Dirichlet regularity and degenerate diffusion

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Abstract

Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set and let $m : \Omega \to (0, \infty)$ be measurable and locally bounded. We study a natural realization of the operator $m\triangle$ in $C_0(\Omega) := \{u \in C(\overline{\Omega}) : u_{|\partial\Omega} = 0\}$. If Ω is Dirichlet regular, then the operator generates a positive contraction semigroup on $C_0(\Omega)$ whenever $\frac{1}{m} \in L^p_{loc}(\Omega)$ for some $p > \frac{N}{2}$. If $m(x)$ does not go fast enough to 0 as $x \to \partial\Omega$, then Dirichlet regularity is necessary. However, if $|m(x)| \leq c \cdot dist(x, \partial\Omega)^2$, then we show that $m\triangle_0$ generates a semigroup on $C_0(\Omega)$ without any regularity assumptions on Ω. We show that the condition for degeneration of m near the boundary is optimal.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be open and bounded and let $m \in L^\infty_{loc}(\Omega)$ be strictly positive. The aim of this paper is to investigate when a natural realization of the operator $m\triangle$ in $C_0(\Omega) := \{u \in C(\overline{\Omega}) : u_{|\partial\Omega} = 0\}$ generates a C_0-semigroup. If Ω is Dirichlet regular, then it suffices that $\frac{1}{m} \in L^p_{loc}(\Omega)$ for some $\frac{N}{2} < p \leq \infty$. If $\frac{1}{m} \in L^p(\Omega)$, then Dirichlet regularity is a necessary condition. However if the *diffusion is weak at a point* $z \in \partial\Omega$ in the sense that $m(x) \leq c \cdot dist(x, \partial\Omega)^2$ in a neighbourhood of z, then Dirichlet regularity is not needed.

In fact, these phenomena are of local nature. Our main result (Theorem 7.1) says the following. Let $m \in L^\infty(\Omega)$ be strictly positive such that $\frac{1}{m} \in L^p_{loc}(\Omega)$ for some $\frac{N}{2} < p \leq \infty$. Assume that for each $z \in \partial\Omega$ one of the following conditions is satisfied

(a) z is a regular point (in the sense of Wiener) or

(b) the diffusion is weak at z

Then $m\triangle_0$ generates a positive C_0-semigroup on $C_0(\Omega)$. Here $m\triangle_0$ is the natural realization of $m\triangle$ in $C_0(\Omega)$ (see Section 4).
Our notion of weak diffusion is optimal. We show that it does not suffice that $m(x) \leq c \cdot dist(x, \partial\Omega)^\beta$ for some $\beta < 2$ to ensure that $m\triangle_0$ generates a semigroup.

It is much easier to study the operator in the setting of L^p spaces. In fact, introducing a weight, we use form methods to obtain a symmetric submarkovian semigroup in Section 3. However, there are good reasons to study the operator on the space $C_0(\Omega)$. One reason is that we obtain a Feller semigroup in this way with the corresponding relations to stochastic processes. Another reason concerns

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possible applications to non-linear problems and dynamical systems. For semilinear problems the space $C_0(\Omega)$ is much better suited than $L^p(\Omega)$-spaces since composition with a locally Lipschitz continuous function is locally Lipschitz continuous on $C_0(\Omega)$ but never on $L^p(\Omega)$, see the treatise of Cazanave-Haraux ([9]), for example. Studying arbitrary measurable functions m seems to be useful for possible applications to quasilinear equations.

Operators of the form $m\triangle$ are partially contained in the class of operators considered by Davies ([11]) and by Pang ([19]) who study mainly spectral properties but also invariance of $C_0(\Omega)$ for Ω of class C^∞. However, the investigation of the role of Dirichlet regularity seems to be completely new in this context.

2 Preliminaries

Here we fix some notation and explain frequently used arguments. Let $\Omega \subset \mathbb{R}^N$ be open and bounded. We write $\omega \subset\subset \Omega$ if ω is an open subset of $\mathbb{R}^N$ such that $\overline{\omega} \subset \Omega$. The space $C_c(\Omega)$ denotes continuous functions on Ω with values in $\mathbb{R}$ having compact support. $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$ is the space of all test functions and $\mathcal{D}(\Omega)'$ the space of all distributions.
We denote by $H^1(\Omega) := \{u \in L^2(\Omega) : \ D_j u \in L^2(\Omega), \ j = 1, ..., d\}$ the first Sobolev space and by $H_0^1(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$. We let

$$L^p_{loc}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \quad \text{measurable:} \ \int_\omega |u(x)|^p \, dx < \infty \quad \text{whenever } \omega \subset\subset \Omega \right\},$$

where $1 \le p < \infty$. Similarly,

$$H^1_{loc}(\Omega) := \left\{ u \in L^2_{loc}(\Omega) : \ D_j u \in L^2_{loc}(\Omega) \quad \text{for } j = 1, ..., d \right\}.$$

We let

$$C_0(\Omega) := \left\{ u \in C(\overline{\Omega}) : \ u_{|\partial\Omega} = 0 \right\}$$

where $\partial\Omega$ is the boundary of Ω.
Then $H^1(\Omega) \cap C_0(\Omega) \subset H_0^1(\Omega)$, but

$$H_0^1(\Omega) \cap C(\overline{\Omega}) \subset C_0(\Omega) \quad \text{if and only if } \Omega \text{ is regular in capacity}$$

(see [8]). The spaces $H_0^1(\Omega)$ and $H^1(\Omega)$ are sublattices of $L^2(\Omega)$. More precisely

$$u \in H^1(\Omega) \quad \text{implies } D_j u^+ \in H^1(\Omega) \text{ and } D_j u^+ = \chi_{\{u>0\}} D_j u \quad j = 1, ..., d,$$

where by χ_A we denote the characteristic function of a set A.
If $u \in H_0^1(\Omega)$, then also $u^+ \in H_0^1(\Omega)$.

If $u \in L^1_{loc}(\Omega)$, then the Laplacian $\triangle u$ is a distribution. By

$$-\triangle u \le 0 \quad \text{in } \mathcal{D}(\Omega)'$$

we mean that

$$-\langle \triangle u, v \rangle \le 0 \qquad \text{whenever } 0 \le v \in \mathcal{D}(\Omega).$$

If $u \in H^1_{loc}(\Omega)$, this is equivalent to

$$\int_\Omega \nabla u(x) \nabla v(x) \, dx \le 0 \quad \text{for } 0 \le v \in \mathcal{D}(\Omega) \tag{1}$$

and if $u \in H^1(\Omega)$, both inequalities remain true for all $0 \le u \in H_0^1(\Omega)$. In fact, the cone $\mathcal{D}(\Omega)_+$ of all positive test functions is dense in $H_0^1(\Omega)_+ := \left\{u \in H_0^1(\Omega) :\ u \ge 0\right\}$.

We frequently use the following **Maximum principle:** Let $u \in H^1(\Omega)$ such that

$$-\triangle u \le 0.$$

If $u^+ \in H_0^1(\Omega)$, then $u \le 0$.

In fact, taking $v = u$ in (1) we obtain $\int_\Omega |\nabla u(x)^+|^2\, dx \le 0$. By Poincaré's inequality, this implies that $u^+ = 0$.

3 The semigroup on $L^2(\Omega, \frac{dx}{m(x)})$.

Let $m : \Omega \to (0,\infty)$ be measurable such that $\frac{1}{m} \in L^1_{loc}(\Omega)$. We consider the Hilbert space $L^2(\Omega, \frac{dx}{m(x)})$ with the scalar product

$$\langle u|v\rangle = \int_\Omega u(x)v(x)\,\frac{dx}{m(x)}.$$

On $L^2(\Omega, \frac{dx}{m(x)})$ we define the operator $m\triangle_2$ by

$$\begin{aligned}\mathcal{D}(m\triangle_2) &:= \left\{u \in H_0^1(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)}) : \exists f \in L^2(\Omega, \frac{dx}{m(x)}) \text{ such that } \triangle u = \frac{f}{m}\right\}\\ (m\triangle_2)u &:= f\end{aligned}$$

Note that $\frac{f}{m} \in L^1_{loc}(\Omega)$ since for $\omega \subset\subset \Omega$

$$\int_\omega \frac{|f(x)|}{m(x)}\, dx \le \Big(\int_\omega |f(x)|^2 \frac{dx}{m(x)}\Big)^{\frac{1}{2}} \Big(\int_\omega \frac{dx}{m(x)}\Big)^{\frac{1}{2}}.$$

Thus the identity $\triangle u = \frac{f}{m}$ is well-defined in $\mathcal{D}(\Omega)'$. The expression $m\triangle_2$ is purely symbolic and has to be understood in the sense of the above definition. In fact, in general $\triangle u$ is merely in $\mathcal{D}(\Omega)'$ and $m\triangle u$ cannot be defined as a distribution.
We will prove the following theorem.

