LINEAR EVOLUTION OPERATORS ON SPACES OF PERIODIC FUNCTIONS

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Abstract. Given a family $A(t)$ of closed unbounded operators on a UMD Banach space X with common domain W, we investigate various properties of the operator $D_A := \frac{d}{dt} - A(\cdot)$ acting from $\mathcal{W}^p_{per} := \{u \in W^{1,p}(0, 2\pi; X) \cap L^p(0, 2\pi; W) : u(0) = u(2\pi)\}$ into $\mathcal{X}^p := L^p(0, 2\pi; X)$ when $p \in (1, \infty)$. The primary focus is on the Fredholmness and index of D_A, but a number of related issues are also discussed, such as the independence of the index and spectrum of D_A upon p or upon the pair (X, W) as well as sufficient conditions ensuring that D_A is an isomorphism. Motivated by applications when D_A arises as the linearization of a nonlinear operator, we also address similar questions in higher order spaces, which amounts to proving (nontrivial) regularity properties. Since we do not assume that $\pm A(t)$ generates any semigroup, approaches based on evolution systems are ruled out. In particular, we do not make use of any analog or generalization of Floquet's theory. Instead, some arguments, which rely on the autonomous case (for which results have only recently been made available) and a partition of unity, are more reminiscent of the methods used in elliptic PDE theory with variable coefficients.

1. Introduction

Throughout this paper, we assume some familiarity with the concepts of Banach space with UMD (unconditionality of martingale differences) and of randomized (a.k.a. Rademacher) boundedness, henceforth abbreviated as r-boundedness. The expositions in the monograph by Denk, Hieber and Prüss [11] or alternatively in any of the papers [3] , [8], [25], [32], are sufficient for our purposes.

If X is a complex Banach space and $p \in [1, \infty]$, recall that $W^{1,p}(0, 2\pi; X)$ is the subspace of $L^p(0, 2\pi; X)$ of those functions whose derivatives in the sense of X-valued distributions are in $L^p(0, 2\pi; X)$. As is well known, $W^{1,p}(0, 2\pi; X) \hookrightarrow C^0([0, 2\pi], X)$, so that $u(0)$ and $u(2\pi)$ are unambiguously defined in X and depend continuously on $u \in W^{1,p}(0, 2\pi; X)$. Thus, the subspace

$$W^{1,p}_{per}(0, 2\pi; X) := \{u \in W^{1,p}(0, 2\pi; X) : u(0) = u(2\pi)\}, \tag{1.1}$$

is well defined and closed in $W^{1,p}(0, 2\pi; X)$.

As usual, if L is an unbounded linear operator on a Banach space, $\sigma(L)$ and $R(\lambda, L) := (L - \lambda I)^{-1}$ denote the spectrum and resolvent of L, respectively and $\rho(L) := \mathbb{C}\backslash\sigma(L)$ is the resolvent set of L. Our starting point is the following result by Arendt and Bu [3, Theorem 2.3] (rephrased):

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Theorem 1.1. *Let X be a Banach space with UMD and let A be a closed unbounded operator on X with domain W equipped with the graph norm. Then, given $p \in (1,\infty)$, the operator $D_A := \frac{d}{dt} - A$ is an isomorphism of $W^{1,p}_{per}(0,2\pi;X) \cap L^p(0,2\pi;W)$ onto $L^p(0,2\pi;X)$ if and only if $\sigma(A) \cap i\mathbb{Z} = \emptyset$ and the set $\{kR(ik,A) : k \in \mathbb{Z}\}$ is r-bounded in $\mathcal{L}(X)$.*

It is noteworthy that, in Theorem 1.1, the operator A need not generate a semigroup. In fact, $\sigma(A)$ may not even be contained in any half-plane. Nonetheless, the case when $\pm A$ does generate a semigroup is of course important in the applications.

In this paper, we consider the more general case when $A = A(t)$ is 2π -periodic with t-independent domain W and discuss various extensions and complements of the "if" part of Theorem 1.1. If $W = X$ is finite-dimensional, it is an easy by-product of Floquet's theory (see for instance Farkas [13]) that the operator $D_A := \frac{d}{dt} - A(\cdot)$ is similar to an operator with constant coefficients, so that everything boils down to applying Theorem 1.1.

On the other hand, if $W = X$ is infinite dimensional, then Floquet's theory usually breaks down, even in Hilbert space. Furthermore, its validity depends upon properties of the monodromy operator which are rarely verifiable in practice, or place drastic limitations on the size of $||A(t)||$ (Massera and Schäffer [22]). See however Chow, Lu and Mallet-Paret [7] for the case of scalar parabolic equations in one space variable. We also point out that "obvious" variants of the condition $\sigma(A) \cap i\mathbb{Z} = \emptyset$ in Theorem 1.1 do not provide an adequate substitute, even in the finite dimensional case when the r-boundedness condition is vacuous. This can be seen on the simple scalar example $X = W = \mathbb{C}$ and $A(t) = iae^{it}$ with $a \in \mathbb{R}\backslash\mathbb{Z}$. Clearly, $\sigma(A(t)) \cap i\mathbb{Z} = \emptyset$ for all t, yet $\ker D_A$ contains $u(t) := e^{ae^{it}}$.

We shall follow a much different route and consider the broader issue of finding sufficient conditions for D_A to be a Fredholm operator. Index considerations and spectral properties are discussed in detail as well. Eventually, isomorphism theorems will be obtained in the t-dependent case, but not under hypotheses fully generalizing those of Theorem 1.1.

We shall always assume that X is a Banach space with UMD, that the operators $A(t)$ have a common domain W and that the embedding $W \hookrightarrow X$ is *compact.* The latter is not required in Theorem 1.1, but it is essential in our approach (see Remark 3.2). In particular, our assumptions rule out the case $W = X$ when $\dim X = \infty$ but they are compatible with $A(t)$ being a differential operator acting between Sobolev spaces. The specific hypotheses are listed in Section 2, where some (mostly known) preliminary results are also collected for convenience.

A sufficient condition for $D_A : W^{1,p}_{per}(0,2\pi;X) \cap L^p(0,2\pi;W) \to L^p(0,2\pi;X)$ to be semi-Fredholm is given in Section 3 (Theorem 3.6). Under our assumptions, both the forward and backward Cauchy problems for D_A are ill-posed in general, so that evolution systems cannot be used and there is no monodromy operator. In particular, it does not even make sense to ask whether a generalization of Floquet's theory is available. Instead, the method consists in obtaining suitable a priori estimates via Theorem 1.1 and a partition of unity. This line of arguments follows Rabier [27], where $(0,2\pi)$ is replaced by the whole line, and has further roots in the work of Robbin and Salamon [29]. As a corollary, we obtain that D_A has compact resolvent and index 0 when $\sigma(D_A) \neq \mathbb{C}$ (Corollary 3.7).

By reduction to the constant coefficient case, it is easily seen that D_A has index 0 when $\dim X < \infty$ (see also Remark 9.1) and we know of no example when

$\sigma(D_A) = \mathbb{C}$. Accordingly, we have no explicit example when $\operatorname{index} D_A \neq 0$, but Theorem 3.6 does not rule out their existence and even allows for problems with index $-\infty$. In Section 4, we show, by a duality argument and under the exact same hypotheses, that D_A is actually Fredholm, i.e., of finite index (Theorem 4.1). Various conditions ensuring that D_A has index 0 without the help of any spectral information are also given.

Section 5 addresses various spectral and related questions. The main results there are that $\sigma(D_A)$ and $\operatorname{index} D_A$ are p-independent (Theorem 5.2). Corollary 5.3 is especially relevant when $A(t)$ is an elliptic operator with boundary conditions on a bounded domain.

The Fredholmness of linear operators is important in its own right (Fredholm alternative), but it is also the key to using degree arguments in nonlinear problems, especially when the index is 0. For such matters, see Benevieri and Furi [6], Pejsachowicz and Rabier [24] and the references therein. However, there is a technical difficulty in using the results of Sections 4 and 5 in nonlinear problems, which is explained at the beginning of Section 6 and motivates studying the operator D_A acting between the higher order spaces $W^{2,p}_{per}(0, 2\pi; X) \cap W^{1,p}_{per}(0, 2\pi; W)$ and $W^{1,p}_{per}(0, 2\pi; X)$.

The problem in higher order spaces is investigated in Sections 6 and 7. In Section 6, we mostly focus on extending many (but not all) results to the new functional setting by relying on the previously developed theory or by repeating more or less the same arguments. The purpose of Section 7 is to show that $\operatorname{index} D_A$ and $\sigma(D_A)$ (and even the multiplicity of the isolated eigenvalues) are not affected by passing to the higher order spaces. This is substantially more demanding and is essentially done by proving several regularity results for D_A. Thus, while higher order spaces are better suited to nonlinear problems, the verification of the Fredholm and spectral properties can safely be confined to the simpler original setting. Naturally, no specific nonlinear application is described in this paper. Also, we did not discuss the properties of D_A from $W^{k+1,p}_{per}(0, 2\pi; X) \cap W^{k,p}_{per}(0, 2\pi; W)$ to $W^{k,p}_{per}(0, 2\pi; X)$ when $k > 1$, which can be done by the same methods.

In concrete problems, especially those of PDE type, it often happens that the operators $A(t)$ act between whole families of spaces (W, X) rather than just a single pair of such spaces (for instance, a differential operator with boundary conditions acts between many pairs of Sobolev spaces). It makes then sense to ask whether $\operatorname{index} D_A$ and $\sigma(D_A)$ depend upon the pair (W, X). Their independence of (W, X) is proved in Section 8, under some natural "compatibility" conditions between such pairs. The main regularity result of Section 7 (Lemma 7.2) is instrumental in the proof of the (W, X)-independence.

Unlike the index of Fredholm operators, the compact resolvent property is not stable by arbitrary compact (or even finite rank) perturbations. Thus, prior to Section 9, the only infinite dimensional case when this property is known (t-independent case; see Theorem 4.3) is of limited use in other problems. In Section 9, we give a sufficient condition for $D_{\pm(A-\lambda I)}$ to be an isomorphism when $\operatorname{Re}\lambda$ is large enough (Theorem 9.2). In particular, $D_{\pm A}$ has compact resolvent. The method of proof does not reveal what extra condition could ensure the isomorphism property when $\lambda = 0$, that is, for $D_{\pm A}$. However, such an extra condition (dissipativity) is given in Corollary 9.3. Then, by using the (W, X)-independence results of Section

8, the isomorphism property can next be extended to suitable pairs $(\widetilde{W}, \widetilde{X})$ without requiring the dissipativity in that setting.

All the statements regarding Fredholmness, nullity or deficiency (and hence also index or invertibility) remain true in *real* Banach spaces, for these concepts are unaffected by replacing X and W by their complexifications.

The spectral and index independence questions for evolution problems, especially abstract ones, have been studied little, although partial results (p-independence of the index) can be found in [27] for problems on the whole line and the half line. Therefore, it is difficult to put this paper in the perspective of earlier works, which partly explains its length. On the other hand, spectral independence in elliptic PDEs has been investigated extensively. It is often a simple corollary to elliptic regularity on "good" bounded domains, but a more delicate matter on unbounded ones (see Hempel and Voigt [15], Arendt [1], the survey by Davies [9], Leopold and Schrohe [19], Hieber and Schrohe [17], among others). Still for elliptic problems, the index independence goes back to Geymonat [14] when the domain is bounded. It fails when the domain is $\mathbb{R}^N$ in the weighted spaces considered by McOwen [23] and others, but positive results in non-weighted Sobolev spaces can be found in Rabier [26], [28]. (Much earlier, Seeley [30] proved the index independence for a class of elliptic singular integral operators on L^p for which ellipticity is equivalent to Fredholmness, but this requirement is not met by the operators arising from PDEs on the whole space.)

The notations used throughout are standard. We only mention explicitly that, as is customary, a "dot" is often used to denote t-differentiation.

2. Preliminaries

From now on, X is a Banach space with UMD, $W \subset X$ is a Banach space and $(A(t))_{t\in[0,2\pi]} \subset \mathcal{L}(W, X)$. In particular, $A(t)$ may also be viewed as an unbounded operator on X with domain W and it thus make sense to refer to the spectrum resolvent, etc., of $A(t)$.

In the sequel, we shall frequently retain some or all of the following hypotheses.

(H1) The embedding $W \hookrightarrow X$ is compact.

(H2) $A \in C^0_{per}([0, 2\pi], \mathcal{L}(W, X))$,
i.e., $A \in C^0([0, 2\pi], \mathcal{L}(W, X))$ and $A(0) = A(2\pi)$.

(H3) For every $t \in [0, 2\pi]$, there is $\kappa(t) \in \mathbb{N}$ such that

$$\{kR(ik, A(t)) : k \in \mathbb{Z}, |k| \geq \kappa(t)\},$$

is r-bounded in $\mathcal{L}(X)$ when $A(t)$ is viewed as an unbounded operator on X with domain W.

Remark 2.1. *None of the above hypotheses is affected by changing $A(t)$ into $-A(t)$.*

The following preliminary result will be used in various places later on. The (easy) proof can be found in [27, Theorem 2.1], in a slightly different context.

Lemma 2.1. *Suppose that the embedding $W \hookrightarrow X$ is continuous. The following properties hold for every $t \in [0, 2\pi]$:*
(i) If $\lambda \in \rho(A(t))$, then, $A(t) - \lambda I \in GL(W, X)$.
(ii) If $\rho(A(t)) \neq \emptyset$, the norm of W is equivalent[1] *to the graph norm of $A(t)$ (hence*

[1] If (H2) also holds, it is not difficult to see (by the compactness of $[0, 2\pi]$) that the equivalence of norms is actually uniform in $t \in [0, 2\pi]$.

$A(t)$ is a closed operator on X with domain W).
(iii) If $\rho(A(t)) \neq \emptyset$ and (H1) *holds, then $A(t)$ has compact resolvent (hence $\sigma(A(t))$ is discrete and consists of isolated eigenvalues of finite algebraic multiplicity).*
(iv) W is a Banach space with UMD.

All the conditions required in Lemma 2.1 are fulfilled if (H1) and (H3) hold. In particular, from (H3) and Lemma 2.1 (i), it follows that $R(ik, A(t)) \in \mathcal{L}(X, W)$ for $k \in \mathbb{Z}$ and $|k| \geq \kappa(t)$. This yields an equivalent formulation of (H3) which will be useful in Section 4:

Lemma 2.2. *Suppose that the embedding $W \hookrightarrow X$ is continuous. Condition* (H3) *holds if and only if for every $t \in [0, 2\pi]$, there is $\kappa(t) \in \mathbb{N}$ such that*

$$\{R(ik, A(t)) : k \in \mathbb{Z}, |k| \geq \kappa(t)\})$$

is r-bounded in $\mathcal{L}(X, W)$.

Proof. Suppose first that (H3) holds and let $k \in \mathbb{Z}$ be such that $|k| \geq \kappa(t)$. From the relation

$$ikR(ik, A(t)) = I + A(t)R(ik, A(t)), \tag{2.1}$$

the set $\{A(t)R(ik, A(t)) : k \in \mathbb{Z}, |k| \geq \kappa(t)\}$ is r-bounded in $\mathcal{L}(X)$. On the other hand, (H3) also implies that $\{R(ik, A(t)) : k \in \mathbb{Z}, |k| \geq \kappa(t)\}$ is r-bounded in $\mathcal{L}(X)$. Since, by Lemma 2.1 (ii), the norm of W is equivalent to the graph norm of $A(t)$, it follows that $\{R(ik, A(t)) : k \in \mathbb{Z}, |k| \geq \kappa(t)\}$ is r-bounded in $\mathcal{L}(X, W)$.

Conversely, if $\{R(ik, A(t)) : k \in \mathbb{Z}, |k| \geq \kappa(t)\})$ is r-bounded in $\mathcal{L}(X, W)$, then $\{A(t)R(ik, A(t)) : k \in \mathbb{Z}, |k| \geq \kappa(t)\}$ is r-bounded in $\mathcal{L}(X)$ and (2.1) implies that $\{kR(ik, A(t)) : k \in \mathbb{Z}, |k| \geq \kappa(t)\})$ is r-bounded in $\mathcal{L}(X)$, so that (H3) holds. ∎

The technical property that r-boundedness conditions such as (2.1) are unaffected by some perturbations will be very useful. Several results of this type are available in the literature. In particular, by (a straightforward variant of) [25, Theorem 3.5] and since the relatively compact operators have relative bound[2] 0, we obtain

Lemma 2.3. *Let X be a Banach space and let A_0 be a closed unbounded operator on X with domain W equipped with the graph norm. Suppose that there is $\kappa_0 \in \mathbb{N} \cup \{0\}$ such that $\{kR(ik, A_0) : k \in \mathbb{Z}, |k| \geq \kappa_0\}$ is r-bounded in $\mathcal{L}(X)$, that is,*

$$r_{\mathcal{L}(X)}(\{kR(ik, A_0) : k \in \mathbb{Z}, |k| \geq \kappa_0\}) < \infty. \tag{2.2}$$

Then, for every $K \in \mathcal{K}(W, X)$ (compact operators), there is $\kappa \in \mathbb{N} \cup \{0\}$ such that

$$r_{\mathcal{L}(X)}(\{kR(ik, A_0 + K) : k \in \mathbb{Z}, |k| \geq \kappa\}) < \infty.$$

(Of course, this implies that $R(ik, A_0 + K)$ exists if $|k| \geq \kappa$.)

It follows from Lemma 2.3 that, if (H3) holds, then it also holds when A is replaced by $A+K$, provided that $K(t) \in \mathcal{K}(W, X)$ for every $t \in [0, 2\pi]$. In particular, if (H1) and (H3) hold, then (H3) also holds when A is replaced by $A - \lambda I$ for any $\lambda \in \mathbb{C}$. This will be used repeatedly and often implicitly.

In Section 7, we shall also need the following "stability" result.

