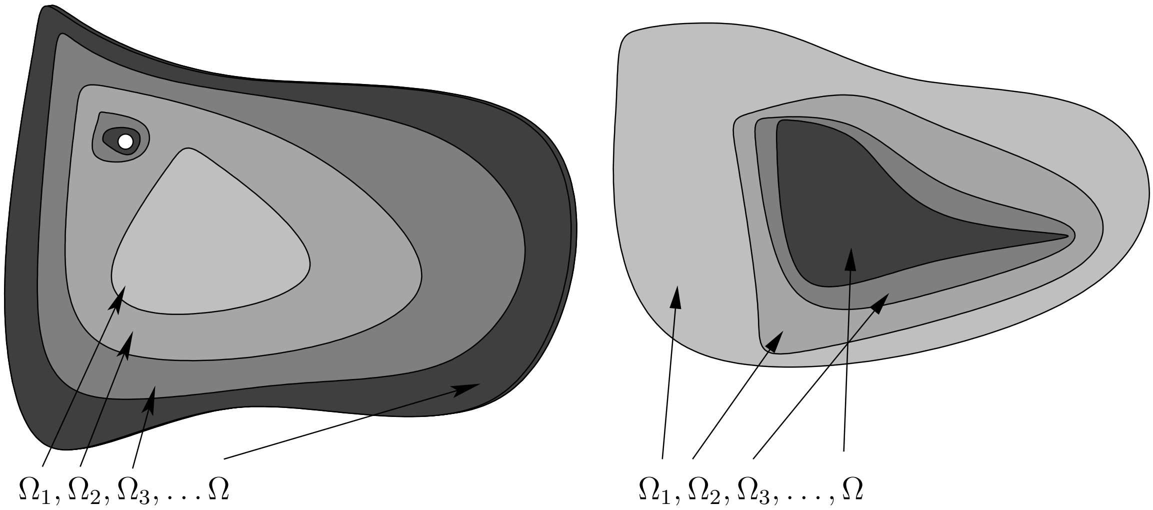

FIGURE 1. Approximation from within (left) and from the outside (right)

THEOREM 4.10. *Let $\Omega \subset \mathbb{R}^d$ be open and $\mathfrak{a}$ be an elliptic form defined on $D(\mathfrak{a}) \subset H_0^1(\Omega)$. Let Ω_n be a sequence of open sets increasing to Ω and let $\mathfrak{a}_{\Omega_n}$ be the elliptic form with the same coefficients as $\mathfrak{a}$ but defined on $D(\mathfrak{a}_n) \subset H_0^1(\Omega_n)$. Assume furthermore, that there exists a core D for $\mathfrak{a}$ which contains only functions with compact support. Then we have that*

$$e^{-t\mathfrak{a}_{\Omega_n}} \longrightarrow e^{-t\mathfrak{a}_\Omega}$$

strongly.

REMARK. We have seen, that $\mathcal{D}(\Omega)$ is a core for the principal part of an elliptic form in corollary 4.2. But the remark at the end of section 2 shows, that this is a core, even if a positive locally integrable potential (which could be unbounded) is added. Thus it is a consequence of the Theorems 4.6 and 4.8 that also if the coefficients of the drift satisfy the conditions in either of these theorems the testfunctions form a core. Thus in all those cases this theorem can be applied.

PROOF. We want to apply Theorem 2.10. We clearly have that $D(\mathfrak{a}_n) \subset D(\mathfrak{a})$ for any n. For the second condition simply note that $\tilde{\mathfrak{a}}_n = \mathfrak{a}_{\Omega_n} - \mathfrak{a} = 0$, which is trivially uniformly sectorial. We show that we can take D for the third condition:
If $f \in D$ then the support of f is covered by the Ω_n. Since the support is compact, there is a finite subcovering. But since the sequence of the Ω_n is increasing, we have that $\text{supp} f \subset \Omega_n$ from some n_0 on. So $f \in \underline{\lim} D(\mathfrak{a}_{\Omega_n})$ and furthermore $\mathfrak{a}_{\Omega_n}[f] \equiv \mathfrak{a}_{\Omega_{n_0}}[f]$ for all $n \geq n_0$. This proves the third condition. Now Theorem 2.10 proves $\mathfrak{a}_{\Omega_n} \xrightarrow{R} \mathfrak{a}_\Omega$ and now Theorem 2.3 finishes the proof. □

Finally, we present a result concerning the approximation of Ω from the outside. We have

THEOREM 4.11. *Let $\Omega \subset \mathbb{R}^d$ be an open set, and $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of open sets shrinking to Ω. Assume that $\tilde{H}_0^1(\Omega) = H_0^1(\Omega)$. Let $\mathfrak{a}_{\Omega_1}$ be an elliptic form defined on $D(\mathfrak{a}_{\Omega_1}) \subset H_0^1(\Omega_1)$ and let $\mathfrak{a}_{\Omega_n}$ and $\mathfrak{a}$ be the elliptic forms with the same coefficients defined on $D(\mathfrak{a}_{\Omega_n}) \subset H_0^1(\Omega_n)$. Then we have that*

$$e^{-t\mathfrak{a}_{\Omega_n}} \to e^{-t\mathfrak{a}_{\Omega}}$$

strongly.

PROOF. Here of course, we will apply Theorem 2.11. We clearly have that $D(\mathfrak{a}_{\Omega_{n+1}}) \subset D(\mathfrak{a}_{\Omega_n})$. Also the second condition (even the strong second condition without the modulus) is obvious, since $\mathfrak{a}_{\Omega_m} - \mathfrak{a}_{\Omega_n} = 0$ for $m \geq n$. Also, we have that $D(\mathfrak{a}) = \bigcap D(\mathfrak{a}_{\Omega_n})$. Here, the inclusion $\subset$ is obvious and the other inclusion uses that $\tilde{H}_0^1(\Omega) = H_0^1(\Omega)$. It follows that $\mathfrak{a}_{\Omega_n} \to \mathfrak{a}$. Now the Theorems 2.11 and 2.3 prove the assertion. $\square$

Notes and References for Chapter 4: As form methods are the usual means of establishing differential operators as semigroup generators and as there is a wealth of results concerning differential operators, we could give plenty of references here. However we mention:

OUHABAZ [**14**, chapter 4] and ARENDT ET AL [**4**, chapter 7] for general elliptic forms, ARENDT, BATTY [**3**] and DAVIES [**8**, chapter 8] for Schrödinger operators.

Theorem 4.6 was suggested by R. CHILL, whereas unbounded drifts were considered by ARENDT, METAFUNE, PALLARA in [**5**]. However, the technique used there is different and leads to different results.

Domain approximation as in section 4, was considered by ARENDT [**2**] but there were also different methods used.

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