Theorem 3.1 *The operator $m\triangle_2$ is self-adjoint and generates a positive, contractive C_0-semigroup T_2 on $L^2(\Omega, \frac{dx}{m(x)})$. Moreover, the semigroup is submarkovian.*

Here, an operator S on $L^2(\Omega, \frac{dx}{m(x)})$ is called *submarkovian* if $f(x) \le 1$ a.e. implies $Sf(x) \le 1$ a.e. This is equivalent to saying that S is positive and

$$\|Sf\|_\infty \le \|f\|_\infty \text{ for all } f \in L^2(\Omega, \frac{dx}{m(x)}) \cap L^\infty(\Omega).$$

To say that the semigroup T_2 is *submarkovian* means that each $T_2(t)$, $t \ge 0$ is submarkovian.

We set $V := H_0^1(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)})$. We let $\mathcal{D}(\Omega)_+ := \{v \in \mathcal{D}(\Omega) : v \ge 0\}$ and $V_+ := \{u \in V : u \ge 0 \text{ a.e.}\}$.

Proposition 3.2 *$\mathcal{D}(\Omega)$ is dense in V and $\mathcal{D}(\Omega)_+$ is dense in V_+.*

Proof. We prove the second assertion. The first assertion then follows since $V = V_+ - V_+$.
a) Let $u \in V_+$. There exists a sequence $\varphi_n \in \mathcal{D}(\Omega)$ s.t. $\varphi_n \to u$ in $H^1(\Omega)$. Let $u_n := (\varphi_n \wedge u) \vee 0$. Then $0 \le u_n \le u$ and $u_n \to u$ in $H^1(\Omega)$. Moreover $u_n \to u$ a.e. (for a subsequence which we denote also by u_n). Hence $u_n \to u$ in $L^2(\Omega, \frac{dx}{m(x)})$ by the dominated convergence theorem. We have shown that $V_+ \cap L_c^\infty(\Omega)$ is dense in V_+, where

$$L_c^\infty(\Omega) := \{u \in L^\infty(\Omega) : \text{supp } u \subset \Omega \text{ is compact}\}$$

b) Let $u \in V_+ \cap L_c^\infty(\Omega), u_n := \rho_n * u$, where ρ_n is a mollifier. Then $u_n \in \mathcal{D}(\Omega)$, supp $u_n \subset K \subset\subset \Omega$ (for $n \ge n_0$) and $\|u_n\|_\infty \le c$ (for $n \ge n_0$), $u_n \to u$ in $H^1(\Omega)$ and $u_n \to u$ a. e. after choosing a subsequence. Hence $u_n \to u$ in $L^2(\Omega, \frac{dx}{m(x)})$. □

Proof of the Theorem 3.1.
Let $a : V \times V \to \mathbb{R}$ be given by

$$a(u, v) = \int_\Omega \nabla u(x) \nabla v(x)\, dx.$$

Then a is continuous, symmetric and bilinear. Moreover, a is *accretive*, i.e., $a(u, u) \ge 0$ for all $u \in V$ and *elliptic* with respect to $L^2(\Omega, \frac{dx}{m(x)})$, i.e.,

$$a(u, u) + \omega \|u\|^2_{L^2(\Omega, \frac{dx}{m(x)})} \ge \alpha \|u\|_V^2$$

for some $\omega \in \mathbb{R}$ and $\alpha > 0$.
This follows from Poincaré's inequality, which asserts that $\sqrt{\int_\Omega |\nabla u(x)|^2\, dx}$ defines an equivalent norm on $H_0^1(\Omega)$.
Let A be the operator associated with a . Then A is self-adjoint and $-A$ generates a contractive semigroup T_2 on $L^2(\Omega, \frac{dx}{m(x)})$. We show that $-m\triangle_2 = A$. In fact, for $u, f \in L^2(\Omega, \frac{dx}{m(x)})$ we have by definition,

$$\begin{aligned} u \in \mathcal{D}(A) \text{ and } -Au = f \qquad & \text{if and only if} \\ a(u, v) = -\int_\Omega f(x) v(x) \frac{dx}{m(x)} \qquad & \text{for all } v \in V. \end{aligned}$$

Taking $v \in \mathcal{D}(\Omega)$, this implies that $\triangle u = \frac{f}{m}$. Hence $u \in \mathcal{D}(m\triangle_2)$ and $(m\triangle_2)u = f$. Conversely, if $u \in \mathcal{D}(m\triangle_2)$ and $(m\triangle_2)u = f$, then $\triangle u = \frac{f}{m}$ in $\mathcal{D}(\Omega)'$. Since $u \in H_0^1(\Omega)$, this implies that

$$\int_\Omega \nabla u(x) \nabla v(x)\, dx = -\langle \triangle u, v\rangle = -\int_\Omega f(x) v(x) \frac{dx}{m(x)}$$

for all $v \in \mathcal{D}(\Omega)$. Since $\mathcal{D}(\Omega)$ is dense in V it follows that $u \in \mathcal{D}(A)$ and $Au = f$.
It follows from the Beurling-Deny criterion ([10], Theorem 1.3.3) or ([17], Corollary 2.17) that the semigroup is submarkovian. □

As a consequence we find a consistent family T_p, $1 \le p \le \infty$, of semigroups on $L^p(\Omega, \frac{dx}{m(x)})$, such that T_2 is the given semigroup generated by $m\triangle_2$. Here T_p is a positive, contractive C_0-semigroup for $1 \le p < \infty$ and $T_\infty(t) = T_1'(t)$ for all $t \ge 0$. We denote the generator of T_p by $m\triangle_p$. Thus $m\triangle_\infty = (m\triangle_1)'$.

We note that the consistency of semigroups implies the consistency of the resolvents. In particular

$$R(\lambda, m\triangle_\infty) f = R(\lambda, m\triangle_2) f \tag{2}$$

for all $\lambda > 0$, $f \in L^\infty(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)})$. We also note that

$$R(\lambda, m\triangle_\infty) \geq 0 \quad \text{ for all } \lambda > 0.$$

Finally, we will frequently use the following local regularity of the Laplacian.
Let $\frac{N}{2} < p \leq \infty$. Then

$$u \in L^1_{loc}(\Omega),\ \triangle u \in L^p_{loc}(\Omega) \quad \text{implies } u \in C(\Omega). \tag{3}$$

See ([12], II.3 Proposition 6). To avoid confusion in the case $N = 1$ we shall tacitly assume $p \geq 1$ throughout the paper.

If $m \equiv 1$, then the operator $\triangle_p = m\triangle_p$ is just the Dirichlet Laplacian on $L^p(\Omega)$. We need the following properties of this operator.