Lemma 2.4. *Suppose that A satisfies* (H2) *and* (H3)*. Then,* (H3) *also holds for every $B \in C^0_{per}([0, 2\pi], \mathcal{L}(W, X))$ with $\sup_{t \in [0, 2\pi]} ||B(t) - A(t)||_{\mathcal{L}(W,X)} > 0$ small enough.*

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[2]See Hess [16] since X is reflexive.

Proof. Let $t_0 \in [0, 2\pi]$ and $\varepsilon > 0$ be given. If $t \in [0, 2\pi]$ and $x \in W$, then

$$||(B(t) - A(t_0))x||_X \\ \leq \left(||(B(t) - A(t))||_{\mathcal{L}(W,X)} + ||(A(t) - A(t_0))||_{\mathcal{L}(W,X)}\right) ||x||_W \leq \varepsilon ||x||_W,$$

if $|t - t_0| < \delta$ with $\delta > 0$ is small enough and $\sup_{s\in[0,2\pi]} ||B(s) - A(s)||_{\mathcal{L}(W,X)} < \frac{\varepsilon}{2}$. From the equivalence of the norm of W and the graph norm of $A(t_0)$, there is a constant $c_0 > 0$ depending only upon $A(t_0)$ (and the norms of X and W) such that $||x||_W \leq c_0||A(t_0)x||_X + c_0||x||_X$. Therefore,

$$||(B(t) - A(t_0))x||_X \leq c(t_0)\varepsilon||A(t_0)x||_X + c(t_0)\varepsilon||x||_X,$$

for every $x \in W$ and every $t \in J_{t_0} := (t_0 - \delta, t_0 + \delta) \cap [0, 2\pi]$.

Thus, if $\varepsilon > 0$ is chosen small enough in the first place, it follows from (H3) with $t = t_0$ and from [25, Theorem 3.5 and Remark 3.5], that there is $\kappa(\varepsilon) \in \mathbb{N}$ such that $r_{\mathcal{L}(X)}(\{kR(ik, B(t)) : k \in \mathbb{Z}, |k| \geq \kappa(\varepsilon)\}) < \infty$ for every $t \in J_{t_0}$. The conclusion follows by covering $[0, 2\pi]$ with finitely many intervals J_{t_0}. ∎

The next lemma reveals an important by-product of the hypothesis (H3).

Lemma 2.5. *If A satisfies* (H3), *then W is dense in X.*

Since r-boundedness implies boundedness and Banach spaces with UMD are reflexive, Lemma 2.5 follows from the more general result below, presumably not new but for which we have found no reference in the literature.

Lemma 2.6. *Let Z be a reflexive complex Banach space and let L be an unbounded linear operator on Z such that there is a sequence $(\lambda_n) \subset \mathbb{C}$ with $\lim_{n\to\infty} |\lambda_n| = \infty$ and $\sup_n ||\lambda_n R(\lambda_n, L)|| < \infty$. Then, the domain $D(L)$ of L is dense in Z.*

Proof. Let $\mu \in \rho(L)$ be chosen once and for all. Since the hypotheses of the lemma readily imply $\lim_{n\to\infty} R(\lambda_n, L) = 0$ in $\mathcal{L}(Z)$, it follows that

$$\lambda_n R(\lambda_n, L)R(\mu, L) = \\ \frac{\lambda_n}{\lambda_n - \mu} (R(\mu, L) - R(\lambda_n, L)) \to R(\mu, L) \text{ in } \mathcal{L}(Z) \text{ as } n \to \infty. \tag{2.3}$$

Let $x \in Z$ be given. By the boundedness of the sequence $\lambda_n R(\lambda_n, L)x$ and the reflexivity of Z, we may assume with no loss of generality that there is $y \in Z$ such that $\lambda_n R(\lambda_n, L)x \overset{w}{\rightharpoonup} y$. Thus, $\lambda_n R(\lambda_n, L)R(\mu, L)x \overset{w}{\rightharpoonup} R(\mu, L)y$. On the other hand, by (2.3), $\lambda_n R(\lambda_n, L)R(\mu, L)x \to R(\mu, L)x$ in norm, so that $R(\mu, L)y = R(\mu, L)x$ and hence $y = x$. This shows that $\lambda_n R(\lambda_n, L)x \overset{w}{\rightharpoonup} x$. Evidently, $\lambda_n R(\lambda_n, L)x \in D(L)$, whence some convex combination of the points $\lambda_n R(\lambda_n, L)x$ (also in $D(L)$) tends to x in norm by Mazur's lemma. This completes the proof. ∎

3. Semi-Fredholmness

In Theorem 3.6 below, we show that if $p \in (1, \infty)$ and the hypotheses (H1) to (H3) hold, the operator

$$D_A := \frac{d}{dt} - A(\cdot) : W^{1,p}_{per}(0, 2\pi; X) \cap L^p(0, 2\pi; W) \to L^p(0, 2\pi; X), \tag{3.1}$$

has closed range and finite dimensional null-space, i.e., is semi-Fredholm of index $\nu \in \mathbb{Z} \cup \{-\infty\}$.

For simplicity of notation, we shall set

$$\mathcal{W}^p_{per} := W^{1,p}_{per}(0,2\pi;X)\cap L^p(0,2\pi;W) \text{ and } \mathcal{X}^p := L^p(0,2\pi;X), \tag{3.2}$$

so that D_A does map $\mathcal{W}^p_{per}$ into $\mathcal{X}^p$. The (natural) norms on $\mathcal{W}^p_{per}$ and $\mathcal{X}^p$ will be denoted by $||\cdot||_{\mathcal{W}^p_{per}}$ and $||\cdot||_{\mathcal{X}^p}$, respectively. Both spaces are Banach spaces for these norms. In addition, by (H1), it follows from Simon [31, Theorem 1] that the embedding $W^{1,p}(0,2\pi;X)\cap L^p(0,2\pi;W)\hookrightarrow L^p(0,2\pi;X)$ is compact (see also Aubin [5]). As a result, the embedding

$$\mathcal{W}^p_{per}\hookrightarrow \mathcal{X}^p \tag{3.3}$$

is compact. As the proof will show, Theorem 3.6 is then a simple by-product of this compactness together with the inequality

$$||u||_{\mathcal{W}^p_{per}} \le M\left(||D_A u||_{\mathcal{X}^p}+||u||_{\mathcal{X}^p}\right), \qquad \forall u\in \mathcal{W}^p_{per}, \tag{3.4}$$

where $M>0$ is a constant independent of u.

Most of this section is devoted to proving the validity of (3.4), which is done in Lemma 3.5. This will follow from the case when u has compact support contained in $(0,2\pi)$ (Lemma 3.4), although the corresponding subspace is not dense in $\mathcal{W}^p_{per}$. In turn, the method of proof of Lemma 3.4 is loosely inspired by the classical procedure to obtain a priori estimates for elliptic PDEs, by freezing the coefficients and partition of unity.

Lemma 3.1. *Suppose that* (H1) *to* (H3) *hold and that* $p\in(1,\infty)$. *Let* $s_0\in[0,2\pi]$ *be given and let* $\lambda_0\in\mathbb{C}$ *be such that* $\sigma(A(s_0)-\lambda_0 I)\cap i\mathbb{Z}=\emptyset$. *Then, there are an open interval* J_0 *about* s_0 *and a constant* $C(s_0)>0$ *such that* $D_{A(s)-\lambda_0 I}\in GL(\mathcal{W}^p_{per},\mathcal{X}^p)$ *for every* $s\in J_0\cap[0,2\pi]$ *and that* $||D^{-1}_{A(s)-\lambda_0 I}||_{\mathcal{L}(\mathcal{X}^p,\mathcal{W}^p_{per})}\le C(s_0)$ *for every* $s\in J_0\cap[0,2\pi]$.
Note: *Since* s *is fixed,* $D_{A(s)-\lambda_0 I}$ *above is the operator* $\frac{d}{dt}-(A(s)-\lambda_0 I)$, *with* constant *coefficients.*

Proof. By Lemma 2.1 (ii), the operator $A(s_0)$ is a closed unbounded operator on X with domain W equipped with a norm equivalent to the graph norm of $A(s_0)$ and hence equivalent to the graph norm of $A(s_0)-\lambda_0 I$. Next, by (H3), $A(s_0)$ satisfies the condition (2.2) of Lemma 2.3, so that, by (H1), there is $\kappa\in\mathbb{N}\cup\{0\}$ such that the set $\{kR(ik,A(s_0)-\lambda_0 I): k\in\mathbb{Z}, |k|\ge\kappa\}$ is r-bounded. Since $\sigma(A(s_0)-\lambda_0 I)\cap i\mathbb{Z}=\emptyset$, the set $\{kR(ik,A(s_0)-\lambda_0 I): k\in\mathbb{Z},|k|<\kappa\}$ is well defined and finite (hence r-bounded). Therefore, the set $\{kR(ik,A(s_0)-\lambda_0 I):k\in\mathbb{Z}\}$ is r-bounded (union of two r-bounded sets). As a result, $D_{A(s_0)-\lambda_0 I}\in GL(\mathcal{W}^p_{per},\mathcal{X}^p)$ by Theorem 1.1 for $A(s_0)-\lambda_0 I$.

By (H2), the mapping $s\in[0,2\pi]\mapsto D_{A(s)-\lambda_0 I}\in\mathcal{L}(\mathcal{W}^p_{per},\mathcal{X}^p)$ is continuous. Since $GL(\mathcal{W}^p_{per},\mathcal{X}^p)$ is open in $\mathcal{L}(\mathcal{W}^p_{per},\mathcal{X}^p)$, it follows that $D_{A(s)-\lambda_0 I}\in GL(\mathcal{W}^p_{per},\mathcal{X}^p)$ with $||D^{-1}_{A(s)-\lambda_0 I}||_{\mathcal{L}(\mathcal{X}^p,\mathcal{W}^p_{per})}$ bounded by a constant $C(s_0)>0$ if $s\in J_0$ and J_0 is a small enough open interval about s_0. This completes the proof. ∎

Lemma 3.2. *Suppose that* (H1) *to* (H3) *hold and that* $p\in(1,\infty)$. *There are a finite set* $\Lambda\subset[-1,1]$ *and a constant* $C>0$ *with the following property: For every* $s\in[0,2\pi]$, *there is* $\lambda\in\Lambda$ *such that* $D_{A(s)-\lambda I}\in GL(\mathcal{W}^p_{per},\mathcal{X}^p)$ *and that* $\left\|D^{-1}_{A(s)-\lambda I}\right\|_{\mathcal{L}(\mathcal{X}^p,\mathcal{W}^p_{per})}\le C$.

Proof. Given $s_0 \in [0, 2\pi]$, it follows from Lemma 2.1 (iii) that $\sigma(A(s_0))$ is discrete. As a result, the projection of $\sigma(A(s_0))$ onto the real axis consists of countably many points. If $\lambda_0 \in [-1, 1]$ is chosen in the complement of this countable set, then $\sigma(A(s_0) - \lambda_0 I) \cap i\mathbb{R} = \emptyset$, so that $\sigma(A(s_0) - \lambda_0 I) \cap i\mathbb{Z} = \emptyset$. Thus, by Lemma 3.1, there are a constant $C(s_0) > 0$ and an open interval J_0 about λ_0 such that $D_{A(s)-\lambda_0 I} \in GL(\mathcal{W}^p_{per}, \mathcal{X}^p)$ for every $s \in J_0 \cap [0, 2\pi]$ and that $||D^{-1}_{A(s)-\lambda_0 I}||_{\mathcal{L}(\mathcal{X}^p, \mathcal{W}^p_{per})} \leq C(s_0)$ for every $s \in J_0 \cap [0, 2\pi]$. The lemma follows by covering $[0, 2\pi]$ by finitely many such intervals $J_\ell, 1 \leq \ell \leq N$, corresponding to points $s_\ell \in [0, 2\pi]$ and values $\lambda_\ell \in [-1, 1]$. Clearly, $\Lambda := \{\lambda_1, \cdots, \lambda_N\}$ and $C := \max_{1 \leq \ell \leq N} C(s_\ell)$ satisfy the required conditions. ∎

Lemma 3.3. *Given $p \in (1, \infty)$, there is $\varepsilon > 0$ such that, for every $\psi \in C_0^\infty(0, 2\pi)$ and every $u \in \mathcal{W}^p_{per}$,*

$$\sup_{s,t \in \text{Supp } \psi} ||A(s) - A(t)||_{\mathcal{L}(W,X)} \leq \varepsilon \Rightarrow \tag{3.5}$$
$$||\psi u||_{\mathcal{W}^p_{per}} \leq \varepsilon^{-1}(||\psi D_A u||_{\mathcal{X}^p} + ||\dot{\psi} u||_{\mathcal{X}^p} + ||\psi u||_{\mathcal{X}^p}).$$

Proof. Let $u \in \mathcal{W}^p_{per}$ be given and set $f := D_A u$. The multiplication of both sides by $\psi \in C_0^\infty(0, 2\pi)$ yields $D_A(\psi u) = \dot{\psi} u + \psi f$. Pick $s_0 \in \text{Supp } \psi$ and let Λ and $\lambda_0 \in \Lambda$ be given by Lemma 3.2. Then,

$$D_{A(s_0)-\lambda_0 I}(\psi u) = (A - A(s_0))\psi u + \psi f + \dot{\psi} u + \lambda_0 \psi u$$

and hence, by Lemma 3.2 (see Remark 3.1 below) and since $|\lambda_0| \leq 1$,

$$||\psi u||_{\mathcal{W}^p_{per}} \leq C(||(A - A(s_0))\psi u||_{\mathcal{X}^p} + ||\psi f||_{\mathcal{X}^p} + ||\dot{\psi} u||_{\mathcal{X}^p} + ||\psi u||_{\mathcal{X}^p}), \tag{3.6}$$

where $C > 0$ is a constant independent of s_0, u and ψ. By writing

$$||(A - A(s_0))\psi u||_{\mathcal{X}^p} = \left(\int_{\text{Supp } \psi} ||(A(t) - A(s_0))\psi(t)u(t)||_X^p dt \right)^{\frac{1}{p}},$$

we obtain the estimate

$$\|(A - A(s_0))\, \psi u\|_{\mathcal{X}^p} \leq \sup_{t \in \text{Supp } \psi} ||A(t) - A(s_0)||_{\mathcal{L}(W,X)} ||\psi u||_{L^p(0,2\pi;W)}$$
$$\leq \sup_{s,t \in \text{Supp } \psi} ||A(s) - A(t)||_{\mathcal{L}(W,X)} ||\psi u||_{\mathcal{W}^p_{per}}.$$

By substitution into (3.6), we get

$$||\psi u||_{\mathcal{W}^p_{per}} \leq C \sup_{s,t \in \text{Supp } \psi} ||A(s) - A(t)||_{\mathcal{L}(W,X)} ||\psi u||_{\mathcal{W}^p_{per}}$$
$$+ C(||\psi f||_{\mathcal{X}^p} + ||\dot{\psi} u||_{\mathcal{X}^p} + ||\psi u||_{\mathcal{X}^p}),$$

which yields (3.5) with $\varepsilon = \frac{1}{2C}$ independent of u and ψ since $f := D_A u$. ∎

Remark 3.1. *It is trivial, yet crucial to the above proof, that $\psi u \in \mathcal{W}^p_{per}$ because $(\psi u)(0) = (\psi u)(2\pi)(= 0)$. In particular,* Lemma 3.2 *cannot be used if ψ is a cut-off function that does not vanish at $t = 0$ or $t = 2\pi$ since the multiplication by ψ does not preserve periodicity in this case.*

We are now in a position to prove the validity of the estimate (3.4). We proceed in two steps.

Lemma 3.4. *Suppose that* (H1) *to* (H3) *hold. Then, for every compact interval* $Q \subset (0, 2\pi)$, *there is a constant* $M(Q) > 0$ *such that*

$$||u||_{\mathcal{W}^p_{per}} \leq M(Q) \left(||D_A u||_{\mathcal{X}^p} + ||u||_{\mathcal{X}^p}\right), \tag{3.7}$$

for every $u \in \mathcal{W}^p_{per}$ *with* $\operatorname{Supp} u \subset Q$.

Proof. Let $\varepsilon > 0$ be given by Lemma 3.3. By the uniform continuity of A on $[0, 2\pi]$, there is $\delta > 0$ such that $||A(s) - A(t)||_{\mathcal{L}(W,X)} < \varepsilon$ whenever $|s - t| < \delta$. Cover Q with finitely many open intervals $I_j \subset [0, 2\pi]$ such that $|I_j| < \delta, 1 \leq j \leq n$, and choose n functions $\psi_j \in C_0^\infty(0, 2\pi)$ such that $\operatorname{Supp} \psi_j \subset I_j$ and $\sum_{j=1}^n \psi_j = 1$ on Q.

If u is as above, then $u = \sum_{j=1}^n \psi_j u$, whence $||u||_{\mathcal{W}^p_{per}} \leq \sum_{j=1}^n ||\psi_j u||_{\mathcal{W}^p_{per}}$ and so, by Lemma 3.3,

$$||u||_{\mathcal{W}^p_{per}} \leq \sum_{j=1}^n \varepsilon^{-1}(||\psi_j D_A u||_{\mathcal{X}^p} + ||\dot{\psi}_j u||_{\mathcal{X}^p} + ||\psi_j u||_{\mathcal{X}^p}).$$

This implies (3.7) with $M(Q) := 2\varepsilon^{-1} \sum_{j=1}^n (\max_{t \in [0,2\pi]} |\psi_j(t)| + |\dot{\psi}_j(t)|)$. ∎

Lemma 3.5. *Suppose that* (H1) *to* (H3) *hold. Then, there is a constant* $M > 0$ *such that*

$$||u||_{\mathcal{W}^p_{per}} \leq M \left(||D_A u||_{\mathcal{X}^p} + ||u||_{\mathcal{X}^p}\right), \qquad \forall u \in \mathcal{W}^p_{per}. \tag{3.8}$$

Proof. Extend A to all of $\mathbb{R}$ by periodicity and note that (H1) to (H3) are not affected by changing $[0, 2\pi]$ into $[a, a + 2\pi]$ where $a \in \mathbb{R}$ is arbitrary. Thus, (3.7) in Lemma 3.4 remains true when Q is a compact subinterval of $(a, a+2\pi)$, the spaces $\mathcal{W}^p_{per}$ and $\mathcal{X}^p$ are replaced by $W^{1,p}_{per}(a, a + 2\pi; X) \cap L^p(a, a + 2\pi; W)$ and $L^p(a, a + 2\pi; X)$, respectively and $\operatorname{Supp} u \subset Q$. Here, membership of u to $W^{1,p}_{per}(a, a + 2\pi; X)$ means that $u \in W^{1,p}(a, a + 2\pi; X)$ and that $u(a) = u(a + 2\pi)$.

Given $u \in \mathcal{W}^p_{per}$, extend u to all of $\mathbb{R}$ by periodicity. For $j \in \{-1, 0, 1\}$, let $\varphi_j \in C_0^\infty(\mathbb{R})$ be such that $\operatorname{Supp} \varphi_j \subset Q_j \subset (j\pi, (j + 2)\pi)$ where Q_j is a compact interval, $[-1, 2\pi+1] \subset \cup_{j=-1}^1 Q_j$ and $\sum_{j=-1}^1 \varphi_j = 1$ on $[0, 2\pi]$. Then $u = \sum_{j=-1}^1 \varphi_j u$ on $[0, 2\pi]$, so that

$$||u||_{\mathcal{W}^p_{per}} \leq \sum_{j=-1}^1 ||\varphi_j u||_{W^{1,p}(0,2\pi;X) \cap L^p(0,2\pi;W)}. \tag{3.9}$$

Now, $||\varphi_{-1} u||_{W^{1,p}(0,2\pi;X) \cap L^p(0,2\pi;W)} = ||\varphi_{-1} u||_{W^{1,p}(0,\pi;X) \cap L^p(0,\pi;W)}$ since $\varphi_{-1} u = 0$ in $[\pi, 2\pi]$ and $||\varphi_{-1} u||_{W^{1,p}(0,\pi;X) \cap L^p(0,\pi;W)} \leq ||\varphi_{-1} u||_{W^{1,p}(-\pi,\pi;X) \cap L^p(-\pi,\pi;W)}$. Therefore,

$$||\varphi_{-1} u||_{W^{1,p}(0,2\pi;X) \cap L^p(0,2\pi;W)} \leq ||\varphi_{-1} u||_{W^{1,p}(-\pi,\pi;X) \cap L^p(-\pi,\pi;W)}. \tag{3.10}$$

Since $\operatorname{Supp} \varphi_{-1} u \subset Q_{-1}$, it follows from (3.7) with $[0, 2\pi]$ replaced by $[-\pi, \pi]$ (see the discussion at the beginning of the proof) that there is $M(Q_{-1}) > 0$ such that

$$||\varphi_{-1} u||_{W^{1,p}(-\pi,\pi;X) \cap L^p(-\pi,\pi;W)} \leq M(Q_{-1}) \left(||D_A(\varphi_{-1} u)||_{L^p(-\pi,\pi;X)} + ||\varphi_{-1} u||_{L^p(-\pi,\pi;X)}\right).$$

By using $D_A(\varphi_{-1} u) = \varphi_{-1} D_A u + \dot{\varphi}_{-1} u$ and (3.10), this yields

$$||\varphi_{-1} u||_{W^{1,p}(0,2\pi;X) \cap L^p(0,2\pi;W)} \leq 2M(Q_{-1})(\max_{t \in \mathbb{R}} |\varphi_{-1}(t)| + |\dot{\varphi}_{-1}(t)|) \left(||D_A u||_{L^p(-\pi,\pi;X)} + ||u||_{L^p(-\pi,\pi;X)}\right).$$

By the periodicity of u and A, $||D_A u||_{L^p(-\pi,\pi;X)} = ||D_A u||_{\mathcal{X}^p}$ and $||u||_{L^p(-\pi,\pi;X)} = ||u||_{\mathcal{X}^p}$, so that

$$||\varphi_{-1}u||_{W^{1,p}(0,2\pi;X)\cap L^p(0,2\pi;W)} \leq 2M(Q_{-1})(\max_{t\in\mathbb{R}} |\varphi_{-1}(t)| + |\dot{\varphi}_{-1}(t)|)\,(||D_A u||_{\mathcal{X}^p} + ||u||_{\mathcal{X}^p})\,.$$

Similar inequalities hold when $j = 0$ and $j = 1$ in (3.9), which yields (3.8) with $M := 2\sum_{j=-1}^{1} M(Q_j)(\max_{t\in\mathbb{R}} |\varphi_j(t)| + |\dot{\varphi}_j(t)|)$. ∎

Theorem 3.6. *Suppose that* (H1) *to* (H3) *hold. Then, the operator* $D_A : \mathcal{W}^p_{per} \to \mathcal{X}^p$ *(see* (3.2)*) has closed range and finite dimensional null-space for every* $p \in (1, \infty)$.

Proof. By the well known Yood criterion (Deimling [10], Yood [33]), it suffices to show that D_A is proper on the closed bounded subsets of $\mathcal{W}^p_{per}$, i.e., that if $(u_n) \subset \mathcal{W}^p_{per}$ is a bounded sequence such that $(D_A u_n)$ converges in $\mathcal{X}^p$, then (u_n) contains a $\mathcal{W}^p_{per}$-convergent subsequence.

By the compactness of the embedding (3.3), we may assume that (u_n) is convergent in $\mathcal{X}^p$ with no loss of generality. That (u_n) actually converges in $\mathcal{W}^p_{per}$ thus follows from (3.7) with u replaced by $u_n - u_m$. ∎

Another way to state Theorem 3.6 is to say that D_A is semi-Fredholm of index in $\mathbb{Z} \cup \{-\infty\}$.

Remark 3.2. *In contrast to* Theorem 1.1, Theorem 3.6 *is false if* (H1) *is dropped. For instance, if* $X = W$ *is an infinite dimensional Hilbert space and* $A = 0$, *then* X *has the UMD property and* (H2) *and* (H3) *hold trivially. Yet,* $\ker D_A = X$ *(constant functions) is not finite dimensional. (If* (H1) *holds, then* $\sigma(A(t)) = \mathbb{C}$ *when* $A = 0$ *has domain* W *and* (H3) *fails.)*

We now show that under the hypotheses of Theorem 3.6, D_A is a closed operator on $\mathcal{X}^p$ with domain $\mathcal{W}^p_{per}$ for every $p \in (1, \infty)$.

Corollary 3.7. *Suppose that* (H1) *to* (H3) *hold. Then, the operator* D_A *is a closed operator on* $\mathcal{X}^p$ *with domain* $\mathcal{W}^p_{per}$ *for every* $p \in (1, \infty)$. *In addition, either* $\sigma(D_A) = \mathbb{C}$ *or* D_A *has compact resolvent and (hence)* $\operatorname{index} D_A = 0$.

Proof. Let $(u_n) \subset \mathcal{W}^p_{per}$ be a sequence such that $u_n \to u$ in $\mathcal{X}^p$ and $D_A u_n \to f$ in $\mathcal{X}^p$. By Theorem 3.6, $\dim\ker D_A < \infty$, so that there is a continuous projection $P \in \mathcal{L}(\mathcal{W}^p_{per})$ onto $\ker D_A$. Furthermore, still by Theorem 3.6, $\operatorname{rge} D_A$ is closed in $\mathcal{X}^p$, so that D_A is an isomorphism of $\ker P$ onto $\operatorname{rge} D_A$. By writing $u_n = Pu_n + (I-P)u_n$ and since $D_A(I-P)u_n = D_A u_n \to f_n$, it follows that $(I-P)u_n$ is convergent in $\mathcal{W}^p_{per}$ and hence also in $\mathcal{X}^p$. Since $u_n \to u$ in $\mathcal{X}^p$, it follows that $Pu_n = u_n-(I-P)u_n$ is convergent in $\mathcal{X}^p$. Since $\dim\ker D_A < \infty$, this amounts to saying that Pu_n is convergent in $\mathcal{W}^p_{per}$. Therefore, u_n is convergent in $\mathcal{W}^p_{per}$ and its limit coincides with its limit u in $\mathcal{X}^p$. This shows that $u \in \mathcal{W}^p_{per}$. Then, by the continuity of $D_A : \mathcal{W}^p_{per} \to \mathcal{X}^p$, it follows that $D_A u_n \to D_A u$ in $\mathcal{X}^p$, so that $D_A u = f$. This completes the proof that D_A is closed.

Since the embedding $\mathcal{W}^p_{per} \hookrightarrow \mathcal{X}^p$ is compact, D_A has compact resolvent if $\rho(D_A) \neq \emptyset$. If so, pick $\lambda_0 \in \rho(D_A)$. The relation $D_A - \lambda_0 I = D_{A+\lambda_0 I}$ shows that $D_{A+\lambda_0 I}$ is an isomorphism of $\mathcal{W}^p_{per}$ onto $\mathcal{X}^p$. On the other hand, for every $\lambda \in \mathbb{C}$, the operator $A + \lambda I$ satisfies (H1) to (H3) (use (H1) and Lemma 2.3) and $D_{A+\lambda I} \in \mathcal{L}(\mathcal{W}^p_{per}, \mathcal{X}^p)$ depends continuously upon λ. Thus, $D_{A+\lambda I}$ is semi-Fredholm for every $\lambda \in \mathbb{C}$ by Theorem 3.6 and so its index is independent of $\lambda \in \mathbb{C}$. In particular, $\operatorname{index} D_A = \operatorname{index} D_{A+\lambda_0 I} = 0$. ∎

In Corollary 3.7, $\sigma(D_A)$ refers to the spectrum of D_A as an unbounded operator on $\mathcal{X}^p$ with domain $\mathcal{W}^p_{per}$ for the chosen value of $p \in (1,\infty)$. As we shall see in Section 5, this value may be left unspecified since $\sigma(D_A)$ turns out to be independent of p.

4. Fredholmness

By Corollary 3.7, D_A has index 0 if $\sigma(D_A) \neq \mathbb{C}$. Regardless of any spectral condition, we now prove that D_A is Fredholm and not merely semi-Fredholm without any additional assumption.

Theorem 4.1. *Suppose that* (H1) *to* (H3) *hold. Then, the operator* $D_A := \frac{d}{dt} - A(\cdot) : \mathcal{W}^p_{per} \to \mathcal{X}^p$ *(see* (3.2)*) is Fredholm for every* $p \in (1,\infty)$.

Proof. Call $j : W \hookrightarrow X$ the embedding, so that $j^* : X^* \to W^*$ is the mapping $j^*(x^*) = (x^*)_{|W}$. By Lemma 2.5, j^* is one to one and thus a continuous embedding of X^* in W^*. More specifically, this means that X^* can be identified with the subset of W^* of those forms that are continuous for the topology of X. Also, j^* is compact since j is compact by (H1) and W^* is a Banach space with UMD since this is true of W (Lemma 2.1 (iv)).

From (H2), $A^* \in C^0([0,2\pi], \mathcal{L}(X^*, W^*))$ and $A^*(0) = A^*(2\pi)$. In addition, given $\lambda \in \mathbb{C}$, then $A^*(t) - \lambda j^*$ is invertible if and only if $A(t) - \lambda j$ is invertible, which shows that $R(\lambda, A^*(t)) = R(\lambda, A(t))^*$. Now, by (H3) and Lemma 2.2, the set $\{R(ik, A(t)), |k| \geq \kappa(t)\}$ is r-bounded in $\mathcal{L}(X, W)$ for every $t \in [0, 2\pi]$. Since X is a Banach space with UMD, it follows from [25, Lemma 2.3 and Remark 3.1] that $\{R(ik, A(t))^*, |k| \geq \kappa(t)\} = \{R(ik, A^*(t)), |k| \geq \kappa(t)\}$ is r-bounded in $\mathcal{L}(W^*, X^*)$. In turn, by another application of Lemma 2.2 with X and W replaced by W^* and X^*, respectively, it follows that $\{kR(ik, A^*(t)), |k| \geq \kappa(t)\}$ is r-bounded in $\mathcal{L}(W^*)$.

In summary, W^* is a Banach space with UMD and the hypotheses (H1) to (H3) hold with X and W replaced by W^* and X^*, respectively and $A(t)$ replaced by $A^*(t)$, and hence also when replaced by $-A^*(t)$ (Remark 2.1). As a result, by Theorem 3.6, the operator $D_{-A^*} : \mathcal{X}^{p'}_{*per} \to \mathcal{W}^{p'}_{*}$ has finite dimensional null-space for every $p \in (1,\infty)$, where $p' := \frac{p}{p-1}$ and (compare with (3.2))

$$\mathcal{X}^{p'}_{*per} := W^{1,p'}_{per}(0,2\pi; W^*) \cap L^{p'}(0,2\pi; X^*) \text{ and } \mathcal{W}^{p'}_{*} := L^{p'}(0,2\pi; W^*). \tag{4.1}$$

Since we already know by Theorem 3.6 that D_A has closed range, the Fredholmness of D_A is equivalent to the finite dimensionality of $\ker(D_A)^*$. However, a direct approach is faced with the difficulty of characterizing $(D_A)^*$ (there is no simple description of the dual of $\mathcal{W}^p_{per}$).

Instead, we shall rely on the finite dimensionality of $\ker D_{-A^*}$ just proved above and show that $(\operatorname{rge} D_A)^\perp \subset \ker D_{-A^*}$, so that $\operatorname{rge} D_A$ has finite codimension in $L^p(0,2\pi; X)$. Note that $(\operatorname{rge} D_A)^\perp \subset (L^p(0,2\pi; X))^* = L^{p'}(0,2\pi; X^*)$, the latter by the reflexivity of X (see Edwards [12]), whereas $\ker D_{-A^*} \subset \mathcal{X}^{p'}_{*per} \subsetneq L^{p'}(0,2\pi; X^*)$. Thus, the claim that $(\operatorname{rge} D_A)^\perp \subset \ker D_{-A^*}$ is in fact an abstract regularity result for the members of $(\operatorname{rge} D_A)^\perp$.

The precise meaning of the relation $(L^p(0,2\pi;X))^* = L^{p'}(0,2\pi;X^*)$ is that every continuous linear form on $L^p(0,2\pi;X)$ is given by

$$f \in L^p(0,2\pi;X) \mapsto \int_0^{2\pi} \langle f(t), v^*(t)\rangle_{X,X^*} dt,$$

for some $v^* \in L^{p'}(0,2\pi;X^*)$. If now $v^* \in (\operatorname{rge} D_A)^\perp$, then

$$\int_0^{2\pi} \langle \dot{u}(t) - A(t)u(t), v^*(t)\rangle_{X,X^*}\, dt = 0, \tag{4.2}$$

for every $u \in \mathcal{W}^p_{per}$. In particular, given any $x \in W$ and any $\varphi \in C_0^\infty(0,2\pi)$, we may choose $u = \varphi \otimes x$ in (4.2), so that

$$\int_0^{2\pi} \langle \dot{\varphi}(t)x - A(t)\varphi(t)x, v^*(t)\rangle_{X,X^*} dt = 0.$$

This may be rewritten as

$$\int_0^{2\pi} \langle x, \dot{\varphi}(t)v^*(t)\rangle_{X,X^*} dt - \int_0^{2\pi} \langle x, \varphi(t)A^*(t)v^*(t)\rangle_{W,W^*} dt = 0$$

and, since the Bochner integral commutes with duality pairings, also as

$$\left\langle x, \int_0^{2\pi} \dot{\varphi}(t)v^*(t)dt \right\rangle_{X,X^*} - \left\langle x, \int_0^{2\pi} \langle x, \varphi(t)A^*(t)v^*(t)dt \right\rangle_{W,W^*} = 0. \tag{4.3}$$

Now, observe that if $x^* \in X^*$ and $x \in W$, then $\langle x, x^*\rangle_{X,X^*} = \langle x, x^*\rangle_{W,W^*}$ since $x^* \in X^*$ is simply identified with its restriction to W when it is viewed as a member of W^*. Thus, (4.3) also reads

$$\left\langle x, \int_0^{2\pi} \dot{\varphi}(t)v^*(t)dt \right\rangle_{W,W^*} - \left\langle x, \int_0^{2\pi} \varphi(t)A^*(t)v^*(t)dt \right\rangle_{W,W^*} = 0.$$

Since $x \in W$ is arbitrary, it follows that $\int_0^{2\pi} \dot{\varphi}(t)v^*(t)dt - \int_0^{2\pi} \varphi(t)A^*(t)v^*(t)dt = 0$ in W^*. In turn, because $\varphi \in C_0^\infty$ is also arbitrary, this means that

$$\dot{v}^* + A^*v^* = 0, \tag{4.4}$$

as a distribution with values in W^*. Since $A^* \in C^0([0,2\pi], \mathcal{L}(X^*, W^*))$ and $v^* \in L^{p'}(0,2\pi;X^*) \subset L^{p'}(0,2\pi;W^*)$, it follows that $\dot{v}^* = -A^*v^* \in L^{p'}(0,2\pi;W^*)$ and hence that $v^* \in W^{1,p'}(0,2\pi;W^*) \cap L^{p'}(0,2\pi;X^*)$.