Proposition 3.3 *The operator $\triangle_p$ is invertible. Moreover, for $\frac{N}{2} < p \leq \infty$ the following holds:*

(a) $\mathcal{D}(\triangle_p) = \left\{u \in H^1_0(\Omega) : \triangle u \in L^p(\Omega)\right\}$ *and* $\triangle_p u = \triangle u$ *in* $\mathcal{D}(\Omega)'$ *for all* $u \in \mathcal{D}(\triangle_p)$

(b) $\mathcal{D}(\triangle_p) \subset C^b(\Omega) := \{u : \Omega \to \mathbb{R} : u \textit{ is bounded and continuous } \}$

Proof. The invertibility follows from ([10], Theorem 1.6.3), for example. Note that for $\frac{N}{2} < p \leq \infty$

$$\|T_p(t)\|_{\mathcal{L}(L^p(\Omega), L^\infty(\Omega))} \leq ct^{-\frac{N}{2p}} e^{-\omega t} \quad (t \geq 0)$$

for some $c > 0$, $\omega > 0$ ([17] Lemma 6.5). Thus

$$R(0, \triangle_p) = \int_0^\infty T_p(t)\, dt \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega)).$$

Let $f \in L^p(\Omega)$, $u = R(0, \triangle_p)f$. Then $u \in L^\infty(\Omega)$. Moreover, $-\triangle u = f$ in $\mathcal{D}(\Omega)'$. In fact, let $f_k \to f$ in $L^p(\Omega)$ where $f_k \in L^2(\Omega) \cap L^p(\Omega)$. Then $u_k := R(0, \triangle_p)f_k \to u$ in $L^\infty(\Omega)$. Moreover, since $R(0, \triangle_p)f_k = R(0, \triangle_2)f_k$, one has $u_k \in H^1_0(\Omega)$ and $-\triangle u_k = f_k$ in $\mathcal{D}(\Omega)'$. Since $u_k \to u$ in $L^\infty(\Omega) \hookrightarrow \mathcal{D}(\Omega)'$, it follows that $\triangle u_k \to \triangle u$ in $\mathcal{D}(\Omega)'$. Thus $-\triangle u = f$. It follows from (3) that $u \in C(\Omega)$. Finally, by the definition of $\triangle_2$, one has

$$\int_\Omega |\nabla u_k(x)|^2\, dx = \int_\Omega f_k(x) u_k(x)\, dx \leq \|f_k\|_{L^p(\Omega)} \|u_k\|_{L^\infty(\Omega)} |\Omega|^{\frac{1}{p'}}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Thus $(u_k)_{k\in\mathbb{N}}$ is bounded in $H^1_0(\Omega)$. Taking a subsequence, we may assume that $u_k \rightharpoonup w \in H^1_0(\Omega)$. Since $u_k \to u \in L^\infty(\Omega)$, it follows that $u = w \in H^1_0(\Omega)$. Thus (b) and one inclusion in (a) are proved.
Let $u \in H^1_0(\Omega)$ such that $f := \triangle u \in L^p(\Omega)$. It remains to show that $u \in \mathcal{D}(\triangle_p)$ and $\triangle_p u = \triangle u$. Let $w = R(0, \triangle_p)f$. Then $w \in H^1_0(\Omega)$ and $-\triangle w = f$ by what has been proved above. Thus $u + w \in H^1_0(\Omega)$ and $\triangle(u + w) = 0$. By the maximum principle (see Introduction) this implies $u + w = 0$. □

Now we can add the following local regularity of the Laplacian.
Let $\frac{N}{2} < p \leq \infty$. Then

$$u \in L^1_{loc}(\Omega),\ \triangle u \in L^p_{loc}(\Omega) \text{ implies } u \in H^1_{loc}(\Omega). \tag{4}$$

In fact, let $u \in L^1_{loc}(\Omega)$ such that $\triangle u \in L^p_{loc}(\Omega)$. Let $\omega \subset\subset \Omega$ be arbitrary and $f = \triangle u_{|\omega} \in L^p(\omega)$. Consider the operator $\triangle_p$ on $L^p(\omega)$. Then $w := \triangle_p^{-1} f \in H^1_0(\omega)$

by Proposition 3.3. Since $\triangle w = f = \triangle u$ in $\mathcal{D}(\Omega)'$, the function $u - w$ is harmonic and hence in $C^\infty(\omega)$. Thus $u \in H^1(\omega)$.

In the following we consider again a function $m : \Omega \to (0,\infty)$ satisfying $\frac{1}{m} \in L^1_{loc}(\Omega)$. We first show how $m\triangle_\infty$ operates on functions.

Proposition 3.4 **(a)** *Let $u \in \mathcal{D}(m\triangle_\infty)$, $f = (m\triangle_\infty)u$. Then*

$$\triangle u = \frac{f}{m} \quad \text{in } \mathcal{D}(\Omega)'.$$

(b) *If $\frac{1}{m} \in L^p_{loc}(\Omega)$ for some $p > \frac{N}{2}$, then*

$$\mathcal{D}(m\triangle_\infty) \subset C^b(\Omega) \cap H^1_{loc}(\Omega).$$

(c) *If $m \in L^\infty_{loc}(\Omega)$, then $\mathcal{D}(\Omega) \subset \mathcal{D}(m\triangle_\infty)$ and $(m\triangle_\infty)u = m \cdot \triangle u$ for $u \in \mathcal{D}(\Omega)$.*

Proof. *(a)* Let $\lambda > 0$. Define $g := \lambda u - f \in L^\infty(\Omega)$. Then $u = R(\lambda, m\triangle_\infty)g$. If $g \in L^\infty(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)})$, then the claim follows from the fact that $R(\lambda, m\triangle_\infty)g = R(\lambda, m\triangle_2)g$. In the general case there exist $g_k \in L^\infty(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)})$ such that $g_k \to g$ for $\sigma(L^\infty(\Omega), L^1(\Omega, \frac{dx}{m(x)}))$. Let $u_k = R(\lambda, m\triangle_\infty)g_k$. Then

$$-\triangle u_k = \frac{g_k - \lambda u_k}{m}$$

Now we use that $R(\lambda, m\triangle_\infty) = R(\lambda, m\triangle_1)'$ is continuous for the *weak$*$*-topology $\sigma(L^\infty(\Omega), L^1(\Omega, \frac{dx}{m(x)}))$. Hence $u_k \to u$ for $\sigma(L^\infty(\Omega), L^1(\Omega, \frac{dx}{m(x)}))$. Since $\mathcal{D}(\Omega) \subset L^1(\Omega, \frac{dx}{m(x)})$ we conclude that $u_k \to u$ in $\mathcal{D}(\Omega)'$. Hence $\triangle u_k \to \triangle u$ in $\mathcal{D}(\Omega)'$. Since $g_k - \lambda u_k \to g - \lambda u$ for $\sigma(L^\infty(\Omega), L^1(\Omega, \frac{dx}{m(x)}))$, it follows that $\frac{g_k - \lambda u_k}{m} \to \frac{g - \lambda u}{m}$ in $\mathcal{D}(\Omega)'$. Thus

$$-\triangle u = \frac{g - \lambda u}{m} = -\frac{f}{m}.$$

The proof of *(a)* is complete.
(b) This follows now from (3) and (4).
(c) Assume that $m \in L^\infty_{loc}(\Omega)$. Let $u \in \mathcal{D}(\Omega)$, $f = m \cdot \triangle u$. Then $u \in H^1_0(\Omega)$, $f \in L^2(\Omega, \frac{dx}{m(x)})$ and $\triangle u = \frac{f}{m}$. Thus $u \in \mathcal{D}(m\triangle_2)$ and $(m\triangle_2)u = f$. Let $\lambda > 0$ and set $g := \lambda u - f$. Then $g \in L^\infty(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)})$ and $R(\lambda, m\triangle_\infty)g = R(\lambda, m\triangle_2)g = u$. Thus $u \in \mathcal{D}(m\triangle_\infty)$ and $\lambda u - (m\triangle_\infty)u = g = \lambda u - f$, i.e., $(m\triangle_\infty)u = f$. □
In Proposition 3.4, the boundary condition is not incorporated. But if $\frac{1}{m} \in L^1(\Omega)$, then $L^\infty(\Omega) \subset L^2(\Omega, \frac{dx}{m(x)})$ and the operator $m\triangle_\infty$ is just the part of $m\triangle_2$ in $L^\infty(\Omega)$. Thus, if $\frac{1}{m} \in L^1(\Omega)$, then

$$\mathcal{D}(m\triangle_\infty) = \left\{ u \in H^1_0(\Omega) \cap L^\infty(\Omega) : \exists f \in L^\infty(\Omega) \text{ s.t. } \triangle u = \frac{f}{m} \right\} \quad (5)$$
$$(m\triangle_\infty)u = f.$$

If $\frac{1}{m} \in L^p(\Omega)$ for some $\infty \geq p > \frac{N}{2}$, we can even assert more.