To complete the proof it suffices to show that $v^*(2\pi) = v^*(0)$ (well defined in W^* since $v^* \in W^{1,p'}(0,2\pi;W^*)$), for then $v^* \in \mathcal{X}^{p'}_{*per}$ (see (4.1)) while $D_{-A^*}v^* = 0$ by (4.4). To see this, let $x \in W$ be given. The constant function $u = 1 \otimes x$ is in $\mathcal{W}^p_{per}$ and so, by (4.2), $\int_0^{2\pi} \langle A(t)x, v^*(t)\rangle_{X,X^*} dt = 0$, that is,

$$\int_0^{2\pi} \langle x, A^*(t)v^*(t)\rangle_{W,W^*} dt = 0.$$

By (4.4), this amounts to $\int_0^{2\pi} \langle x, \dot{v}^*(t)\rangle_{W,W^*} dt = 0$, i. e., $\int_0^{2\pi} \frac{d}{dt}(\langle x, v^*\rangle_{W,W^*})(t)dt = 0$. Since $\langle x, v^*\rangle_{W,W^*} \in W^{1,p'}(0,2\pi)$, we infer that $\langle x, v^*(2\pi)\rangle_{W,W^*} = \langle x, v^*(0)\rangle_{W,W^*}$ and hence that $v^*(2\pi) = v^*(0)$ since $x \in W$ is arbitrary. ∎

Although the relation $(\text{rge}\, D_A)^\perp \subset \ker D_{-A^*}$ suffices in the above proof, it is easily seen (by reversing the arguments and by the denseness of $(C_0^\infty(0,2\pi)\oplus\mathbb{C})\otimes W$ in $\mathcal{W}^p_{per}$) that the stronger relation $(\text{rge}\, D_A)^\perp = \ker D_{-A^*}$ holds. As a result, we obtain

$$\text{index}\, D_A = \dim\ker D_A - \dim\ker D_{-A^*}, \tag{4.5}$$

under the assumptions of Theorem 4.1. Note that even though this formula is established by viewing D_A and D_{-A^*} acting from $\mathcal{W}^p$ to $\mathcal{X}^p$ and from $\mathcal{X}^{p'}_*$ to $\mathcal{W}^{p'}_*$, respectively, it follows from Theorem 5.2 in the next section (invariance of the null-space) that it remains true when D_A and D_{-A^*} act from $\mathcal{W}^p$ to $\mathcal{X}^p$ and from $\mathcal{X}^q_*$ to $\mathcal{W}^q_*$, respectively, for any choice of $p,q\in(1,\infty)$ (in particular, when $p=q$). This is useful in some arguments.

In its general form, Corollary 4.2 below does *not* follow from the well known property that compact perturbations of Fredholm operators do not affect the index.

Corollary 4.2. *Suppose that* (H1) *to* (H3) *hold. If* $K\in C^0_{per}([0,2\pi],\mathcal{K}(W,X))$, *then,* D_{A+K} *is Fredholm from* $\mathcal{W}^p_{per}$ *to* $\mathcal{X}^p$ *for every* $p\in(1,\infty)$ *and* $\text{index}\, D_{A+K} = \text{index}\, D_A$.

Proof. For $s\in[0,1]$, set $A_s(t) := A(t)+sK(t)$, so that $A_0 = A$ and $A_1 = A+K$. For every $s\in[0,1]$, A_s satisfies (H1) to (H3) (for the latter, see Lemma 2.3). Thus, D_{A_s} is Fredholm for every $s\in[0,1]$ by Theorem 4.1 and $\text{index}\, D_{A_1} = \text{index}\, D_{A_0}$ by the local constancy of the index (Kato [18], Lindenstrauss and Tzafriri [20]). ∎

In Corollary 4.2, the compactness of $K(t)$ does not suffice to ascertain that D_{A+K} is a compact perturbation of D_A. However, this is true provided that $K\in C^0([0,2\pi],\mathcal{L}(Z,X))$, where Z is a Banach space such that $W\subset Z\subset X$ and the embedding $W\hookrightarrow Z$ is compact, for then the embedding $\mathcal{W}^p_{per}\hookrightarrow \mathcal{Z}^p := L^p(0,2\pi;Z)$ is compact by [31] and the multiplication by K is continuous from $\mathcal{Z}^p$ to $\mathcal{X}^p$. In this case, Corollary 4.2 remains true even if $K(2\pi)\neq K(0)$, so that $A+K$ is *not* periodic (note that the condition $A(0)=A(2\pi)$ is not needed for D_A to map $\mathcal{W}^p_{per}$ into $\mathcal{X}^p$).

The question whether the index of D_A is always 0 in Theorem 4.1 is open (except in the finite dimensional case). On the other hand, it can be shown that the index is 0 under various extra conditions. A few are discussed in the remainder of this section, that are derived from the "basic" case is when $A(t)$ is t-independent:

Theorem 4.3. *Suppose that, in* Theorem 4.1, $A(t)=A$ *is* t*-independent (so that* (H2) *is vacuous). Then* $D_A:\mathcal{W}^p_{per}\to\mathcal{X}^p$ *has index* 0 *and compact resolvent for every* $p\in(1,\infty)$. *Furthermore:*
(i) $\ker D_A = \oplus_{\{k\in\mathbb{Z}: ik\in\sigma(A)\}} e_k\otimes\ker(A-ikI)$, *where* e_k *denotes the function* e^{ikt}.
(ii) $\text{rge}\, D_A = \{f\in\mathcal{X}^p : \widehat{f}(k) := \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}dt \in \text{rge}(A-ikI), \forall k\in\mathbf{Z}\}$.

Proof. By Lemma 2.1 (iii), $\sigma(A)$ is discrete, so that there is $\lambda\in\mathbb{R}$ such that $\sigma(A-\lambda I)\cap i\mathbb{Z}=\emptyset$. Furthermore, since λI is compact when viewed as an operator from W to X by (H1), it follows from (H3) and Lemma 2.3 that there is $\kappa\in\mathbb{N}$ such that the family $\{kR(ik,A-\lambda I): k\in\mathbb{Z}, |k|\geq\kappa\}$ is r-bounded. Since $\{kR(ik,A-\lambda I): k\in\mathbb{Z}, |k|<\kappa_n\}$ is finite (and well defined), hence r-bounded, it follows that $\{kR(ik,A-\lambda I): k\in\mathbb{Z}\}$ is r-bounded. By Theorem 1.1, the operator $D_{A-\lambda I}$ is an isomorphism of $\mathcal{W}^p_{per}$ to $\mathcal{X}^p$ for every $p\in(1,\infty)$. But $D_{A-\lambda I} = D_A+\lambda I$, which shows that $-\lambda \in \rho(D_A)$. Thus, by Corollary 3.7, D_A has compact resolvent and index 0.

We now prove the characterizations of $\ker D_A$ and $\operatorname{rge} D_A$ given in (i) and (ii) of the theorem.

(i) It is readily checked that if $u := \sum_{\{k\in\mathbb{Z}, ik\in\sigma(A)\}} e_k \otimes x_k$ with $x_k \in \ker(A - ikI) \subset W$, then $u \in \mathcal{W}^p_{per}$ and $D_A u = 0$. Conversely, if $u \in \ker D_A$, then $(A - ik)\widehat{u}(k) = 0$ for every $k \in \mathbb{Z}$, where $\widehat{u}(k) := \frac{1}{2\pi}\int_0^{2\pi} u(t)e^{-ikt}dt \in W$ (see [3, Lemma 2.1]). Thus, $\widehat{u}(k) = 0$ if $ik \in \rho(A)$, i.e., for all but finitely many indices k by (H3), and then Fejér's theorem (see [2] for the vector-valued case) shows that $u = \sum_{\{k\in\mathbb{Z}, ik\in\sigma(A)\}} e_k \otimes \widehat{u}(k)$ (finite sum). This proves the claim.

(ii) If $f \in \operatorname{rge} D_A$, so that $D_A u = f$ for some $u \in \mathcal{W}^p$, then $(A - ikI)\widehat{u}(k) = \widehat{f}(k)$ for every $k \in \mathbb{Z}$, whence $\widehat{f}(k) \in \operatorname{rge}(A - ikI)$ for every $k \in \mathbb{Z}$. This is a restriction only if $ik \in \sigma(A)$. For every such k, and since $A - ikI$ is Fredholm of index 0, we have $\operatorname{codim} \operatorname{rge}(A - ikI) = \dim\ker(A - ikI) := d_k$. Choose a complement Z_k of $\operatorname{rge}(A - ikI)$ in X (so that $\dim Z_k = d_k$) and call $P_k \in \mathcal{L}(X)$ the projection onto Z_k associated with the direct sum $X = \operatorname{rge}(A - ikI) \oplus Z_k$. Next, define $\widehat{P}_k : \mathcal{X}^p \to X$ by

$$\widehat{P}_k(f) := P_k\widehat{f}(k).$$

Since $f \in \mathcal{X}^p \mapsto \widehat{f}(k) \in X$ is onto, it follows that $\operatorname{rank} \widehat{P}_k = \operatorname{rank} P_k = d_k$. Equivalently, $\operatorname{codim} \ker \widehat{P}_k = d_k$. With this notation, the necessary condition $\widehat{f}(k) \in \operatorname{rge}(A - ikI)$ whenever $ik \in \sigma(A)$ for f to be in $\operatorname{rge} D_A$ reads

$$\operatorname{rge} D_A \subset \cap_{ik\in\sigma(A)} \ker \widehat{P}_k. \tag{4.6}$$

From part (i) and since D_A has index 0, it follows that $d := \operatorname{codim} \operatorname{rge} D_A = \sum_{ik\in\sigma(A)} d_k$. On the other hand, if $\left(\cap_{ik\in\sigma(A)} \ker \widehat{P}_k\right) \cap \mathcal{Z} = \{0\}$ for some d-dimensional subspace $\mathcal{Z}$ of $\mathcal{X}^p$, then $\operatorname{codim} \cap_{ik\in\sigma(A)} \ker \widehat{P}_k f \geq d$. If so, equality must hold in (4.6), whence $\operatorname{rge} D_A = \{f \in \mathcal{X}^p : \widehat{f}(k) \in \operatorname{rge}(A - ikI), \forall k \in \mathbf{Z}\}$.

Now, the space $\mathcal{Z} := \bigoplus_{ik\in\sigma(A)} e_k \otimes Z_k$ has dimension d. If $f = \sum_{ik\in\sigma(A)} e_k \otimes z_k \in \mathcal{Z}$ $(z_k \in Z_k)$, then $\widehat{P}_j f = z_j$, so that $f \in \cap_{ik\in\sigma(A)} \ker \widehat{P}_k f$ only if $z_k = 0$ for every $ik \in \sigma(A)$, and then $f = 0$. This completes the proof. ∎

Remark 4.1. *By replacing A by $A + \lambda I$ in* Theorem 4.3, *it follows that $\lambda \in \sigma(D_A)$ if and only if $\lambda + ik \in \sigma(A)$ for some $k \in \mathbb{Z}$. Equivalently, $\sigma(D_A) = \sigma(A) + i\mathbb{Z}$.*

Corollary 4.4. *Suppose that, in* Theorem 4.1, *$A(t) - A(0) \in \mathcal{K}(W, X)$. Then $D_A : \mathcal{W}^p_{per} \to \mathcal{X}^p$ has index 0 for every $p \in (1, \infty)$.*

Proof. Use Theorem 4.3 and Corollary 4.2. ∎

From the comments after Corollary 4.2, if $A - A(0) \in C^0([0, 2\pi], \mathcal{L}(Z, X))$ where Z is a Banach space such that $W \subset Z \subset X$ and the embedding $W \hookrightarrow Z$ is compact, then Corollary 4.4 remains true when $A(0) \neq A(2\pi)$. We complete this section with the remark that a very different property (symmetry) also implies that the index is 0.

Corollary 4.5. *Suppose that, in* Theorem 4.1, *$A(2\pi - t) = A(t)$ for $t \in [0, 2\pi]$. Then $D_A : \mathcal{W}^p_{per} \to \mathcal{X}^p$ has index 0 for every $p \in (1, \infty)$.*

Proof. For $s \in [0,\pi]$, set

$$A_s(t) := \begin{cases} A(t) \text{ if } t \in [0,s] \cup [2\pi - s, 2\pi], \\ A(s) \text{ if } t \in (s, 2\pi - s). \end{cases}$$

It is obvious that A_s satisfies (H1) to (H3), so that $D_{A_s} : \mathcal{W}^p_{per} \to \mathcal{X}^p$ is semi-Fredholm for every $p \in (1,\infty)$ and every $s \in [0,\pi]$ by Theorem 3.6. Also, $D_{A_s} \in \mathcal{L}(\mathcal{W}^p_{per}, \mathcal{X}^p)$ depends continuously upon s, so that $\operatorname{index} D_{A_\pi} = \operatorname{index} D_{A_0}$. But $A_\pi = A$ and $A_0 = A(0)$. Thus, $\operatorname{index} D_A = 0$ by Theorem 4.3 for $A(0)$. ∎

More generally, by combining Corollaries 4.4 and 4.5, D_A has index 0 if $A(2\pi - t) - A(t)$ is compact for every $t \in [0,2\pi]$.

5. The p-independence of the index and spectrum

In Corollary 3.7, we proved that (H1) to (H3) imply that D_A is a closed operator on $\mathcal{X}^p$ with domain $\mathcal{W}^p_{per}$ for every $p \in (1,\infty)$. In this section, we show that $\operatorname{index} D_A$ and $\sigma(D_A)$ as well as the multiplicity of the isolated eigenvalues of D_A are independent of p. The proof relies on the following "consistency" property.

Lemma 5.1. *Suppose that* (H1) *to* (H3) *hold. If $p,q \in (1,\infty)$ and $f \in \mathcal{X}^p \cap \mathcal{X}^q$, every $u \in \mathcal{W}^p_{per}$ such that $D_A u = f$ is in $\mathcal{W}^p_{per} \cap \mathcal{W}^q_{per}$.*

Proof. Since the result is obvious if $p \geq q$, we assume $p < q$. By an argument similar to the one used in the proof of Lemma 3.5 (extension of A, u and f by periodicity), and by noticing that $\mathcal{W}^p_{per} \subset \mathcal{X}^p \cap \mathcal{X}^q$, it suffices to prove the result when $\operatorname{Supp} u$ is contained in a compact subinterval Q of $(0,2\pi)$.

Let $t_0 \in Q$ be given. Since $\sigma(A(t_0)) \neq \mathbb{C}$ by (H3), choose $\lambda_0 \in \mathbb{C}$ such that $A(t_0) - \lambda_0 I \in GL(W,X)$, so that $D_{A(t_0)-\lambda_0 I}$ is an isomorphism of $\mathcal{W}^p_{per}$ to $\mathcal{X}^p$ and of $\mathcal{W}^q_{per}$ to $\mathcal{X}^q$ by Theorem 1.1. For $\varepsilon > 0$, define $A_\varepsilon(t)$ by

$$A_\varepsilon(t) := \begin{cases} A(t_0 - \varepsilon) \text{ if } t \in [0, t_0 - \varepsilon), \\ A(t) \text{ if } t \in [t_0 - \varepsilon, t_0 + \varepsilon], \\ A(t_0 + \varepsilon) \text{ if } t \in (t_0 - \varepsilon, 2\pi]. \end{cases}$$

Clearly, $A_\varepsilon \in C^0([0,2\pi], \mathcal{L}(W,X))$ and $\lim_{\varepsilon\to 0} \sup_{t\in[0,2\pi]} ||A_\varepsilon(t) - A(t_0)||_{\mathcal{L}(W,X)} = 0$. As a result, $D_{A_\varepsilon - \lambda_0 I} \to D_{A(t_0)-\lambda_0 I}$ in $\mathcal{L}(\mathcal{W}^p_{per}, \mathcal{X}^p)$ and in $\mathcal{L}(\mathcal{W}^q_{per}, \mathcal{X}^q)$, so that $D_{A_\varepsilon - \lambda_0 I}$ is an isomorphism of $\mathcal{W}^p_{per}$ to $\mathcal{X}^p$ and of $\mathcal{W}^q_{per}$ to $\mathcal{X}^q$ if $\varepsilon > 0$ is small enough. ($A_\varepsilon(0)$ need not equal $A_\varepsilon(2\pi)$, but this is irrelevant since $D_{A_\varepsilon - \lambda_0 I}$ still maps $\mathcal{W}^p_{per}$ into $\mathcal{X}^p$.)

Now, let $\psi \in C_0^\infty$ be such that $\operatorname{Supp}\psi \subset (t_0 - \varepsilon, t_0 + \varepsilon)$. Since $D_A u = f$, it follows that $D_{A-\lambda_0 I} u = f + \lambda_0 u$ and so

$$D_{A-\lambda_0 I}(\psi u) = \psi f + \lambda_0 \psi u - \dot{\psi} u.$$

Observe that $D_{A-\lambda_0 I}(\psi u) = D_{A_\varepsilon - \lambda_0 I}(\psi u)$ since $\operatorname{Supp}\psi \subset (t_0 - \varepsilon, t_0 + \varepsilon)$ and since $A_\varepsilon(t)$ coincides with $A(t)$ for $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$. Therefore,

$$D_{A_\varepsilon - \lambda_0 I}(\psi u) = \psi f + \lambda_0 \psi u - \dot{\psi} u.$$

Above, the right-hand side is in $\mathcal{X}^p \cap \mathcal{X}^q$ since $f \in \mathcal{X}^p \cap \mathcal{X}^q$ by hypothesis and since $u \in \mathcal{W}^p_{per} \subset \mathcal{X}^p \cap \mathcal{X}^q$. From the above, ψu is the unique solution $v \in \mathcal{W}^p_{per}$ of the equation $D_{A_\varepsilon - \lambda_0 I} v = \psi f + \lambda_0 \psi u - \dot{\psi} u$. But this equation also has a unique solution in $\mathcal{W}^q_{per}$. Since $\mathcal{W}^q_{per} \subset \mathcal{W}^p_{per}$ (because $p < q$), its solutions in $\mathcal{W}^p_{per}$ and $\mathcal{W}^q_{per}$ coincide. This shows that $\psi u \in \mathcal{W}^p_{per} \cap \mathcal{W}^q_{per}$.

That u (with $\operatorname{Supp} u \subset Q \subset (0,2\pi)$) is in $\mathcal{W}^p_{per} \cap \mathcal{W}^q_{per}$ follows easily by covering Q by finitely many intervals $(t_0 - \varepsilon, t_0 + \varepsilon)$ as above and using a partition of unity on Q (as in the proof of Lemma 3.4). ∎

Theorem 5.2. *Suppose that* (H1) *to* (H3) *hold and, for* $p \in (1,\infty)$, *view* D_A *as a closed unbounded operator on* $\mathcal{X}^p$ *with domain* $\mathcal{W}^p_{per}$ *(*Corollary 3.7*). Then,* $\ker D_A$, $\operatorname{index} D_A$ *(see* Theorem 4.1*) and* $\sigma(D_A)$ *are independent of* $p \in (1,\infty)$. *Moreover, if* $\sigma(D_A) \neq \mathbb{C}$, *every* $\lambda \in \sigma(D_A)$ *is an isolated eigenvalue whose multiplicity is independent of* $p \in (1,\infty)$.

Proof. First, it follows at once from Lemma 5.1 that $\ker D_A$ is independent of p. Thus, the injectivity of D_A is independent of p. Below, we show that, given $p, q \in (1,\infty)$ and $k \in \mathbb{N} \cup \{0\}$, the condition $\operatorname{codim} \operatorname{rge} D_A \geq k$ when $D_A : \mathcal{W}^p_{per} \to \mathcal{X}^p$ implies $\operatorname{codim} \operatorname{rge} D_A \geq k$ when $D_A : \mathcal{W}^q_{per} \to \mathcal{X}^q$, so that $\operatorname{codim} \operatorname{rge} D_A$ is independent of p. In particular, whether D_A is onto $\mathcal{X}^p$ is independent of p. Thus, the index of D_A and its invertibility are independent of p. Upon replacing A by $A - \lambda I$ in the last statement, it follows that $\sigma(D_A)$ is independent of p.

Suppose then that $\operatorname{codim} \operatorname{rge} D_A \geq k$ when $D_A : \mathcal{W}^p_{per} \to \mathcal{X}^p$. There is a k -dimensional subspace Z_k of $\mathcal{X}^p$ such that $Z_k \cap D_A(\mathcal{W}^p_{per}) = \{0\}$. Since Z_k is finite-dimensional and $D_A(\mathcal{W}^p_{per}) \subset \mathcal{X}^p$ is closed, the condition $Z_k \cap D_A(\mathcal{W}^p_{per}) = \{0\}$ is unaffected by small enough perturbations of Z_k. In particular, by the denseness of $C^\infty_0(0,2\pi) \otimes X$ in $\mathcal{X}^p$, it is not restrictive to assume $Z_k \subset C^\infty_0(0,2\pi) \otimes X$ (if $k > 0$, just approximate a basis of Z_k by members of $C^\infty_0(0,2\pi) \otimes X$). If so, $Z_k \subset \mathcal{X}^p \cap \mathcal{X}^q$ and if $g \in Z_k$ and $D_A u = g$ for some $u \in \mathcal{W}^q_{per}$, then $u \in \mathcal{W}^p_{per}$ by Lemma 5.1. It follows that $g \in Z_k \cap D_A(\mathcal{W}^p_{per}) = \{0\}$, i.e., $g = 0$. Thus, $Z_k \cap D_A(\mathcal{W}^q_{per}) = \{0\}$, so that $\operatorname{codim} \operatorname{rge} D_A \geq k$ when $D_A : \mathcal{W}^q_{per} \to \mathcal{X}^q$. This completes the proof that $\operatorname{index} D_A$ and $\sigma(D_A)$ are independent of p.

If $\sigma(D_A) \neq \mathbb{C}$, every $\lambda \in \sigma(D_A)$ is an isolated eigenvalue of finite multiplicity (Corollary 3.7). Given $p \in (1,\infty)$, the multiplicity m_p of λ when $D_A : \mathcal{W}^p_{per} \to \mathcal{X}^p$is the (finite) dimension of the space $P_p(\mathcal{X}^p)$, where

$$P_p := -\frac{1}{2\pi i} \int_\Gamma R_p(\zeta, D_A) d\zeta,$$

$R_p(\zeta, D_A) := (D_A - \zeta I)^{-1} \in \mathcal{L}(\mathcal{X}^p)$ and Γ is a small circle around λ lying entirely in $\rho(D_A)$ (independent of p from the above). By the denseness of $\mathcal{X}^p \cap \mathcal{X}^q$ in $\mathcal{X}^p$ and the finite dimensionality of $P(\mathcal{X}^p)$, it follows that $m_p = \dim P_p(\mathcal{X}^p \cap \mathcal{X}^q)$. Likewise, $m_q = \dim P_q(\mathcal{X}^p \cap \mathcal{X}^q)$. But, by Lemma 5.1, $R_p(\zeta, D_A)$ and $R_q(\zeta, D_A)$ coincide in $\mathcal{X}^p \cap \mathcal{X}^q$, so that $P_p(\mathcal{X}^p \cap \mathcal{X}^q) = P_q(\mathcal{X}^p \cap \mathcal{X}^q)$ and hence $m_p = m_q$. ∎

In the above proof, the result that $\ker D_A$ is independent of p is also true when A is replaced by $A - \lambda I$ with $\lambda \in \mathbb{C}$. Therefore, all the eigenspaces of D_A are p-independent (irrespective of $\sigma(D_A)$ being the whole plane or not).

Remark 5.1. *It is readily checked that* $\sigma(D_A)$ *is invariant by* $i\mathbb{Z}$ *translations, i.e.,* $\sigma(D_A) = \sigma(D_A) + i\mathbb{Z}$. *This is obvious if* $\sigma(D_A) = \mathbb{C}$. *Otherwise, every* $\lambda \in \sigma(D_A)$ *is an eigenvalue of* D_A. *If* u *is a corresponding eigenfunction and* $k \in \mathbb{Z}$, *then* $\lambda + ik$ *is an eigenvalue associated with the eigenfunction* $v := e^{ikt}u$.

The next corollary is especially relevant when $\pm A(t)$ is a differential operator. Recall that a closed operator A_0 on X with domain W is said to be *sectorial* if both W and $A_0(W)$ are dense in X and if $(-\infty, 0) \subset \rho(A_0)$ with $\{\zeta R(-\zeta, A_0) : \zeta > 0\}$

bounded in $\mathcal{L}(X)$. If the set $\{\zeta R(-\zeta, A_0) : \zeta > 0\}$ is not only bounded but also r-bounded in $\mathcal{L}(X)$, then A_0 is said to be *r-sectorial*.

If A_0 is an r-sectorial operator, then for $\theta > 0$ small enough, the set $\{\zeta R(-\zeta, A_0) : |\arg \zeta| \leq \theta\}$ is r-bounded in $\mathcal{L}(X)$ (see for instance [11, p. 43]). The *r-angle* $\phi^r_{A_0}$ of A_0 is the infimum of those $\theta \in (0, \pi)$ such that the set $\{\zeta R(-\zeta, A_0) : |\arg \zeta| \leq \pi - \theta\}$ is r-bounded. The value of Corollary 5.3 below when $\pm A(t)$ is an elliptic operator - possibly a system- associated with suitable homogeneous boundary conditions[3] on a domain with compact boundary, is that there are known sufficient conditions about the coefficients ensuring that $A(t) + \mu_t I$ is r-sectorial with r-angle $\phi^r_{A(t)+\mu_t I} < \frac{\pi}{2}$ for some $\mu_t \geq 0$ ([11, Theorem 8.2, p. 102]). As the proof of Corollary 5.3 will show, this condition is stronger than (H3).

Corollary 5.3. *Suppose that* (H1) *and* (H2) *hold and that, for every* $t \in [0, 2\pi]$, *there is* $\mu_t \geq 0$ *such that* $A(t) + \mu_t I$ *is r-sectorial with r-angle* $\phi^r_{A(t)+\mu_t I} < \frac{\pi}{2}$. *Then,* $D_{\pm A} : \mathcal{W}^p_{per} \to \mathcal{X}^p$ *is Fredholm for every* $p \in (1, \infty)$ *and its index is independent of* p.
Note: *We shall prove later* (Corollary 9.3) *that, among other things,* $\operatorname{index} D_{\pm A} = 0$.

Proof. Upon increasing μ_t by any amount, it is not restrictive to assume that $A(t) + \mu_t I$ is invertible. This does not affect r-sectoriality and does not increase the r-angle (see for instance Proposition 4.3 in [11] with $B = 0$ and $\alpha = \beta = 0$ in that proposition). Then, since $\phi^r_{A(t)+\mu_t I} < \frac{\pi}{2}$, it follows that $\{\xi R(-i\xi, A(t) + \mu_t I) : \xi \in \mathbb{R}\}$ is r-bounded in $\mathcal{L}(X)$. Since this set is invariant upon changing ξ into $-\xi$, this amounts to saying that $\{\xi R(i\xi, A(t) + \mu_t I) : \xi \in \mathbb{R}\}$ is r-bounded in $\mathcal{L}(X)$. In particular, $\{kR(ik, A(t) + \mu_t I) : k \in \mathbb{Z}\}$ is r-bounded in $\mathcal{L}(X)$. By (H1) and Lemma 2.3, it follows that there is $\kappa(t) \in \mathbb{N}$ such that $\{kR(ik, A(t)) : k \in \mathbb{Z}, |k| \geq \kappa(t)\}$ is r-bounded, so that (H3) holds. Thus, the conclusion for D_A follows from Theorem 4.1.

Next, observe that the hypotheses of the corollary are unchanged by changing $A(t)$ into $B(t) := A(2\pi - t)$. Thus, $D_B : \mathcal{W}^p_{per} \to \mathcal{X}^p$ is Fredholm from the above. Now, the change of variable $t = 2\pi - s$ changes D_B into $-D_{-A}$, so that $-D_{-A}$, and hence also D_{-A}, is Fredholm from $\mathcal{W}^p_{per}$ to $\mathcal{X}^p$ (with p-independent index). ∎

6. Higher order spaces I

This section is motivated by the applications of the Fredholm theory for D_A to nonlinear problems, notably in PDEs. Typically, such applications involve a nonlinear mapping $\mathcal{F} : \mathcal{W}^p \to \mathcal{X}^p$. Since $\mathcal{W}^p$ and $\mathcal{X}^p$ are spaces of functions on $(0, 2\pi)$ with values in W and X, respectively, many such mappings arise from some $F : [0, 2\pi] \times W \to X$ via substitution, that is, defined by $\mathcal{F}(u)(t) := F(t, u(t))$ where $u \in \mathcal{W}^p$. (Incidentally, the part $-A(t)u$ in D_A is also of this form.) Of course, the properties of F must ensure that $\mathcal{F}(u) \in \mathcal{X}^p$ whenever $u \in \mathcal{W}^p$.

Now, it is intuitively clear and widely corroborated by numerous examples, that there are many more nonlinear mappings (and with better properties) defined on a Banach algebra rather than just on a Banach space. Since $\mathcal{W}^p \subset W^{1,p}(0, 2\pi; X) \cap$

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[3] In practice, the boundary conditions are incorporated to the definition of W, so that these boundary conditions must be t-independent; this is one of the limitations induced by the hypothesis that the domain W of $A(t)$ is t-independent.

$L^p(0, 2\pi; W)$, the only obvious way for $\mathcal{W}^p$ to embed in a Banach algebra[4] is when either $W^{1,p}(0, 2\pi; X)$ or $L^p(0, 2\pi; W)$ is contained in such an algebra. The case $W = \mathbb{C}$ already shows that this is hopeless for the latter space, so that the only option is that $W^{1,p}(0, 2\pi; X)$ is contained in a Banach algebra. This will indeed happen when X is contained in a Banach algebra. However, keeping in mind that X must also be UMD, hence reflexive, the most useful case in PDE applications is when X is a (closed subspace of a) Lebesgue space $L^q(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$. Unfortunately, $L^q(\Omega)$ is a Banach algebra only when $q = \infty$, a case ruled out in virtually all PDE applications.

On the other hand, it is typical that W is a closed subspace of some Sobolev space $W^{m,q}(\Omega)$ with $m \geq 1$, which is a Banach algebra when $mq > N$. If so, $W^{1,p}(0, 2\pi; W) \hookrightarrow W^{1,p}(0, 2\pi; W^{m,q}(\Omega))$ and the latter space is then a Banach algebra for all $p \geq 1$. This provides a motivation to look into the Fredholm properties of D_A now acting from the space

$$\mathcal{W}^{1,p}_{per} := W^{2,p}_{per}(0, 2\pi; X) \cap W^{1,p}_{per}(0, 2\pi; W), \tag{6.1}$$

where

$$W^{2,p}_{per}(0, 2\pi; X) := \{u \in W^{2,p}(0, 2\pi; X) : u(0) = u(2\pi), \dot{u}(0) = \dot{u}(2\pi)\}, \tag{6.2}$$

into the space

$$\mathcal{X}^{1,p}_{per} := W^{1,p}_{per}(0, 2\pi; X). \tag{6.3}$$

It will be useful to notice that an equivalent definition of $\mathcal{W}^{1,p}_{per}$ is

$$\mathcal{W}^{1,p}_{per} := \{u \in \mathcal{W}^p_{per} : \dot{u} \in \mathcal{W}^p_{per}\}. \tag{6.4}$$

As we shall see in this section and the next one, the Fredholm and spectral properties of D_A in the above setting can be obtained by using a combination of the previous results or arguments together with “regularity” properties that we shall establish along the way. The first one is a variant of Theorem 1.1 (t-independent case) in this new functional framework.

Theorem 6.1. *Let A be a closed unbounded operator on*[5] *X with domain W equipped with the graph norm. If $\sigma(A) \cap i\mathbb{Z} = \emptyset$ and the set $\{kR(ik, A) : k \in \mathbb{Z}\}$ is r-bounded in $\mathcal{L}(X)$, then, given $p \in (1, \infty)$, the operator $D_A := \frac{d}{dt} - A$ is an isomorphism of $\mathcal{W}^{1,p}_{per}$ onto $\mathcal{X}^{1,p}_{per}$.*

Proof. Clearly, D_A is one to one on $\mathcal{W}^{1,p}_{per}$ since it is already one to one on $\mathcal{W}^p_{per}$ by Theorem 1.1. To prove the surjectivity of D_A, let $f \in \mathcal{X}^{1,p}_{per}$ be given. Since $f \in \mathcal{X}^p$ and $\dot{f} \in \mathcal{X}^p$, it follows from Theorem 1.1 that there are $u, v \in \mathcal{W}^p_{per}$ such that $D_A u = f$ and $D_A v = \dot{f}$. Thus, by (6.4), it suffices to show that $v = \dot{u}$. This is obvious if f is an X-valued trigonometric polynomial, for then $f = \sum_{k=-n}^{n} e_k \otimes \widehat{f}(k)$ for some $n \in \mathbb{N} \cup \{0\}$ (where, as before, e_k is the function e^{ikt}) and $u = -\sum_{k=-n}^{n} e_k \otimes R(ik, A)\widehat{f}(k)$, so that $\dot{u} \in \mathcal{W}^p_{per}$ and $D_A \dot{u} = \ddot{u} - A\dot{u} = \dot{f}$.

In general, that $v = \dot{u}$ follows from the above and the denseness of the X-valued trigonometric polynomials in $\mathcal{X}^{1,p}_{per}$ (see below). Indeed, if (f_n) is a sequence of such polynomials such that $f_n \to f$ in $\mathcal{X}^{1,p}_{per}$ and if $u_n := (D_A)^{-1} f_n$, then $u_n \to u$ in

[4]There may be nonobvious ways, depending upon X and W. For example, if $X = L^p(0, 1)$ and $W = W^{2,p}(0, 1)$, then $\mathcal{W}^p \subset W^{1,p}((0, 2\pi) \times (0, 1))$, which is a Banach algebra when $p > 2$.

[5]Recall that in this paper, X has the UMD property.

$\mathcal{W}^p_{per}$ and $\dot{u}_n = (D_A)^{-1}\dot{f}_n \to v$ in $\mathcal{W}^p_{per}$. Since also $\dot{u}_n$ tends to $\dot{u}$ as a distribution with values in X, it follows that $\dot{u} = v$.

The proof of the denseness claim is the same as in the familiar scalar case: Given $m, n \in \mathbb{N} \cup \{0\}$, set

$$g_m := \sum_{k=-m}^{m} e_k \otimes \widehat{f}(k), \quad \widetilde{g}_m := \sum_{k=-m}^{m} e_k \otimes (ik\widehat{f}(k))$$

and

$$f_n := \frac{1}{n+1}\sum_{m=0}^{n} g_m, \quad \widetilde{f}_n := \frac{1}{n+1}\sum_{m=0}^{n} \widetilde{g}_m.$$

By Fejér's theorem, $f_n \to f$ in $\mathcal{X}^p$ and $\widetilde{f}_n \to \dot{f}$ in $\mathcal{X}^p$ (the latter since the Fourier coefficients of $\dot{f}$ are $(ik\widehat{f}(k))$ by the periodicity of f). On the other hand, it is obvious that $\widetilde{g}_m = \dot{g}_m$, so that $\widetilde{f}_n = \dot{f}_n$. Therefore, $f_n \to f$ in $\mathcal{X}^{1,p}_{per}$. ∎

In what follows, we shall use the notation

$$A \in C^1_{per}([0, 2\pi], \mathcal{L}(W, X)) \tag{6.5}$$

when $A, \dot{A} \in C^0_{per}([0, 2\pi], \mathcal{L}(W, X))$. Thus, (6.5) is a strengthening of (H2). This will be used repeatedly without further mention.

By using Theorem 6.1 instead of Theorem 1.1, we obtain the following variant of Theorem 3.6:

Theorem 6.2. *Suppose that* (H1) *and* (H3) *hold and that* $A \in C^1_{per}([0, 2\pi], \mathcal{L}(W, X))$. *Then, for every* $p \in (1, \infty)$, *the operator* $D_A := \frac{d}{dt} - A(\cdot) : \mathcal{W}^{1,p}_{per} \to \mathcal{X}^{1,p}_{per}$ *(see* (6.1) *and* (6.3)*) is well defined and has closed range and finite dimensional null-space.*

Proof. If $u \in \mathcal{W}^{1,p}_{per} \subset \mathcal{W}^p_{per}$, then $D_A u \in \mathcal{X}^p$ and $\frac{d}{dt}(D_A u) = \ddot{u} - \dot{A}u - A\dot{u} \in \mathcal{X}^p$ since $\ddot{u} \in \mathcal{X}^p$, $A\dot{u} \in \mathcal{X}^p$ and $\dot{A}u \in \mathcal{X}^p$ (from the assumption that A is C^1). Thus, $D_A u \in W^{1,p}(0, 2\pi; X)$. Furthermore, $D_A u(0) = \dot{u}(0) - A(0)u(0) = \dot{u}(2\pi) - A(2\pi)u(2\pi)$, whence $D_A u \in \mathcal{X}^{1,p}_{per}$ (see (6.3)). This shows that D_A maps $\mathcal{W}^{1,p}_{per}$ into $\mathcal{X}^{1,p}_{per}$.