Proposition 3.5 *Assume that $\frac{1}{m} \in L^p(\Omega)$ where $\frac{N}{2} < p \leq \infty$. Then $m\triangle_\infty$ is invertible.*

Proof. Let $f \in L^\infty(\Omega)$. Then $\frac{f}{m} \in L^p(\Omega)$. Thus by Proposition 3.3 there exists $u \in H^1_0(\Omega)$ such that $\triangle u = \frac{f}{m}$. This shows that $m\triangle_\infty$ is surjective. If $u \in \mathcal{D}(m\triangle_\infty)$, $(m\triangle_\infty)u = 0$, then by (5) we have $u \in H^1_0(\Omega)$ and $\triangle u = 0$. This implies that $u = 0$. Thus $(m\triangle_\infty)$ is injective. Since the operator is closed, the proof is finished.

□

The positive semigroups T_p generated by $m\triangle_p$ on $L^p(\Omega, \frac{dx}{m(x)})$ have many interesting properties. We just mention that they are always irreducible if Ω is connected (where we assume only $0 < m,\ \frac{1}{m} \in L^1_{loc}(\Omega)$ as before). This means that

$$(e^{t(m\triangle_p)}f)(x) > 0 \text{ a.e. for all } 0 \leq f \in L^p(\Omega, \frac{dx}{m(x)}),\ f \neq 0, \text{ and for all } t > 0.$$

For $p = 2$ this follows from Ouhabaz' simple criterion that

$$\chi_C \cdot H^1_0(\Omega) \subset H^1_0(\Omega) \quad \text{implies } |C| = 0 \text{ or } |\Omega \backslash C)| = 0$$

for each Borel set $C \subset \Omega$ (see [17], Section 4.2 or [3]). For another proof of irreducibility we refer to [13], and for consequences to [4].

4 The operator $m\triangle_0$ on $C_0(\Omega)$

Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Let $m : \Omega \to (0, \infty)$ be a measurable function such that $m \in L^\infty_{loc}(\Omega)$ and $\frac{1}{m} \in L^p_{loc}$ where $p > \frac{N}{2}$. We want to define a maximal realization of $m\triangle$ in $C_0(\Omega)$. If $u \in C_0(\Omega)$ then $\triangle u \in \mathcal{D}(\Omega)'$, but $m\triangle u$ may not be defined as a distribution. Thus the following definition is natural.

Definition 4.1 *We define the operator $m\triangle_0$ on $C_0(\Omega)$ by*

$$\begin{aligned} \mathcal{D}(m\triangle_0) &:= \left\{ u \in C_0(\Omega) :\ \exists f \in C_0(\Omega) \ s.t.\ \triangle u = \frac{f}{m} \right\} \\ (m\triangle_0)u &:= f \end{aligned}$$

Since $\frac{f}{m} \in L^1_{loc} \subset \mathcal{D}(\Omega)'$ this definition makes sense. The notation $(m\triangle_0)$ is purely symbolic. But if $u \in C_0(\Omega) \cap C^2(\Omega)$ such that $m \cdot \triangle u \in C_0(\Omega)$, then $u \in \mathcal{D}(m\triangle_0)$ and $(m\triangle_0)u = m \cdot \triangle u$.

Proposition 4.2 *The operator $m\triangle_0$ is closed and dissipative. Moreover, if $R(\lambda_0, m\triangle_\infty)C_0(\Omega) \subset C_0(\Omega)$ for some $\lambda_0 > 0$, then $m\triangle_0$ generates a C_0-semigroup of positive contractions on $C_0(\Omega)$. In that case*

$$\begin{aligned} &(0, \infty) \subset \rho(m\triangle_0) \\ &R(\lambda, m\triangle_\infty)C_0(\Omega) \subset C_0(\Omega) \quad \text{for all } \lambda > 0 \qquad \text{and} \\ &R(\lambda, m\triangle_0) = R(\lambda, m\triangle_\infty)_{|C_0(\Omega)}. \end{aligned}$$

Note that in general, $\mathcal{D}(\Omega) \not\subseteq \mathcal{D}(m\triangle_0)$ since we do not assume that m is continuous. Thus in Proposition 4.2 density of the domain (which is necessary for the generation property) needs a separate argument.
Since $m\triangle_0$ is dissipative, it follows in particular that no proper restriction of $m\triangle_0$ may generate a C_0-semigroup on $C_0(\Omega)$.
We first prove dissipativity.

Lemma 4.3 *Let $\lambda > 0,\ u = \mathcal{D}(m\triangle_0),\ f = \lambda u - (m\triangle_0)u$. Let $c > 0$ be such that*

$$f(x) \leq c \quad \text{for all } x \in \Omega.$$

Then $\lambda u(x) \leq c$ for all $x \in \Omega$.

Proof. By the definition of the operator we have

$$\lambda \frac{u}{m} - \triangle u = \frac{f}{m} \leq \frac{c}{m}.$$

Since by (4) $u \in H^1_{loc}(\Omega)$, this implies that for $0 \leq v \in \mathcal{D}(\Omega)$

$$\int_\Omega \frac{(\lambda u(x) - c)}{m(x)} v(x)\, dx + \int_\Omega \nabla u(x) \nabla v(x)\, dx \leq 0. \tag{6}$$

Since $u \in C_0(\Omega)$, $(\lambda u - c)^+$ has compact support. Let $\omega \subset\subset \Omega$ such that supp $(\lambda u - c)^+ \subset \omega$. Then $(\lambda u - c)^+ \in H^1_0(\omega)$ and $(\lambda u - c) \in H^1(\omega)$. Now (6) implies that

$$\int_\omega \frac{(\lambda u(x) - c)}{m(x)} v(x)\, dx + \frac{1}{\lambda} \int_\omega \nabla(\lambda u(x) - c) \nabla v(x)\, dx \leq 0$$

for all $0 \leq v \in H^1_0(\omega)$. Taking in particular, $v := (\lambda u - c)^+$ we see that

$$\int_\omega \frac{(\lambda u(x) - c)^{+2}}{m(x)}\, dx + \frac{1}{\lambda} \int_\omega |\nabla(\lambda u(x) - c)^+|^2\, dx \leq 0$$

This implies that $(\lambda u - c)^+ = 0$, i.e., $\lambda u \leq c$. □

Applying Lemma 4.3 to $\pm u$, we see that

$$\|\lambda u\|_{L^\infty(\Omega)} \leq \|\lambda u - (m\triangle_0)u\|_\infty$$

for all $u \in \mathcal{D}(m\triangle_0)$, i.e., $m\triangle_0$ is dissipative. But in fact, Lemma 4.3 shows that the operator $m\triangle_0$ is *dispersive*. We refer to ([5], [16],Chapter II) for this notion.

Proof of Proposition 4.2. The dissipativity has been proved above and the closedness is easy to see. Let now $R(\lambda, m\triangle_\infty)C_0(\Omega) \subset C_0(\Omega)$ for some $\lambda > 0$. We show that $\lambda \in \rho(m\triangle_0)$ and $R(\lambda, m\triangle_0) = R(\lambda, m\triangle_\infty)_{|C_0(\Omega)}$. Let $f \in C_0(\Omega)$ and consider $u = R(\lambda, m\triangle_\infty) f \in C_0(\Omega)$. Then (by Proposition 3.4)

$$\lambda \frac{u}{m} - \triangle u = \frac{f}{m} \quad \text{in } \mathcal{D}(\Omega)'.$$

It follows that $u \in \mathcal{D}(m\triangle_0)$ and $(\lambda u - (m\triangle_0)u) = f$. We have shown that $\lambda - (m\triangle_0)$ is surjective. Since the injectivity of $(\lambda - m\triangle_0)$ follows from the dissipativity of $m\triangle_0$, the closed graph theorem implies now that $\lambda \in \rho(m\triangle_0)$. The calculation above shows also that $R(\lambda, m\triangle_0)f = u = R(\lambda, m\triangle_\infty)f$.
By the resolvent identity (see ([1], Proposition 3.II.2)) we have for $0 \leq f \in C_0(\Omega)$ and $\lambda > \lambda_0$

$$0 \leq R(\lambda, m\triangle_\infty)f \leq R(\lambda_0, m\triangle_\infty)f \in C_0(\Omega).$$