The remainder of the proof follows the proof of Theorem 3.6: Lemma 3.1 remains true with $\mathcal{W}^p_{per}$ and $\mathcal{X}^p$ replaced by $\mathcal{W}^{1,p}_{per}$ and $\mathcal{X}^{1,p}_{per}$, respectively, upon using Theorem 6.1 instead of Theorem 1.1 and the proof of Lemma 3.2 can be repeated verbatim.

The generalization of Lemma 3.3 with the new spaces $\mathcal{W}^{1,p}_{per}$ and $\mathcal{X}^{1,p}_{per}$ is slightly more delicate and requires the condition $\sup_{s,t \in \mathrm{Supp}\,\psi} ||A(s) - A(t)||_{\mathcal{L}(W,X)} \leq \varepsilon$ in (3.5) to be replaced by the stronger requirement

$$\sup_{s,t \in \mathrm{Supp}\,\psi} ||A(s) - A(t)||_{\mathcal{L}(W,X)} + \sup_{s,t \in \mathrm{Supp}\,\psi} ||\dot{A}(s) - \dot{A}(t)||_{\mathcal{L}(W,X)} \leq \varepsilon.$$

However, by the uniform continuity of A and $\dot{A}$ on $[0, 2\pi]$, this is not an obstacle to reproducing the proof of Lemma 3.4 with obvious modifications and Lemma 3.5 remains valid because of the assumptions made about A (in particular, $\dot{A}(0) = \dot{A}(2\pi)$ ensures that the periodic extension of A is C^1). Then, the argument used in proof of Theorem 3.6 yields the desired result. Observe that the repetition of this argument makes use of the compactness of the embedding

$$\mathcal{W}^{1,p}_{per} \hookrightarrow \mathcal{X}^{1,p}_{per}, \tag{6.6}$$

which follows at once from (6.4) and the compactness of the embedding $\mathcal{W}^p_{per} \hookrightarrow \mathcal{X}^p$. ∎

It is readily checked that Corollary 3.7 is still true in the $\mathcal{W}^{1,p}_{per}$ - $\mathcal{X}^{1,p}_{per}$ setting, provided that $A \in C^1_{per}([0,2\pi],\mathcal{L}(W,X))$.

Because the proof of Theorem 4.1 relies heavily on the fact that the dual of $L^p(0,2\pi;X)$ is $L^{p'}(0,2\pi;X^*)$, it cannot be repeated when $\mathcal{W}^p_{per}$ and $\mathcal{X}^p$ are replaced by $\mathcal{W}^{1,p}_{per}$ and $\mathcal{X}^{1,p}_{per}$, respectively. This is a serious difficulty. To prove the validity of Theorem 4.1 in this setting, we shall show in the next section that $D_A : \mathcal{W}^{1,p}_{per} \to \mathcal{X}^{1,p}_{per}$ has the same index as $D_A : \mathcal{W}^p_{per} \to \mathcal{X}^p$ (Theorem 7.3), so that the Fredholm property follows from Theorem 4.1 itself. In fact, the material developed to prove this property will yield much more than the $\mathcal{W}^{1,p}_{per}$ - $\mathcal{X}^{1,p}_{per}$ variant of Theorem 4.1: It will also enable us to show that the spectrum of $D_A : \mathcal{W}^{1,p}_{per} \to \mathcal{X}^{1,p}_{per}$ coincides with the spectrum of $D_A : \mathcal{W}^p_{per} \to \mathcal{X}^p_{per}$ and, under suitable additional conditions, that $\ker D_A$ is the same in both functional settings (Lemma 7.2 and Corollary 7.8).

After the Fredholm property has been established in the $\mathcal{W}^{1,p}_{per}$-$\mathcal{X}^{1,p}_{per}$ setting, a routine check reveals that Corollary 4.2 remains true in that setting if it is also assumed that $A \in C^1_{per}([0,2\pi],\mathcal{L}(W,X))$ and $K \in C^1_{per}([0,2\pi],\mathcal{K}(W,X))$. (In fact, the multiplication by $K \in C^0_{per}([0,2\pi],\mathcal{L}(W,Z)) \cap C^1([0,2\pi],\mathcal{L}(W,Z))$ is compact from $\mathcal{W}^{1,p}_{per}$ to $\mathcal{X}^{1,p}_{per}$ if $Z \subset X$ is a Banach space such that the embedding $Z \hookrightarrow X$ is compact. Thus, the condition $\dot{K}(0) = \dot{K}(2\pi)$ is not needed in this case.). Likewise, Theorem 4.3 (constant A) as well as Corollaries 4.4 and 4.5 are still valid, provided that $A \in C^1_{per}([0,2\pi],\mathcal{L}(W,X))$ in the latter two.

Lemma 5.1 remains true as well, but the (simple) proof must be given. This is done below.

Lemma 6.3. *Suppose that* (H1) *and* (H3) *hold and that* $A \in C^1_{per}([0,2\pi],\mathcal{L}(W,X))$. *If* $p,q \in (1,\infty)$ *and* $f \in \mathcal{X}^{1,p}_{per} \cap \mathcal{X}^{1,q}_{per}$, *then every* $u \in \mathcal{W}^{1,p}_{per}$ *such that* $D_A u = f$ *is in* $\mathcal{W}^{1,p}_{per} \cap \mathcal{W}^{1,q}_{per}$.

Proof. Since the result is trivial if $p \geq q$, it suffices to consider the case $p < q$. By Lemma 5.1, we already have that $u \in \mathcal{W}^q_{per}$. Thus, by (6.4), it remains to show that $\dot{u} \in \mathcal{W}^q_{per}$.

By differentiation of the relation $D_A u := \dot{u} - Au = f$, we get $\ddot{u} - A\dot{u} = \dot{f} + \dot{A}u$ as distributions. Since $\dot{f} \in \mathcal{X}^q$ and $u \in \mathcal{W}^q_{per}$, the right-hand side is in $\mathcal{X}^p \cap \mathcal{X}^q$, so that $\dot{u} \in \mathcal{W}^q_{per}$ by another application of Lemma 5.1. ∎

By using Lemma 6.3 instead of Lemma 5.1 in the proof of Theorem 5.2, it follows that the null-space, index and spectrum of $D_A : \mathcal{W}^{1,p}_{per} \to \mathcal{X}^{1,p}_{per}$ are independent of $p \in (1,\infty)$ if (H1) to (H3) hold and $A \in C^1_{per}([0,2\pi],\mathcal{L}(W,X))$.

7. Higher order spaces II

We begin with another regularity property (see also Corollary 7.5 later). The question is simple: If $u \in \mathcal{W}^p_{per}$ and $D_A u = f \in \mathcal{X}^{1,p}_{per}$, is it true that $u \in \mathcal{W}^{1,p}_{per}$? Unlike in the ODE case, this does not follow by differentiating $D_A u = f$, because the term $A\dot{u}$ makes no sense if u is only in $\mathcal{W}^p_{per}$ and so the product rule cannot be used with Au. The answer to this question is more involved than one might perhaps expect. In a first step, we resolve the issue under an extra condition about A.

Lemma 7.1. *Suppose that* (H1) *and* (H3) *hold and that* $A \in C^1_{per}([0, 2\pi], \mathcal{L}(W, X)) \cap W^{2,\infty}(0, 2\pi; \mathcal{L}(W, X))$. *Given* $p \in (1, \infty)$, *let* $f \in \mathcal{X}^{1,p}_{per}$ *and* $u \in \mathcal{W}^p_{per}$ *be such that* $D_A u = f$. *Then,* $u \in \mathcal{W}^{1,p}_{per}$.

Proof. It follows easily from (H3) and the continuity of A that if $k \in \mathbb{Z}$ and $|k|$ is large enough, then $A(t) - ikI$ is invertible for every $t \in [0, 2\pi]$. Since $D_A u = f$, we have $D_{A-ikI}u = D_A u + iku = f + iku \in \mathcal{X}^{1,p}_{per}$ since $\mathcal{W}^p_{per} \subset \mathcal{X}^{1,p}_{per}$. Thus, upon replacing A by $A - ikI$, we may and shall assume that $A(t)$ is invertible for every $t \in [0, 2\pi]$. By setting $B := A^{-1}$ for simplicity of notation and observing that $B \in C^1_{per}([0, 2\pi], \mathcal{L}(X, W)) \cap W^{2,\infty}(0, 2\pi; \mathcal{L}(X, W))$, this makes it possible to rewrite $D_A u = f$ in the form[6]

$$u = B(\dot{u} - f) = \frac{d}{dt}(Bu) + \dot{B}u - Bf. \tag{7.1}$$

Let $\omega \in C^\infty_0$ be such that $\omega \geq 0, \operatorname{Supp} \omega \subset (-1, 1)$ and $\int_{\mathbb{R}} \omega = 1$. For $\varepsilon > 0$, set $\omega_\varepsilon(t) := \varepsilon^{-1}\omega(\varepsilon^{-1}t)$. By extending (7.1) to all of $\mathbb{R}$ by periodicity and convolving with ω_ε, we infer that

$$\omega_\varepsilon * u = \dot{\omega}_\varepsilon * (Bu) + \omega_\varepsilon * (\dot{B}u - Bf). \tag{7.2}$$

In the right-hand side, $Bu \in W^{1,p}(0, 2\pi; W)$ and $\dot{B}u - Bf \in W^{1,p}(0, 2\pi; W)$ since $u, f \in W^{1,p}(0, 2\pi; X)$ and $\dot{B} \in W^{1,\infty}(0, 2\pi; \mathcal{L}(X, W))$. As a result, $\omega_\varepsilon * u \in W^{2,p}(0, 2\pi; W)$ and, in fact, $\omega_\varepsilon * u \in W^{2,p}_{per}(0, 2\pi; W)$ since convolution does not affect periodicity. In particular, $\omega_\varepsilon * u \in \mathcal{W}^{1,p}_{per}$.

To complete the proof, it suffices to show that $D_A(\omega_\varepsilon * u)$ is bounded in $\mathcal{X}^{1,p}_{per}$ as $\varepsilon \to 0$. Indeed, $\omega_\varepsilon * u \to u$ in $W^{1,p}(0, 2\pi; X)$, hence in $\mathcal{X}^{1,p}_{per}$, so that it is bounded in $\mathcal{X}^{1,p}_{per}$. Therefore, by the analog of Lemma 3.5 in the $\mathcal{W}^{1,p}_{per}$ - $\mathcal{X}^{1,p}_{per}$ setting (whose validity under the assumptions of the lemma was noticed in the proof of Theorem 6.2), the boundedness of both $\omega_\varepsilon * u$ and $D_A(\omega_\varepsilon * u)$ in $\mathcal{X}^{1,p}_{per}$ implies that $\omega_\varepsilon * u$ is bounded in $\mathcal{W}^{1,p}_{per}$, that is, in $W^{2,p}_{per}(0, 2\pi; X)$ and in $W^{1,p}_{per}(0, 2\pi; W)$. By the reflexivity of these spaces[7], there is a sequence $\varepsilon_n \to 0$ such that $\omega_{\varepsilon_n} * u$ is weakly convergent in $W^{2,p}_{per}(0, 2\pi; X)$ and in $W^{1,p}_{per}(0, 2\pi; W)$. The continuity of the embeddings $W^{2,p}_{per}(0, 2\pi; X) \hookrightarrow \mathcal{X}^{1,p}_{per}$ and $W^{1,p}_{per}(0, 2\pi; W) \hookrightarrow \mathcal{X}^{1,p}_{per}$ shows that both weak limits coincide with the (strong) limit u of $\omega_{\varepsilon_n} * u$ in $\mathcal{X}^{1,p}_{per}$, so that $u \in W^{2,p}_{per}(0, 2\pi; X) \cap W^{1,p}_{per}(0, 2\pi; W) = \mathcal{W}^{1,p}_{per}$.

Accordingly, the remaining step is to prove the boundedness of $D_A(\omega_\varepsilon * u)$ in $\mathcal{X}^{1,p}_{per}$ as $\varepsilon \to 0$, which is the same as boundedness in $W^{1,p}(0, 2\pi; X)$. First, $D_A(\omega_\varepsilon * u) = \dot{\omega}_\varepsilon * u - A(\omega_\varepsilon * u)$, so that, by (7.2),

$$\begin{aligned} D_A(\omega_\varepsilon * u) &= \dot{\omega}_\varepsilon * u - A[\dot{\omega}_\varepsilon * (Bu) + \omega_\varepsilon * (\dot{B}u - Bf)] \\ &= A[B(\dot{\omega}_\varepsilon * u) - \dot{\omega}_\varepsilon * (Bu)] - A[\omega_\varepsilon * (\dot{B}u - Bf)]. \end{aligned} \tag{7.3}$$

Since $\dot{B}u - Bf \in W^{1,p}(0, 2\pi; W)$, it follows that $\omega_\varepsilon * (\dot{B}u - Bf) \to \dot{B}u - Bf$ in $W^{1,p}(0, 2\pi; W)$, whence $A[\omega_\varepsilon * (\dot{B}u - Bf)] \to A(\dot{B}u - Bf)$ in $W^{1,p}(0, 2\pi; X)$. Thus, the term $A[\omega_\varepsilon * (\dot{B}u - Bf)]$ in (7.3) is bounded in $W^{1,p}(0, 2\pi; X)$ and it only remains to show that $A[B(\dot{\omega}_\varepsilon * u) - \dot{\omega}_\varepsilon * (Bu)]$ is bounded as well. In turn, this reduces to showing that

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[6] Note that in contrast with Au, the product rule *is* valid with Bu.

[7] Recall that X is UMD and so is W by Lemma 2.1 (iv).

(i) $A[B(\dot\omega_\varepsilon * u) - \dot\omega_\varepsilon * (Bu)]$ is bounded in $L^p(0, 2\pi; X)$,
(ii) $\dot A[B(\dot\omega_\varepsilon * u) - \dot\omega_\varepsilon * (Bu)]$ is bounded in $L^p(0, 2\pi; X)$,
(iii) $A\frac{d}{dt}[B(\dot\omega_\varepsilon * u) - \dot\omega_\varepsilon * (Bu)]$ is bounded in $L^p(0, 2\pi; X)$.

Both (i) and (ii) follow at once from the boundedness of $B(\dot\omega_\varepsilon * u) - \dot\omega_\varepsilon * (Bu)$ in $L^p(0, 2\pi; W)$, proved below.

Write

$$B(t)(\dot\omega_\varepsilon * u)(t) - (\dot\omega_\varepsilon * (Bu))(t) = \int_{t-\varepsilon}^{t+\varepsilon} \dot\omega_\varepsilon(t-s)(B(t) - B(s))u(s)ds.$$

Since B is C^1 and periodic on $\mathbb{R}$, there is a constant $c > 0$ such that $||B(t) - B(s)||_{\mathcal{L}(X,W)} \leq c|t-s|$. Therefore,

$$\begin{aligned} (7.4)\quad ||B(t)(\dot\omega_\varepsilon * u)(t) - (\dot\omega_\varepsilon * (Bu))(t)||_W &\leq c\int_{t-\varepsilon}^{t+\varepsilon} |t-s|\,|\dot\omega_\varepsilon(t-s)|\,||u(s)||_X ds \\ &\leq c\int_{t-\varepsilon}^{t+\varepsilon} \varepsilon\,|\dot\omega_\varepsilon(t-s)|\,||u(s)||_X ds. \end{aligned}$$

If $t \in [0, 2\pi]$ and $\varepsilon > 0$ is small enough, then $\chi_{[-2\pi,4\pi]}(s) = 1$ whenever $s \in [t-\varepsilon, t+\varepsilon]$, so that (7.4) also reads

$$||B(t)(\dot\omega_\varepsilon * u)(t) - (\dot\omega_\varepsilon * (Bu))(t)||_W \leq c\int_{t-\varepsilon}^{t+\varepsilon} \varepsilon\,|\dot\omega_\varepsilon(t-s)|\,||u(s)||_X \chi_{[-2\pi,4\pi]}(s)ds.$$

Hence, for every $t \in \mathbb{R}$,

$$\begin{aligned} (7.5)\quad ||B(t)(\dot\omega_\varepsilon * u)(t) &- (\dot\omega_\varepsilon * (Bu))(t)||_W \chi_{[0,2\pi]}(t) \\ &\leq c\int_{t-\varepsilon}^{t+\varepsilon} \varepsilon\,|\dot\omega_\varepsilon(t-s)|\,||u(s)||_X \chi_{[-2\pi,4\pi]}(s)ds. \end{aligned}$$

Now, $|\dot\omega_\varepsilon(t)| = \varepsilon^{-2}|\dot\omega(\varepsilon^{-1}t)|$, so that $\varepsilon|\dot\omega_\varepsilon(t)| = \eta_\varepsilon(t)$ with $\eta := |\dot\omega|$ and (7.5) becomes

$$||B(t)(\dot\omega_\varepsilon * u)(t) - (\dot\omega_\varepsilon * (Bu))(t)||_W \chi_{[0,2\pi]}(t) \leq c\eta_\varepsilon * \left(||u||_X \chi_{[-2\pi,4\pi]}\right)(t)$$

and so, by Young's inequality,

$$||B(\dot\omega_\varepsilon * u) - (\dot\omega_\varepsilon * (Bu))||_{L^p(0,2\pi;W)} \leq c||\eta_\varepsilon||_{L^1(\mathbb{R})}||u||_{L^p(-2\pi,4\pi;X)}.$$

Since $||\eta_\varepsilon||_{L^1(\mathbb{R})} = ||\eta||_{L^1(\mathbb{R})}$ and $||u||_{L^p(-2\pi,4\pi;X)} = 3^{\frac{1}{p}}||u||_{L^p(0,2\pi;X)}$ by periodicity, we find

$$||B(\dot\omega_\varepsilon * u) - (\dot\omega_\varepsilon * (Bu))||_{L^p(0,2\pi;W)} \leq 3^{\frac{1}{p}}c||\eta||_{L^1(\mathbb{R})}||u||_{L^p(0,2\pi;X)},$$

which proves the boundedness of $B(\dot\omega_\varepsilon * u) - (\dot\omega_\varepsilon * (Bu))$ in $L^p(0, 2\pi; W)$ as $\varepsilon \to 0$.

In the above arguments, we may replace u by $\dot u$ or B by $\dot B$ (even though $\dot B$ is not C^1, it is $C^{0,1}$ and this is the property of B actually used to obtain (7.4)). Thus, both $B(\dot\omega_\varepsilon * \dot u) - (\dot\omega_\varepsilon * (B\dot u))$ and $\dot B(\dot\omega_\varepsilon * u) - (\dot\omega_\varepsilon * (\dot Bu))$ are bounded in $L^p(0, 2\pi; W)$ as $\varepsilon \to 0$, which implies that $\frac{d}{dt}[B(\dot\omega_\varepsilon * u) - \dot\omega_\varepsilon * (Bu)]$ is bounded in $L^p(0, 2\pi; W)$. But then, $A\frac{d}{dt}[B(\dot\omega_\varepsilon * u) - \dot\omega_\varepsilon * (Bu)]$ is bounded in $L^p(0, 2\pi; X)$. This is the requirement (iii) and the proof is complete. ∎

The method of proof of Lemma 7.1, by mollification and a priori estimates, is a standard way to establish elliptic regularity in PDEs. In evolution problems, it was also used by Robbin and Salamon [29] in a special case ($p = 2, X$ Hilbert, $A(t)$ selfadjoint) for related but different purposes and in a different spirit.

With the help of Lemma 7.1, we can now show that, under the additional condition $A \in W^{2,\infty}(0,2\pi;\mathcal{L}(W,X))$, then $\ker D_A, \operatorname{index} D_A$ and $\sigma(D_A)$ do not change when $\mathcal{W}^p_{per}$ and $\mathcal{X}^p$ are replaced by $\mathcal{W}^{1,p}_{per}$ and $\mathcal{X}^{1,p}_{per}$, respectively. The extra condition about A will be removed later.

Lemma 7.2. *Suppose that* (H1) *and* (H3) *hold and that* $A \in C^1_{per}([0,2\pi],\mathcal{L}(W,X)) \cap W^{2,\infty}(0,2\pi;\mathcal{L}(W,X))$. *Then, for every* $p \in (1,\infty)$, *the operators* $D_A : \mathcal{W}^p_{per} \to \mathcal{X}^p$ *and* $D_A : \mathcal{W}^{1,p}_{per} \to \mathcal{X}^{1,p}_{per}$ *are Fredholm and* $\ker D_A, \operatorname{index} D_A$ *and* $\sigma(D_A)$ *are the same in both settings (and independent of* p*).*

Proof. The semi-Fredholm property is just Theorems 3.6 and 6.2. That $\ker D_A$ is the same follows from $\mathcal{W}^{1,p}_{per} \subset \mathcal{W}^p_{per}$ and Lemma 7.1. If $f \in \mathcal{X}^{1,p}_{per}$ and $f \notin D_A(\mathcal{W}^{1,p}_{per})$, then $f \notin D_A(\mathcal{W}^p_{per})$ by Lemma 7.1. Thus, if $Z \subset \mathcal{X}^{1,p}_{per}$ is a finite dimensional subspace such that $Z \cap D_A(\mathcal{W}^{1,p}_{per}) = \{0\}$, then also $Z \cap D_A(\mathcal{W}^p_{per}) = \{0\}$. This shows that the codimension (finite or infinite) of $D_A(\mathcal{W}^p_{per})$ in $\mathcal{X}^p$ is no less than the codimension of $D_A(\mathcal{W}^{1,p}_{per})$ in $\mathcal{X}^{1,p}_{per}$.

To prove the converse, let now $Z \subset \mathcal{X}^p$ be a finite dimensional subspace such that $Z \cap D_A(\mathcal{W}^p_{per}) = \{0\}$. By the denseness of $C^\infty_0(0,2\pi) \otimes X$ and the closedness of $D_A(\mathcal{W}^p_{per})$ in $\mathcal{X}^p$, it is not restrictive to assume that $Z \subset \mathcal{X}^{1,p}_{per}$. Then, since $D_A(\mathcal{W}^{1,p}_{per}) \subset D_A(\mathcal{W}^p_{per})$, it is obvious that $Z \cap D_A(\mathcal{W}^{1,p}_{per}) = \{0\}$. This proves that the codimension of $D_A(\mathcal{W}^{1,p}_{per})$ in $\mathcal{X}^{1,p}_{per}$ is no less than (and hence equal to) the codimension of $D_A(\mathcal{W}^p_{per})$ in $\mathcal{X}^p$. In particular, by Theorem 4.1, D_A is Fredholm and $\operatorname{index} D_A$ is the same in both settings.

By (H1) (and Lemma 2.3), $A + \lambda I$ satisfies the same hypotheses as A in Lemma 7.2 for every $\lambda \in \mathbb{C}$. Therefore, from the above, $D_A - \lambda I : \mathcal{W}^p_{per} \to \mathcal{X}^p$ and $D_A - \lambda I : \mathcal{W}^{1,p}_{per} \to \mathcal{X}^{1,p}_{per}$ have closed range and fail simultaneously to be one to one or onto, which shows that $\sigma(D_A)$ is the same in both cases. The p -independence of $\ker D_A, \operatorname{index} D_A$ and $\sigma(D_A)$ was observed earlier in both settings, under more general assumptions about A. ∎

Theorem 7.3 below is the variant of Theorem 4.1 announced earlier.

Theorem 7.3. *Suppose that* (H1) *and* (H3) *hold and that* $A \in C^1_{per}([0,2\pi],\mathcal{L}(W,X))$. *Then, the operators* $D_A : \mathcal{W}^p_{per} \to \mathcal{X}^p$ *and* $D_A : \mathcal{W}^{1,p}_{per} \to \mathcal{X}^{1,p}_{per}$ *are Fredholm and have the same* p*-independent index for every* $p \in (1,\infty)$.

Proof. By Theorems 3.6 and 6.2, $\operatorname{index} D_A \in \mathbb{Z} \cup \{-\infty\}$ is well defined in both cases and, if also A is in $W^{2,\infty}(0,2\pi;\mathcal{L}(W,X))$, the result follows from Lemma 7.2.

If A is only in $C^1_{per}([0,2\pi],\mathcal{L}(W,X))$, approximate A in C^1 norm by a sequence $A_n \in C^\infty_{per}([0,2\pi],\ \mathcal{L}(W,X))$ (this can be done by extending A to all of $\mathbb{R}$ by periodicity and convolving by a sequence of mollifiers). Then, $D_{A_n} \to D_A$ in both $\mathcal{L}(\mathcal{W}^p_{per},\mathcal{X}^p)$ and $\mathcal{L}(\mathcal{W}^{1,p}_{per},\mathcal{X}^{1,p}_{per})$, so that the local constancy of the index shows that $\operatorname{index} D_A = \operatorname{index} D_{A_n}$ for n large enough, in both the $\mathcal{W}^p_{per}$ - $\mathcal{X}^p$ and $\mathcal{W}^{1,p}_{per}$ - $\mathcal{X}^{1,p}_{per}$ settings.

Thus, it suffices to show that A_n satisfies (H1) to (H3) (for n large enough), for then the finiteness and independence of $\operatorname{index} D_{A_n}$ upon the functional setting - and then those of $\operatorname{index} D_A$ as well - follow from the first part of the proof. But (H1) and (H2) are not an issue and, if n is large enough, the validity of (H3) is ensured by Lemma 2.4. ∎

For example, Theorem 7.3 yields at once a $\mathcal{W}^{1,p}_{per}$ - $\mathcal{X}^{1,p}_{per}$ variant of Corollary 5.3. Note however that its proof does not show that $\ker D_A$ is the same in both cases (compare with Lemma 7.2). In that regard, see Corollary 7.8 below.

Our next task will be to prove that $\sigma(D_A)$ is independent of the functional setting when A is only C^1 (if A is $W^{2,\infty}$, this was shown in Lemma 7.2). To do this, we need the following abstract lemma; see [28, Lemma 4.3] for a proof.

Lemma 7.4. *Let E and F be complex Banach spaces and let $T \in \mathcal{L}(E, F)$ be Fredholm of index 0 and not invertible. There is an open ball $B(0, \rho) \subset \mathcal{L}(E, F)$ with the following property: Given $H \in B(0, \rho)$ such that $T + H$ is invertible and $\varepsilon > 0$, there is $\delta \in (0, \varepsilon]$ such that if $S \in B(T, \delta) \subset \mathcal{L}(E, F)$, then $S + zH$ is not invertible for some $z \in \mathbb{C}$ with $|z| < \varepsilon$.*

Since the meaning of Lemma 7.4 may be somewhat cryptic on a first reading, it may help to notice that, when $E = F$ and 0 is an isolated eigenvalue of T, then H may be chosen to be a multiple of I. If so, Lemma 7.4 asserts that every operator $S \in \mathcal{L}(E)$ close enough to T has an eigenvalue arbitrarily close to 0. This is of course well known. Lemma 7.4 is a generalization of this property when either $E \neq F$ or 0 is not necessarily an isolated eigenvalue of T. In [28], Lemma 7.4 was already used in connection with spectral independence, but of a different nature and for very different problems (elliptic systems on $\mathbb{R}^N$).

Theorem 7.5. *Suppose that* (H1) *and* (H3) *hold and that $A \in C^1_{per}([0, 2\pi], \mathcal{L}(W, X))$. Then, for every $p \in (1, \infty)$, the operators $D_A : \mathcal{W}^p_{per} \to \mathcal{X}^p$ and $D_A : \mathcal{W}^{1,p}_{per} \to \mathcal{X}^{1,p}_{per}$ are simultaneously invertible.*

Proof. In this proof, it will be convenient to use different notations for the two operators D_A. Accordingly, we set

$$D^0_A := D_A : \mathcal{W}^p_{per} \to \mathcal{X}^p \text{ and } D^1_A := D_A : \mathcal{W}^{1,p}_{per} \to \mathcal{X}^{1,p}_{per}. \tag{7.6}$$

Suppose first that D^0_A is invertible. In particular, D^0_A is Fredholm of index 0 and so D^1_A is Fredholm of index 0 by Theorem 7.3. Since also $\ker D^1_A \subset \ker D^0_A = \{0\}$, it follows that D^1_A is invertible.

Conversely, suppose that D^1_A is invertible and, by contradiction, assume that D^0_A is not invertible. Given $B \in C^1_{per}([0, 2\pi], \mathcal{L}(W, X))$, the operators D^0_B and D^1_B are well defined and it is readily checked that if B and A are close in $C^1([0, 2\pi], \mathcal{L}(W, X))$, then D^1_B is close to D^1_A and D^0_B is close to D^0_A.

In particular, by the openness of linear isomorphisms in $\mathcal{L}(\mathcal{W}^{1,p}_{per}, \mathcal{X}^{1,p}_{per})$ there is $R > 0$ such that D^1_B is invertible whenever $B \in C^1_{per}([0, 2\pi], \mathcal{L}(W, X))$ and $||B - A||_{C^1} < 2R$, where we used the abbreviation $||B - A||_{C^1}$ for the norm of $B - A$ in $C^1([0, 2\pi], \mathcal{L}(W, X))$. By Lemma 7.2,

$$\{B \in C^\infty_{per}([0, 2\pi], \mathcal{L}(W, X)) \text{ and } ||B - A||_{C^1} < 2R\} \Rightarrow D^0_B \text{ invertible}. \tag{7.7}$$

To get a contradiction, choose $E := \mathcal{W}^p_{per}, F := \mathcal{X}^p$ and $T := D^0_A$ in Lemma 7.4 and let $\rho > 0$ be given by that lemma. Upon shrinking $R > 0$ above, we may assume that

$$||B - A||_{C^1} < 2R \Rightarrow ||D^0_B - D^0_A||_{\mathcal{L}(\mathcal{W}^p_{per}, \mathcal{X}^p)} < \rho. \tag{7.8}$$

Choose $A^\dagger \in C^\infty_{per}([0, 2\pi], \mathcal{L}(W, X))$ such that $||A^\dagger - A||_{C^1} < R$, which is possible by the denseness of $C^\infty_{per}([0, 2\pi], \mathcal{L}(W, X))$ in $C^1_{per}([0, 2\pi], \mathcal{L}(W, X))$. By (7.7), $D^0_{A^\dagger}$ is invertible and, by (7.8), $||D^0_{A^\dagger} - D^0_A||_{\mathcal{L}(\mathcal{W}^p_{per}, \mathcal{X}^p)} < \rho$.

Now, $A^\dagger = A + (A^\dagger - A)$ and, by using once again the openness of linear isomorphisms (but now in $\mathcal{L}(\mathcal{W}^p_{per}, \mathcal{X}^p)$) we can approximate $A^\dagger - A \in C^1_{per}([0, 2\pi], \mathcal{L}(W, X))$ by $C \in C^\infty_{per}([0, 2\pi], \mathcal{L}(W, X))$, in such a way that $||C||_{C^1} < R$ (so that $||D^0_{A+C} - D^0_A||_{\mathcal{L}(\mathcal{W}^p_{per}, \mathcal{X}^p)} < \rho$ by (7.8)) and that D^0_{A+C} is invertible.

At this point, let $H := D^0_{A+C} - D^0_A$ and $\varepsilon = 1$ in Lemma 7.4. With $\delta > 0$ given by that lemma, use once again the denseness of $C^\infty_{per}([0, 2\pi], \mathcal{L}(W, X))$ in $C^1_{per}([0, 2\pi], \mathcal{L}(W, X))$ to find $A^\ddagger \in C^\infty_{per}([0, 2\pi], \mathcal{L}(W, X))$ such that $||A^\ddagger - A||_{C^1} < R$ and that $||D^0_{A^\ddagger} - D^0_A||_{\mathcal{L}(\mathcal{W}^p_{per}, \mathcal{X}^p)} < \delta$. Then, Lemma 7.4 with $\varepsilon = 1$ and $S = D^0_{A^\ddagger}$ asserts that there is $z \in \mathbb{C}$ with $|z| < 1$ such that $D^0_{A^\ddagger} + z(D^0_{A+C} - D^0_A) = D^0_{A^\ddagger + zC}$ is not invertible. But $A^\ddagger + zC \in C^\infty_{per}([0, 2\pi], \mathcal{L}(W, X))$ while $||A^\ddagger + zC - A||_{C^1} \leq ||A^\ddagger - A||_{C^1} + |z|\, ||C||_{C^1} < 2R$. Thus, by (7.7), $D^0_{A^\ddagger + zC}$ is invertible. This contradiction completes the proof. ∎

By simply replacing A by $A - \lambda I$ in Theorem 7.5 (recall that $A - \lambda I$ satisfies (H3) by (H1) and Lemma 2.3), it follows at once that

Corollary 7.6. *Suppose that* (H1) *and* (H3) *hold and that* $A \in C^1_{per}([0, 2\pi], \mathcal{L}(W, X))$. *Then, for every* $p \in (1, \infty)$, *the operators* $D_A : \mathcal{W}^p_{per} \to \mathcal{X}^p$ *and* $D_A : \mathcal{W}^{1,p}_{per} \to \mathcal{X}^{1,p}_{per}$, *viewed as closed unbounded operators on the target space with domain the source space, have the same p-independent spectrum* $\sigma(D_A)$.

If $\sigma(D_A) \neq \mathbb{C}$, we can now improve upon the regularity result of Lemma 7.1.

Corollary 7.7. *Suppose that* (H1) *and* (H3) *hold and that* $A \in C^1_{per}([0, 2\pi], \mathcal{L}(W, X))$. *Suppose also that* $\sigma(D_A) \neq \mathbb{C}$ *when* D_A *is viewed as a closed unbounded operator on* $\mathcal{X}^q$ *(or* $\mathcal{X}^{1,q}_{per}$*) with domain* $\mathcal{W}^q_{per}$ *(or* $\mathcal{W}^{1,q}_{per}$*) for some* $q \in (1, \infty)$. *Given* $p \in (1, \infty)$, *let* $f \in \mathcal{X}^{1,p}_{per}$ *and* $u \in \mathcal{W}^p_{per}$ *be such that* $D_A u = f$. *Then,* $u \in \mathcal{W}^{1,p}_{per}$.

Proof. From Corollary 7.6, there is no loss of generality in assuming that $q = p$. Let then $\lambda \in \mathbb{C}$ be such that $D_A - \lambda I$ is (simultaneously, by Corollary 7.6) an isomorphism of $\mathcal{W}^p_{per}$ onto $\mathcal{X}^p$ and an isomorphism of $\mathcal{W}^{1,p}_{per}$ onto $\mathcal{X}^{1,p}_{per}$. Since $D_A u = f$ amounts to $(D_A - \lambda I)u = f - \lambda u \in \mathcal{X}^{1,p}_{per}$ (recall $\mathcal{W}^p_{per} \subset \mathcal{X}^{1,p}_{per}$) and since the equation $(D_A - \lambda I)v = f - \lambda u$ has a unique solution in $\mathcal{W}^p_{per}$ and in $\mathcal{W}^{1,p}_{per} \subset \mathcal{W}^p_{per}$, it follows that this solution is the same in both spaces and thus coincides with u. ∎

We do not know whether Corollary 7.7 is still true when $\sigma(D_A) = \mathbb{C}$ *and* A is not better than C^1 (if also $A \in W^{2,\infty}(0, 2\pi; \mathcal{L}(W, X))$, this is settled in Lemma 7.1).