Since by Proposition 3.3 the function $R(\lambda, m\triangle_\infty)f$ is continuous, it follows from the domination property above that $R(\lambda, m\triangle_\infty)f \in C_0(\Omega)$. Thus $C_0(\Omega)$ is invariant for all $\lambda \geq \lambda_0$. Hence $[\lambda_0, \infty) \subset \rho(m\triangle_0)$.
Next we show that $\mathcal{D}(m\triangle_0)$ is dense in $C_0(\Omega)$. Since $m \in L^\infty_{loc}(\Omega)$, we have $\mathcal{D}(\Omega) \subset \mathcal{D}(m\triangle_\infty)$ by Proposition 3.4. Hence $C_0(\Omega) \subset \overline{\mathcal{D}(m\triangle_\infty)}$. Thus, for $f \in C_0(\Omega)$ one has

$$\lim_{\lambda\to\infty} \lambda R(\lambda, m\triangle_0)f = \lim_{\lambda\to\infty} \lambda R(\lambda, m\triangle_\infty)f = f.$$

Since $\lambda R(\lambda, m\triangle_0)f \in \mathcal{D}(m\triangle_0)$, density of the domain is proved. Now the Lumer-Phillips theorem implies that $m\triangle_0$ generates a contractive C_0-semigroup. Since the resolvent of $m\triangle_0$ is positive, this semigroup is positive. It follows also that $(0, \infty) \subset \rho(m\triangle_0)$. □

We will now consider two cases which imply the invariance given in Proposition 4.2 namely that Ω is Dirichlet regular or that the diffusion coefficient $m(x)$ tends to 0 fast enough as x approaches the boundary. We start discussing Dirichlet regularity.

5 Regular points

Let $\Omega \subset \mathbb{R}^N$ be open, bounded and let $\frac{N}{2} < p \leq \infty$. Let $m : \Omega \to (0, \infty)$ be measurable such that $m \in L^{\infty}_{loc}(\Omega)$ and $\frac{1}{m} \in L^{p}_{loc}(\Omega)$.

Theorem 5.1 *If Ω is Dirichlet regular, then $m\triangle_0$ generates a positive contractive C_0-semigroup on $C_0(\Omega)$.*

Thus in the case of a Dirichlet regular set, no condition on $m(x)$ as x approaches the boundary is needed. We merely impose a (very weak) regularity condition on m in the interior of Ω.
It will be useful to prove an individual version of Theorem 5.1 first. For this we have to recall the notion of regular points.
Consider the Dirichlet problem.

$$\begin{aligned} h &\in C(\overline{\Omega}) \cap C^2(\Omega) \\ \triangle h &= 0 \text{ in } \Omega \\ h_{|\partial\Omega} &= \varphi \end{aligned} \tag{7}$$

where $\varphi \in C(\partial\Omega)$ is given. Recall that Ω is called *Dirichlet regular*, if for each $\varphi \in C(\partial\Omega)$ a (necessarily unique) solution exists. If Ω has Lipschitz boundary then Ω is Dirichlet regular. Much weaker geometric properties of the boundary suffice, though. In dimension $N = 1$ each bounded open subset Ω of $\mathbb{R}$ is Dirichlet regular. If $N = 2$ then each simply connected bounded open set is Dirichlet regular. This is no longer true in $\mathbb{R}^3$. The Lebesgue cusp gives an example of a simply connected domain with continuous boundary, which is not Dirichlet regular (see [6] for more information).
A function $u \in C(\overline{\Omega})$ is called a *subsolution* if

$$-\triangle u \leq 0 \text{ in } \mathcal{D}(\Omega)' \quad \text{and} \quad \limsup_{x \to z,\, x \in \Omega} u(x) \leq \varphi(z) \quad \text{for all } z \in \partial\Omega.$$

A function $u \in C(\overline{\Omega})$ is called a *supersolution* if

$$-\triangle u \geq 0 \text{ in } \mathcal{D}(\Omega)' \quad \text{and} \quad \liminf_{x \to z,\, x \in \Omega} u(x) \geq \varphi(z) \quad \text{for all } z \in \partial\Omega.$$

Theorem 5.2 ***(Perron)***
Let $\varphi \in C(\partial\Omega)$. Then for all $x \in \Omega$

$$h_\varphi(x) := \sup \{u(x) : \quad u \text{ is a subsolution}\}$$

exists. Moreover,

$$h_\varphi(x) = \inf \{v(x) : \quad v \text{ is a supersolution}\}.$$

The function h_φ is harmonic and

$$\inf_{\partial\Omega} \varphi \leq h_\varphi(x) \leq \sup_{\partial\Omega} \varphi$$

for all $x \in \Omega$. If (7) has a solution h, then $h_\varphi = h$.

The function h_φ is called the *Perron solution* of (7).
A point $z \in \partial\Omega$ is called *regular* if

$$\lim_{x \to z,\, x \in \Omega} h_\varphi(x) = \varphi(z)$$

for all $\varphi \in C(\partial\Omega)$. Thus Ω is Dirichlet regular if and only if each point $z \in \partial\Omega$ is regular. It is possible to characterize regular points by the existence of a barrier or by a capacity condition (Wiener's theorem). We refer to [15].
Now we can formulate the local version of Theorem 5.1 which we want to prove.

Theorem 5.3 *Let Ω be bounded and open. Let $z \in \partial\Omega$ be a regular point. Let $\lambda > 0,\ f \in C_0(\Omega),\ u = R(\lambda, m\triangle_\infty)f$. Then*

$$\lim_{x\to z,\, x\in\Omega} u(x) = 0.$$

Thus, if Ω is Dirichlet regular, then $C_0(\Omega)$ is invariant under $R(\lambda, m\triangle_\infty)$ and Theorem 5.1 follows from Proposition 4.2.
For the proof of Theorem 5.3 we use the following variational characterization of the Perron solution (see [7]).

Theorem 5.4 *Let $\Phi \in C(\overline{\Omega})$ be such that $\triangle\Phi \in H^{-1}(\Omega)$. Let $\varphi = \Phi_{|\partial\Omega}$. Let u be the unique solution of*

$$\begin{aligned} u &\in H_0^1(\Omega) \\ -\triangle u &= \triangle\Phi. \end{aligned}$$

Then $h_\varphi = \Phi + u$.

For our purposes the following consequence is important. Recall that by Proposition 3.3 for all $f \in L^p(\Omega)$ there exists a unique $u \in H_0^1(\Omega)$ such that

$$-\triangle u = f \quad \text{in } \mathcal{D}(\Omega)'.$$

In fact, $u = R(0, \triangle_p)f$ where $\triangle_p$ denotes the Dirichlet Laplacian on $L^p(\Omega)$. Moreover, one has $u \in C^b(\Omega)$.

Corollary 5.5 *Let $f \in L^p(\Omega),\ u = R(0, \triangle_p)f$. Then*

$$\lim_{x\to z,\, x\in\Omega} u(x) = 0$$

for each regular point $z \in \partial\Omega$. Thus, if Ω is Dirichlet regular, then $u \in C_0(\Omega)$.

Proof. It follows from the Sobolev embedding theorem that $L^p(\Omega) \subset H^{-1}(\Omega)$. Let $f \in L^p(\Omega)$. Let $\Phi = E * f$, where E is the Newtonian potential. Then (by [12], II.3, Proposition 6) $\Phi \in C(\mathbb{R}^N)$ and in $\mathcal{D}(\Omega)'$ we have

$$\triangle\Phi = f \in L^p(\Omega) \subset H^{-1}(\Omega).$$

Let $u = R(0, \triangle_p)f$. Then it follows from Theorem 5.4 that $h_\varphi = \Phi + u$. Thus

$$\lim_{x\to z,\, x\in\Omega} h_\varphi(x) = \varphi(z) \qquad \text{if } z \in \partial\Omega \text{ is regular.}$$

Consequently[1], $\lim_{x\to z} u(x) = 0$. □

Remark. a) In [2] a more special case of Corollary 5.5 is proved with the help of H^1-barriers (proof of Theorem 3.8 in [2]).
b) Special cases of Theorem 5.4 were obtained before by Hildebrandt [14] and Simader [20].

Proof of Theorem 5.3. (a) Let $\lambda > 0,\ 0 \le f \in C_c(\Omega),\ u = R(\lambda, m\triangle_\infty)f$. Then $u \in H_0^1(\Omega)$ and

$$\lambda\frac{u}{m} - \triangle u = \frac{f}{m} \quad \text{in } \mathcal{D}(\Omega)'.$$

[1] We will sometimes use the notation $\lim_{x\to z} f(x) := \lim_{x\to z,\, x\in\Omega} f(x)$ for $f : \Omega \to \mathbb{R}$

Moreover $0 \le u \in C^b(\Omega)$. Observe that $0 \le \frac{f}{m} \in L^p(\Omega)$. Let $w = R(0, \triangle_p)\frac{f}{m}$. Then we know that $0 \le w \in H_0^1(\Omega) \cap C^b(\Omega)$ and, by Corollary 5.5, $\lim_{x\to z} w(x) = 0$ for all regular points $z \in \partial\Omega$. By definition

$$-\triangle w = \frac{f}{m} \quad \text{in } \mathcal{D}(\Omega)'.$$

Thus $-\triangle(u-w) \le 0$ in $\mathcal{D}(\Omega)'$. Since $u - w \in H^1(\Omega)$ and $(u-w)^+ \in H_0^1(\Omega)$, it follows from the maximum principle that $u \le w$. Hence $\lim_{x\to z} u(x) = 0$ for each regular point $z \in \partial\Omega$.
(b) Let $z \in \partial\Omega$ be a regular point. Then by *(a)*

$$\lim_{x\to z, x\in\Omega} (R(\lambda, m\triangle_\infty)f)(x) = 0$$

for each $0 \le f \in C_c(\Omega)$, hence also for each $f \in C_c(\Omega)$. Since $C_c(\Omega)$ is dense in $C_0(\Omega)$, this remains true for all $f \in C_0(\Omega)$. □

Next we show a converse of Theorem 5.1. If the diffusion coefficient m is not weak enough at the boundary, then Dirichlet regularity is necessary for $m\triangle_0$ to generate a C_0-semigroup. More precisely, the following holds. Recall that $\frac{N}{2} < p \le \infty$.

Theorem 5.6 *Assume that $\frac{1}{m} \in L^p(\Omega)$. Then $m\triangle_0$ generates a C_0-semigroup if and only if Ω is Dirichlet regular.*

For the proof we need the following.

Proposition 5.7 *Let $u \in C_0(\Omega)$ be such that $-\triangle u = f \in L^p(\Omega)$ for some $p > \frac{N}{2}$. Then $u \in H_0^1(\Omega)$, hence $u = R(0, \triangle_p)f$.*

This follows from ([6],Corollary 1.4) since $L^p(\Omega) \subset H^{-1}(\Omega)$.

Proof of Theorem 5.6. Assume that $m\triangle_0$ generates a C_0-semigroup. Since $\frac{1}{m} \in L^p(\Omega)$, we know from Proposition 3.5 that $[0,\infty) \subset \rho(m\triangle_\infty)$ and $R(\lambda, m\triangle_\infty) \ge 0$ for all $\lambda \ge 0$. We now claim that $R(\lambda, m\triangle_\infty)C_0(\Omega) \subset C_0(\Omega)$ and $R(\lambda, m\triangle_0) = R(\lambda, m\triangle_\infty)_{|C_0(\Omega)}$ for any $\lambda > 0$. Let $f \in C_0(\Omega)$, $u = R(\lambda, m\triangle_0)f$. Then

$$-\triangle u = \frac{f}{m} - \lambda\frac{u}{m} \in L^p(\Omega).$$

Since $u \in C_0(\Omega)$, it follows from Proposition 5.7 that $u \in H_0^1(\Omega)$. Since $\frac{1}{m} \in L^p(\Omega)$ we have $L^\infty(\Omega) \subset L^2(\Omega, \frac{dx}{m(x)})$. Thus by (5) we have $u \in \mathcal{D}(m\triangle_\infty)$ and $\lambda u - (m\triangle_\infty)u = f$. Hence $u = R(\lambda, m\triangle_\infty)f$. This proves the claim.
Since $0 \in \rho(m\triangle_\infty)$, the claim implies that

$$\limsup_{\lambda\to 0} \|R(\lambda, m\triangle_0)\|_{\mathcal{L}(C_0(\Omega))} < \infty,$$

hence $0 \in \rho(m\triangle_0)$ and $R(0, m\triangle_0) \ge 0$.
Let $0 \le f \in C_0(\Omega)$, $f(x) > 0$ for all $x \in \Omega$, $u = R(0, m\triangle_0)f$. Then $u \in C_0(\Omega)$ and $-\triangle u = \frac{f}{m}$ in $\mathcal{D}(\Omega)'$. Hence $R(0, \triangle_p)\frac{f}{m} = u \in C_0(\Omega)$ by Proposition 5.7. We deduce that $R(0, \triangle_p)g \in C_0(\Omega)$ for all $g \in L^p(\Omega)$ such that $|g| \le \frac{f}{m}$ for some $0 \le f \in C_0(\Omega)$. The space of all such functions g is dense in $L^p(\Omega)$. Thus $R(0, \triangle_p)L^p(\Omega) \subset C_0(\Omega)$. Now it follows from ([2], Theorem 2.4) that Ω is Dirichlet regular. □

6 Points of weak diffusion

Let $\Omega \subset \mathbb{R}^N$ be open and bounded and let $m : \Omega \to (0, \infty)$ be a bounded measurable function such that $\frac{1}{m} \in L^p_{loc}(\Omega)$ for some $\frac{N}{2} < p \leq \infty$. Instead of regularity we may assume that m is small in a neighbourhood of a boundary point. We say that $z \in \partial\Omega$ is a *point of weak diffusion* (for the operator $m\triangle$) if there exist $r > 0,\ c > 0$ such that

$$m(x) \leq c \cdot dist(x, \partial\Omega)^2 \tag{8}$$

for all $x \in \Omega \cap B(z, r)$. If $z \in \partial\Omega$ is a point of weak diffusion, then we show that

$$\lim_{x \to z,\ x \in \Omega} (R(\lambda, m\triangle_\infty)f)(x) = 0 \tag{9}$$

for all $f \in C_0(\Omega)$. We will also show that the condition (8) is optimal in the sense that

$$m(x) \leq c \cdot dist(x, \partial\Omega)^\alpha$$

for some $0 < \alpha < 2$ does not suffice to enforce (9).
We need the notion of a regularized distance function.

Lemma 6.1 *There exist a constant $c_\sigma > 0$ and a function $\sigma : \Omega \to (0, +\infty)$, which is of class $C^\infty(\Omega)$ and fulfills:*

$$\begin{array}{rcl} c_\sigma^{-1} d(x) \leq & \sigma(x) & \leq c_\sigma d(x) \\ |\nabla\sigma|^2 & \leq & c_\sigma \\ |\sigma\triangle\sigma| & \leq & c_\sigma \end{array}$$

for all $x \in \Omega$, where $d(x) := \inf\left\{\|x - y\|,\ y \in \mathbb{R}^d \backslash \Omega\right\}$.

See ([21],Chapter 6) for a proof based on the Whitney decomposition of Ω.

Since $\sigma \in C_0(\Omega)$ it follows in particular that $\sigma \in H^1_0(\Omega)$. At first we now consider the case $m(x) := \sigma(x)^2$.

Proposition 6.2 *The operator $\sigma^2\triangle_0$ generates a strongly continuous semigroup of positive contractions on $C_0(\Omega)$.*

Proof. Let $\lambda \geq c_\sigma + 1$ where c_σ is a constant from Lemma 6.1. Set $u = R(\lambda, \sigma^2\triangle_\infty)\sigma$. Since $\sigma \in L^2(\Omega, \frac{dx}{\sigma(x)^2})$ it follows from (2) that $0 \leq u \in H^1_0(\Omega) \cap L^2(\Omega, \frac{dx}{\sigma(x)^2})$ and

$$\lambda\frac{u}{\sigma^2} - \triangle u = \frac{\sigma}{\sigma^2} \quad \text{in } \mathcal{D}(\Omega)'.$$

Since $\sigma\triangle\sigma \leq c_\sigma$, it follows that $\sigma \leq \lambda\sigma - c_\sigma\sigma \leq \lambda\sigma - \sigma^2\triangle\sigma$. Thus

$$\lambda\frac{u}{\sigma^2} - \triangle u = \frac{1}{\sigma^2}\sigma \leq \lambda\frac{\sigma}{\sigma^2} - \triangle\sigma \quad \text{in } \mathcal{D}(\Omega)'.$$

Hence

$$\lambda\frac{(u - \sigma)}{\sigma^2} - \triangle(u - \sigma) \leq 0 \quad \text{in } \mathcal{D}(\Omega)'.$$

Since $u - \sigma \in H^1(\Omega)$ and $(u - \sigma)^+ \leq u \in H^1_0(\Omega)$, it follows that $(u - \sigma)^+ \in H^1_0(\Omega)$. Now the maximum principle (see Preliminaries) implies that $(u - \sigma)^+ \leq 0$, i.e., $u \leq \sigma$.
We have shown that

$$R(\lambda, \sigma^2\triangle_\infty)\sigma \leq \sigma \quad (\lambda \geq \lambda_0 := 1 + c_\sigma). \tag{10}$$

Thus, for $f \in C_0(\Omega)$ such that $|f| \le c\sigma$ one has

$$|R(\lambda, \sigma^2 \triangle_\infty) f| \le c R(\lambda, \sigma^2 \triangle_\infty)\sigma \le c\sigma.$$

Consequently, $R(\lambda, \sigma^2\triangle_\infty) f \in C_0(\Omega)$ for $\lambda \ge \lambda_0$. Since functions satisfying $|f| \le c\sigma$ for some $c \ge 0$ are dense in $C_0(\Omega)$ we deduce that $R(\lambda, \sigma^2\triangle_\infty) C_0(\Omega) \subset C_0(\Omega)$ for $\lambda \ge \lambda_0$. Now the claim follows from Proposition 4.2. □

We comment that the result of Proposition 6.2 may be alternatively deduced from ([11], Theorem 5.4). However, our argument given here is quite different from [11].

We need a local extension of the resolvents of $\sigma^2\triangle$. Recall that $\frac{N}{2} < p \le \infty$.

Lemma 6.3 *Let $\omega \subset\subset \Omega$, $\lambda > 0$. There exists an operator*

$$Q(\lambda, \omega) \in \mathcal{L}(L^p(\omega), C_0(\Omega))$$

such that

$$Q(\lambda, \omega) f = R(\lambda, \sigma^2 \triangle_0) f \quad \text{for all } f \in L^p(\omega) \cap C_0(\Omega).$$

For $f \in L^p(\omega)$ the function $u = Q(\lambda, \omega) f$ is the unique solution of

$$\begin{aligned} & u \in C_0(\Omega) \\ & \lambda \frac{u}{\sigma^2} - \triangle u = \frac{f}{\sigma^2} \ \text{in } \mathcal{D}(\Omega)'. \end{aligned} \tag{11}$$

Moreover, $u \in H^1_0(\Omega)$.

Here we consider $L^p(\omega)$ as a subspace of $L^p(\Omega)$ extending functions by 0 outside ω. Similarly, we consider $C_c(\omega) \subset C_0(\omega) \subset C_0(\Omega)$.

Proof. (a) Let $0 \le f \in C_c(\omega)$. There exists $\delta > 0$ such that $\sigma^2 \ge \delta$ on ω. Let $u = R(\lambda, \sigma^2\triangle_0) f = R(\lambda, \sigma^2 \triangle_2) f$. Then $0 \le u \in H^1_0(\Omega)$ and

$$\lambda \frac{u}{\sigma^2} - \triangle u = \frac{f}{\sigma^2} \le \frac{1}{\delta} f.$$

Let $w := \frac{1}{\delta} R(0, \triangle_p) f$, where $\triangle_p$ denotes the Dirichlet Laplacian on $L^p(\Omega)$. Then $w \in H^1_0(\Omega) \cap L^\infty(\Omega)$ and

$$-\triangle w = \frac{1}{\delta} f \quad \text{in } \mathcal{D}(\Omega)'.$$

Moreover $\|w\|_{L^\infty(\Omega)} \le c_1 \|f\|_{L^p(\omega)}$ where $c_1 = \frac{1}{\delta}\|R(0, \triangle_p)\|_{\mathcal{L}(L^p(\Omega), L^\infty(\Omega))}$ (see Proposition 3.3(b)). We show that $u \le w$. In fact, we have

$$\begin{aligned} -\triangle u &\le \lambda \frac{u}{\sigma^2} - \triangle u \le \frac{1}{\delta} f \quad \text{and} \\ -\triangle w &= \frac{1}{\delta} f, \end{aligned}$$

hence $-\triangle(u - w) \le 0$ in $\mathcal{D}(\Omega)'$. Consequently, by the maximum principle (see Preliminaries), $u \le w$. Thus

$$\|u\|_{L^\infty(\Omega)} \le \|w\|_{L^\infty(\Omega)} \le c_1 \|f\|_{L^p(\omega)}.$$

We have shown that

$$\|R(\lambda, \sigma^2 \triangle_0) f\|_{L^\infty(\Omega)} \le c_1 \|f\|_{L^p(\omega)} \tag{12}$$

for $0 \le f \in C_c(\omega)$. Since for arbitrary $f \in C_c(\omega)$,

$$|R(\lambda, \sigma^2 \triangle_0) f| \le R(\lambda, \sigma^2 \triangle_0)|f|,$$

the estimate (12) remains true for all $f \in C_c(\omega)$. By the density of $C_c(\omega)$ in $L^p(\omega)$ the first claim is proved.
(b) In order to prove the second, let $f \in L^p(\omega)$, $u = Q(\lambda, \omega) f$. Let $f_k \in C_c(\omega)$ be such that $f_k \to f$ in $L^p(\omega)$. Then $u_k := Q(\lambda, \omega) f_k \to u$ in $C_0(\Omega)$. We have $u_k \in H_0^1(\Omega) \cap C_0(\Omega)$ and

$$\lambda \frac{u_k}{\sigma^2} - \triangle u_k = \frac{f_k}{\sigma^2} \quad \text{in } \mathcal{D}(\Omega)'. \tag{13}$$

Passing to the limit as $k \to \infty$ shows that (11) holds.
It remains to show that $u \in H_0^1(\Omega)$. Multiplying (13) by u_k and integrating yields

$$\begin{aligned} \lambda \int_\Omega \frac{u_k(x)^2}{\sigma(x)^2} \, dx + \int_\Omega |\nabla u_k(x)|^2 \, dx = \int_\Omega \frac{f_k(x) u_k(x)}{\sigma(x)^2} \, dx \le \\ \le \|u_k\|_{L^\infty(\Omega)} \frac{1}{\delta^2} \cdot |\Omega|^{\frac{1}{p'}} \|f_k\|_{L^p(\Omega)}. \end{aligned}$$

This shows that $(u_k)_{k \in \mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Thus, passing to a subsequence we may assume that $u_k \rightharpoonup w \in H_0^1(\Omega)$. Since $u_k \to u$ in $C_0(\Omega)$, it follows that $u = w \in H_0^1(\Omega)$. □

Now we consider a more general function m satisfying the hypothesis formulated in the beginning of this section. We prove regularity of $m\triangle_\infty$ at points of weak diffusion.

Theorem 6.4 *Let $z \in \partial\Omega$ be a point of weak diffusion (in the sense of (8)). Let $f \in C_0(\Omega)$, $\lambda > 0$, $u = R(\lambda, m\triangle_\infty) f$. Then*

$$\lim_{x \to z,\ x \in \Omega} u(x) = 0.$$

Proof. Let $r_1 > 0$ be a large radius such that $\overline{\Omega} + \overline{B}(0, r) \subset B(0, r_1)$. Consider the open set

$$\tilde{\Omega} := (\Omega \cap B(z, r)) \cup (B(0, r_1) \backslash \overline{B}(z, \frac{r}{2})).$$

Then $\Omega \subset \tilde{\Omega}$ and $\overline{B}(z, \frac{r}{2}) \cap \partial\Omega \subset \partial\tilde{\Omega}$. In particular, $z \in \partial\tilde{\Omega}$. Consider a regularized distance $\tilde{\sigma}$ with respect to $\tilde{\Omega}$. Then there exists a constant $c > 0$ such that

$$m(x) \le c \tilde{\sigma}(x)^2 \quad \text{for all } x \in \Omega. \tag{14}$$

In fact, for $x \in B(z, r) \cap \Omega$ this follows from (8). But for $x \in \Omega \backslash B(z, \frac{3}{4} r)$, one has $dist(x, \partial\tilde{\Omega}) \ge \frac{r}{4}$. Since m is bounded, it follows that

$$m(x) \le c_2 (\frac{r}{4})^2 \le c_2 dist(x, \partial\tilde{\Omega})^2$$

for all $x \in \Omega \backslash B(z, \frac{3}{4} r)$. This shows that (14) is valid for a suitable constant $c > 0$.
Now let $\lambda > 0$. Let $0 \le f \in C_c(\Omega)$, $u = R(\lambda, m\triangle_\infty) f$. Then $u \in C^b(\Omega) \cap H_0^1(\Omega)$ and

$$\lambda \frac{u}{m} - \triangle u = \frac{f}{m} \quad \text{in } \mathcal{D}(\Omega)'.$$

Let $\rho := \frac{m}{\tilde{\sigma}^2}$. Then $0 < \rho \le c$ on Ω and

$$\frac{1}{c} \le \frac{1}{\rho} = \frac{\tilde{\sigma}^2}{m} \in L^p_{loc}(\Omega).$$

Hence

$$\frac{\lambda}{c}\frac{u}{\tilde{\sigma}^2} \leq \frac{\lambda}{\rho}\frac{u}{\tilde{\sigma}^2} = \frac{\lambda u}{m}.$$

Thus

$$\frac{\lambda}{c}\frac{u}{\tilde{\sigma}^2} - \triangle u \leq \frac{f}{m} = \frac{1}{\tilde{\sigma}^2}\frac{f}{\rho}.$$

Let $\omega \subset\subset \Omega$ be such that *supp* $f \subset \omega$. Consider the operator $Q(\lambda, \omega) \in \mathcal{L}(L^p(\omega), C_0(\tilde{\Omega}))$ of Lemma 6.3 defined with respect to $\tilde{\sigma}$. Let $w = Q(\frac{\lambda}{c}, \omega)\frac{f}{\rho}$. Note that w is well defined, since $\frac{f}{\rho} \in L^p(\omega)$. Then $0 \leq w \in C_0(\tilde{\Omega}) \cap H_0^1(\tilde{\Omega})$ and by (11),

$$\frac{\lambda}{c}\frac{w}{\tilde{\sigma}^2} - \triangle w = \frac{1}{\tilde{\sigma}^2}\frac{f}{\rho} \quad \text{ in } \mathcal{D}(\tilde{\Omega})'$$

and hence also in $\mathcal{D}(\Omega)'$. Thus

$$\frac{\lambda}{c}\frac{(u-w)}{\tilde{\sigma}^2} - \triangle(u-w) \leq 0 \quad \text{ in } \mathcal{D}(\Omega)'.$$

Recall that $u \in H_0^1(\Omega) \cap C^b(\Omega)$. Thus $(u-w) \in H^1(\Omega)$. Hence

$$\frac{\lambda}{c}\int_\Omega \frac{(u(x)-w(x))}{\tilde{\sigma}(x)^2}v(x)\,dx + \int_\Omega \nabla(u(x)-w(x))\nabla v(x)\,dx \leq 0 \tag{15}$$

for all $0 \leq v \in \mathcal{D}(\Omega)$. Since $(u-w)^+ \in H^1(\Omega)$ and $(u-w)^+ \leq u \in H_0^1(\Omega)$, it follows that $(u-w)^+ \in H_0^1(\Omega)$.
Since $u = R(\lambda, m\triangle_\infty)f = R(\lambda, m\triangle_2)f$, it follows that

$$u \in L^2(\Omega, \frac{dx}{m(x)}) \subset L^2(\Omega, \frac{dx}{\tilde{\sigma}(x)^2})$$

because of (14). It follows (since also $w \in L^2(\Omega, \frac{dx}{\tilde{\sigma}(x)^2})$ that

$$v_1 := (u-w)^+ \in V := L^2(\Omega, \frac{dx}{\tilde{\sigma}(x)^2}) \cap H_0^1(\Omega).$$

Since $\mathcal{D}(\Omega)_+$ is dense in V_+ by Proposition 3.2, (15) remains true for $v := v_1$. This means that

$$\frac{\lambda}{c}\int_\Omega \frac{(u(x)-w(x))^{+2}}{\tilde{\sigma}(x)^2}\,dx + \int_\Omega |\nabla(u(x)-w(x))^+|^2\,dx \leq 0.$$

This implies that $(u-w)^+ = 0$. Hence $0 \leq u \leq w$.
Since

$$\lim_{x\to z, x\in\tilde{\Omega}} w(x) = 0,$$

it follows that

$$\lim_{x\to z, x\in\Omega} u(x) = 0.$$

We have proved the theorem for the case when $0 \leq f \in C_c(\Omega)$. Hence it is also true for arbitrary $f \in C_c(\Omega)$. Since $R(\lambda, m\triangle_\infty) \in \mathcal{L}(L^\infty(\Omega))$, and $C_c(\Omega)$ is dense in $C_0(\Omega)$ it follows that

$$\lim_{x\to z, x\in\Omega}(R(\lambda, m\triangle_\infty)f)(x) = 0$$

for all $f \in C_0(\Omega)$. □

Corollary 6.5 *Assume that each $z \in \partial\Omega$ is a point of weak diffusion (in the sense of (8)). Then $m\triangle_0$ generates a positive, contractive C_0-semigroup on $C_0(\Omega)$.*

7 Conclusion

We may now formulate the following general generation theorem. Let $\Omega \subset \mathbb{R}^N$ be bounded, open and $\frac{N}{2} < p \leq \infty$. Let $m : \Omega \to (0, \infty)$ be bounded and such that $\frac{1}{m} \in L^p_{loc}(\Omega)$.

Theorem 7.1 *Assume that for each point $z \in \partial\Omega$ one of the following conditions is satisfied:*

(a) *z is a regular point or*

(b) *z is a point of weak diffusion (in the sense of (8)).*

Then $m\triangle_0$ generates a positive, contractive C_0-semigroup on $C_0(\Omega)$.

Proof. Theorem 5.3 and Theorem 6.4 show that $C_0(\Omega)$ is invariant. Thus the claim follows from Proposition 4.2. □

Finally, we show that the condition (8) of being a point of weak diffusion is optimal.

Let $N = 2$ and $\Omega = \{x \in \mathbb{R}^2 : 0 < |x| < 2\}$. Then $\partial\Omega = \mathbb{T} \cup \{0\}$ where $\mathbb{T} = \{x \in \mathbb{R}^2 : |x| = 2\}$. The points in $\mathbb{T}$ are regular but 0 is not regular.
Consider the function d given by $d(x) = |x|,\ x \in \Omega$. Thus $d(x) = dist(x, \partial\Omega)$ for $0 < |x| < \frac{1}{2}$. Then $\frac{1}{d} \in L^q(\Omega)$ if and only if $q < 2$. Now let $0 < \beta < 2$. Then $\frac{1}{d^\beta} \in L^p(\Omega)$ for some $p > 1 = \frac{N}{2}$. Since Ω is not Dirichlet regular, it follows from Theorem 5.6 that $d^\beta\triangle_0$ is not a generator.
On the other hand, if $\beta \geq 2$, then for $m = d^\beta$, the point 0 is of weak diffusion. Since the other boundary points are regular, it follows from Theorem 7.1 that $d^\beta\triangle_0$ generates a C_0-semigroup on $C_0(\Omega)$.

An interesting open set in $\mathbb{R}^3$ with continuous boundary and exactly one singular point is the Lebesgue cusp (see e.g. [7] for a detailed investigation).

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