By using Corollary 7.7, we obtain in turn a refinement of Corollary 7.6:

Corollary 7.8. *Suppose that* (H1) *and* (H3) *hold and that* $A \in C^1_{per}([0, 2\pi], \mathcal{L}(W, X))$. *If* $\sigma(D_A) \neq \mathbb{C}$, *every* $\lambda \in \sigma(D_A)$ *is an isolated eigenvalue of finite multiplicity of* $D_A : \mathcal{W}^p_{per} \to \mathcal{X}^p$ *and of* $D_A : \mathcal{W}^{1,p}_{per} \to \mathcal{X}^{1,p}_{per}$. *Furthermore,* $\ker(D_A - \lambda I)$ *and the multiplicity of* λ *are the same in both cases. (By* Theorem 5.2, *this multiplicity is also independent of* $p \in (1, \infty)$.*)*

Proof. In this proof, it will be convenient to use once again the notation (7.6). If $\sigma(D_A) \neq \mathbb{C}$, it follows from Corollary 7.6 and from Corollary 3.7 and its analog in the $\mathcal{W}^{1,p}_{per}$ - $\mathcal{X}^{1,p}_{per}$ setting (see Section 6) that if $\lambda \in \sigma(D_A)$, then λ is an isolated eigenvalue of finite multiplicity of D^0_A and of D^1_A.

Since it is clear that $\ker(D_A^1 - \lambda I) \subset \ker(D_A^0 - \lambda I)$, it suffices to prove that the converse is true. Let $u \in \ker(D_A^0 - \lambda I) \subset \mathcal{W}_{per}^p$, so that $D_A u = \lambda u \in \mathcal{W}_{per}^p \subset \mathcal{X}_{per}^{1,p}$. Since $\sigma(D_A) \neq \mathbb{C}$, Corollary 7.7 ensures that $u \in \mathcal{W}_{per}^{1,p}$ and so $u \in \ker(D_A^1 - \lambda I)$. Thus, $\ker(D_A^0 - \lambda I) = \ker(D_A^1 - \lambda I)$.

Now, call m^0 and m^1 the (finite) multiplicities of λ as an eigenvalue of D_A^0 and D_A^1, respectively. Then, $m^0 = \dim P^0(\mathcal{X}^p)$ and $m^1 = \dim P^1(\mathcal{X}_{per}^{1,p})$ where for $\ell = 0, 1$,

$$P^\ell := -\frac{1}{2\pi i}\int_\Gamma R(\zeta, D_A^\ell)d\zeta$$

and Γ is a small circle around λ contained in $\rho(D_A^0) = \rho(D_A^1)$. By the denseness of $\mathcal{X}_{per}^{1,p}$ in $\mathcal{X}_{per}^p$ and the finite dimensionality of $\dim P^0(\mathcal{X}^p)$, it follows that $P^0(\mathcal{X}^p) = P^0(\mathcal{X}_{per}^{1,p})$. But $P^0(\mathcal{X}_{per}^{1,p}) = P^1(\mathcal{X}_{per}^{1,p})$ since $R(\zeta, D_A^0) = R(\zeta, D_A^1)$ on $\mathcal{X}_{per}^{1,p}$ by Corollary 7.7, so that $m^0 = m^1$. ∎

8. The (W, X)-independence of the index and spectrum

In this section, $\widetilde{X}$ and $\widetilde{W}$ denote new Banach spaces such that

$$\widetilde{W} \hookrightarrow W \hookrightarrow \widetilde{X} \hookrightarrow X. \tag{8.1}$$

A typical example of (8.1) arises when Ω is a bounded open subset of $\mathbb{R}^N$ and $X := L^q(\Omega), W := W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ for some $q \in (1, \infty)$ and $\widetilde{X} := L^{\tilde{q}}(\Omega), \widetilde{W} := W^{2,\tilde{q}}(\Omega) \cap W_0^{1,\tilde{q}}(\Omega)$ and either $\tilde{q} > q \geq \frac{N}{2}$ or $q < \frac{N}{2}$ and $q < \tilde{q} \leq \frac{Nq}{N-2q}$.

We also assume that (just like X) $\widetilde{X}$ has the UMD property and denote by ($\tilde{\text{H}}$1), ($\tilde{\text{H}}$2) and ($\tilde{\text{H}}$3) the hypotheses (H1), (H2) and (H3) when X and W are replaced by $\widetilde{X}$ and $\widetilde{W}$, respectively. Naturally, these hypotheses make sense only when $A(t) \in \mathcal{L}(W, X) \cap \mathcal{L}(\widetilde{W}, \widetilde{X})$ for every $t \in [0, 2\pi]$, which is implicitly assumed in the sequel. Likewise, the spaces $\widetilde{\mathcal{W}}_{per}^p$ and $\widetilde{\mathcal{X}}^p$ refer to the spaces $\mathcal{W}_{per}^p$ and $\mathcal{X}^p$, respectively, after the same substitution is performed. It is our goal here to show that, under reasonable compatibility conditions, $\operatorname{index} D_A$ and $\sigma(D_A)$ are unchanged upon replacing X and W by $\widetilde{X}$ and $\widetilde{W}$, respectively.

Lemma 8.1. *Suppose that* (H1) *to* (H3) *and* ($\tilde{\text{H}}$1) *to* ($\tilde{\text{H}}$3) *hold. Suppose also that* $A \in C_{per}^1([0, 2\pi], \mathcal{L}(W, X)) \cap W^{2,\infty}(0, 2\pi; \mathcal{L}(W, X))$. *Then, given* $p \in (1, \infty)$, *the operator* D_A *is Fredholm with the same null-space, the same index and the same spectrum when acting from* $\mathcal{W}_{per}^p$ *to* $\mathcal{X}^p$ *and when acting from* $\widetilde{\mathcal{W}}_{per}^p$ *to* $\widetilde{\mathcal{X}}^p$.

Proof. In this proof, it is convenient to agree that D_A acts only from $\mathcal{W}_{per}^p$ to $\mathcal{X}^p$ and to use the notation $\widetilde{D}_A$ when the action is from $\widetilde{\mathcal{W}}_{per}^p$ to $\widetilde{\mathcal{X}}^p$.

By (8.1), it is obvious that $\widetilde{\mathcal{W}}_{per}^p \subset \mathcal{W}_{per}^p$, so that $\ker \widetilde{D}_A \subset \ker D_A$. To prove the equality of the null-spaces, it remains to show that if $u \in \ker D_A$, then $u \in \widetilde{\mathcal{W}}_{per}^p$. By Lemma 7.2, $u \in \mathcal{W}_{per}^{1,p}$. In particular, $u \in W_{per}^{1,p}(0, 2\pi; W) \subset W_{per}^{1,p}(0, 2\pi; \widetilde{X})$ by (8.1). Thus, $u, \dot{u} \in L^p(0, 2\pi; \widetilde{X})$ and so $\dot{u} - iku \in L^p(0, 2\pi; \widetilde{X})$ for every $k \in \mathbb{Z}$. On the other hand, it easily follows from ($\tilde{\text{H}}$2) and ($\tilde{\text{H}}$3) that, if $|k|$ is large enough, then $A(t) - ikI \in \mathcal{L}(\widetilde{W}, \widetilde{X})$ is invertible for every $t \in [0, 2\pi]$. Since $D_A u = 0$ entails $\dot{u} - iku = (A - ikI)u$, it follows that $u = (A - ikI)^{-1}(\dot{u} - iku) \in L^p(0, 2\pi; \widetilde{W})$. Thus, in summary, $u \in W_{per}^{1,p}(0, 2\pi; \widetilde{X}) \cap L^p(0, 2\pi; \widetilde{W}) = \widetilde{\mathcal{W}}_{per}^p$.

Next, we prove that $\operatorname{codim}\operatorname{rge} D_A = \operatorname{codim}\operatorname{rge}\widetilde{D}_A$. To see this, let $Z \subset X$ be a finite dimensional subspace such that $\operatorname{rge} D_A \cap Z = \{0\}$. Since $\operatorname{rge} D_A$ is closed in $\mathcal{X}^p$ and since $C_0^\infty(0,2\pi)\otimes\widetilde{X}$ is dense in $\mathcal{X}^p$ by the denseness of $\widetilde{X}$ in X (by (8.1) and the denseness of W in X; see Lemma 2.5), it is not restrictive to assume -without changing $\dim Z$- that $Z \subset C_0^\infty(0,2\pi)\otimes\widetilde{X} \subset \widetilde{\mathcal{X}}^p$. But then, since $\operatorname{rge}\widetilde{D}_A \subset \operatorname{rge} D_A$, it is obvious that $\operatorname{rge}\widetilde{D}_A \cap Z = \{0\}$. This shows that $\operatorname{codim}\operatorname{rge}\widetilde{D}_A \geq \operatorname{codim}\operatorname{rge} D_A$.

To prove the reverse inequality, let $\widetilde{Z} \subset \widetilde{\mathcal{X}}^p$ be a finite dimensional space such that $\operatorname{rge}\widetilde{D}_A \cap \widetilde{Z} = \{0\}$. Since $\operatorname{rge}\widetilde{D}_A$ is closed in $\widetilde{\mathcal{X}}^p$ (because $\widetilde{D}_A$ is Fredholm by Theorem 4.1) and $C_0^\infty(0,2\pi)\otimes\widetilde{X}$ is dense in $\widetilde{\mathcal{X}}^p$, it is once again not restrictive to assume -without changing $\dim\widetilde{Z}$- that $\widetilde{Z} \subset C_0^\infty(0,2\pi)\otimes\widetilde{X} \subset C_0^\infty(0,2\pi)\otimes X \subset \mathcal{X}^{1,p}_{per}$. Therefore, it follows from Lemma 7.1 that if $u \in \mathcal{W}^p_{per}$ and $D_A u = \widetilde{f} \in \widetilde{Z}$, then $u \in \mathcal{W}^{1,p}_{per}$. In particular, $u \in W^{1,p}_{per}(0,2\pi;W) \subset W^{1,p}_{per}(0,2\pi;\widetilde{X})$ by (8.1) and $\dot{u} - iku - \widetilde{f} \in L^p(0,2\pi;\widetilde{X})$ for every $k \in \mathbb{Z}$. By choosing k as in the first part of the proof and rewriting $D_A u = \widetilde{f}$ as $\dot{u} - iku - \widetilde{f} = (A - ikI)u$, we obtain $u = (A - ikI)^{-1}(\dot{u} - iku - \widetilde{f}) \in L^p(0,2\pi;\widetilde{W})$. Thus, $u \in \widetilde{\mathcal{W}}^p_{per}$, so that $\widetilde{f} \in \widetilde{Z} \cap \operatorname{rge}\widetilde{D}_A = \{0\}$. This shows that $\operatorname{rge} D_A \cap \widetilde{Z} = \{0\}$ and, hence, that $\operatorname{codim}\operatorname{rge} D_A \geq \operatorname{codim}\operatorname{rge}\widetilde{D}_A$. This completes the proof of the equality $\operatorname{codim}\operatorname{rge} D_A = \operatorname{codim}\operatorname{rge}\widetilde{D}_A$.

Together with the relation $\ker D_A = \ker\widetilde{D}_A$, this proves that $\operatorname{index}\widetilde{D}_A = \operatorname{index} D_A$ and that $\widetilde{D}_A$ and D_A are simultaneously invertible. Upon replacing A by $A - \lambda I$ in the latter property, we find that $\widetilde{D}_A$ and D_A have the same spectrum. ∎

We now remove the extra smoothness requirements about A in Lemma 8.1 .

Theorem 8.2. *Suppose that* (H1) *to* (H3) *and* ($\tilde{\text{H}}$1) *to* ($\tilde{\text{H}}$3) *hold. Then, given* $p \in (1,\infty)$, *the operator* D_A *is Fredholm when acting from* $\mathcal{W}^p_{per}$ *to* $\mathcal{X}^p$ *and when acting from* $\widetilde{\mathcal{W}}^p_{per}$ *to* $\widetilde{\mathcal{X}}^p$. *Furthermore,* $\ker D_A, \operatorname{index} D_A$ *and* $\sigma(D_A)$ *are the same in both cases. (By* Theorem 5.2, $\ker D_A$, $\operatorname{index} D_A$ *and* $\sigma(D_A)$ *are also independent of p.)*

Proof. After extending A by periodicity and convolving with a sequence of mollifiers, we obtain a sequence $A_n \in C^\infty_{per}([0,2\pi],\mathcal{L}(W,X)) \cap C^\infty_{per}([0,2\pi],\mathcal{L}(\widetilde{W},\widetilde{X}))$ such that $A_n \to A$ in $C^0_{per}([0,2\pi],\mathcal{L}(W,X))$ and in $C^0_{per}([0,2\pi],\mathcal{L}(\widetilde{W},\widetilde{X}))$. Note that this implies that $D_{A_n} \to D_A$ in both $\mathcal{L}(W,X)$ and $\mathcal{L}(\widetilde{W},\widetilde{X})$.

By Lemma 2.4, A_n also satisfies (H1) to (H3) and ($\tilde{\text{H}}$1) to ($\tilde{\text{H}}$3) for n large enough. Thus, for such indices n, Lemma 8.1 ensures that[8] $\widetilde{D}_{A_n}$ and D_{A_n} have the same index.

By Theorem 4.1, $\widetilde{D}_A$ and D_A are Fredholm , and $\operatorname{index}\widetilde{D}_A = \operatorname{index}\widetilde{D}_{A_n}$, $\operatorname{index} D_A = \operatorname{index} D_{A_n}$ for n large enough by the local constancy of the index. This shows that $\operatorname{index}\widetilde{D}_A = \operatorname{index} D_A$.

The equality of the indexes means that

$$\dim\ker\widetilde{D}_A - \operatorname{codim}\operatorname{rge}\widetilde{D}_A = \dim\ker D_A - \operatorname{codim}\operatorname{rge} D_A. \tag{8.2}$$

Now, the arguments in the proof of Lemma 8.1 showing that $\ker\widetilde{D}_A \subset \ker D_A$ (hence $\dim\ker\widetilde{D}_A \leq \dim\ker D_A$) and that $\operatorname{codim}\operatorname{rge}\widetilde{D}_A \geq \operatorname{codim}\operatorname{rge} D_A$ do not

[8] Here and in what follows, we use the notation introduced in the proof of Lemma 8.1.

require any smoothness of A and therefore remain valid under the weaker assumptions of this theorem. Therefore, since the index is finite, (8.2) shows that

$$\dim\ker \widetilde{D}_A = \dim\ker D_A \text{ and } \operatorname{codim}\operatorname{rge}\widetilde{D}_A = \operatorname{codim}\operatorname{rge} D_A. \tag{8.3}$$

. In particular, $\ker \widetilde{D}_A = \ker D_A$ and $\widetilde{D}_A$ and D_A are simultaneously isomorphisms. By replacing A by $A - \lambda I$ in this statement (which is legitimate by ($\tilde{\text{H}}$1)/(H1) and Lemma 2.3), it follows that the spectra of $\widetilde{D}_A$ and D_A coincide. ∎

A simpler proof of Theorem 8.2, independent of Lemma 8.1, can be given under the additional assumption that D_A has compact resolvent as a closed unbounded operator on $\mathcal{X}^p$ with domain $\mathcal{W}^p_{per}$ *and* as a closed unbounded operator on $\widetilde{\mathcal{X}}^p$ with domain $\widetilde{\mathcal{W}}^p_{per}$. But these are crucial extra hypotheses which, up to this point of our exposition, are only known to be true in the t-independent case (Theorem 4.3), even though this limitation will be substantially reduced in the next section. Nonetheless, part of the value of Theorem 8.2 is that it can be used to *prove* the compact resolvent property with some pair $(\widetilde{W}, \widetilde{X})$ after the same property has been established for another pair (W, X).

Remark 8.1. *In* Theorem 8.2*, the equality of the spectra implies that if D_A -or equivalently $\widetilde{D}_A$- is invertible and if $\widetilde{f} \in \widetilde{\mathcal{X}}^p$ and $u \in \mathcal{W}^p_{per}$ are such that $D_A u = \widetilde{f}$, then $u \in \widetilde{\mathcal{W}}^p_{per}$. This follows at once from $\widetilde{\mathcal{X}}^p \subset \mathcal{X}^p$, $\widetilde{\mathcal{W}}^p_{per} \subset \mathcal{W}^p_{per}$ and the uniqueness of a solution in both $\widetilde{\mathcal{W}}^p_{per}$ and $\mathcal{W}^p_{per}$.*

Corollary 8.3. *Suppose that* (H1) *to* (H3) *and* ($\tilde{\text{H}}$1) *to* ($\tilde{\text{H}}$3) *hold and let $p \in (1, \infty)$. If $\sigma(D_A) \neq \mathbb{C}$, every $\lambda \in \sigma(D_A)$ is an eigenvalue of finite multiplicity of D_A when D_A is viewed as a closed unbounded operator on $\mathcal{X}^p$ with domain $\mathcal{W}^p_{per}$, or as a closed unbounded operator on $\widetilde{\mathcal{X}}^p$ with domain $\widetilde{\mathcal{W}}^p_{per}$. Moreover, the multiplicity of λ is the same in both cases. (By* Theorem 5.2*, this multiplicity is also independent of p.)*

Proof. We continue to use the notation introduced in the proof of Lemma 8.1. Since the (finite) indexes and spectra of $\widetilde{D}_A$ and D_A are the same by Theorem 8.2, it follows from Corollary 3.7 and the assumption $\sigma(D_A) \neq \mathbb{C}$ that $\lambda \in \sigma(D_A)$ is an eigenvalue of $\widetilde{D}_A$ and of D_A of finite multiplicity. It remains to show that this multiplicity is the same in both cases.

Call $\widetilde{m}$ and m the corresponding multiplicities of λ, so that $\widetilde{m} = \dim \widetilde{P}(\widetilde{\mathcal{X}}^p)$ and $m = \dim P(\mathcal{X}^p)$ and

$$\widetilde{P} := -\frac{1}{2\pi i}\int_\Gamma R(\zeta, \widetilde{D}_A)d\zeta \text{ and } P := -\frac{1}{2\pi i}\int_\Gamma R(\zeta, D_A)d\zeta,$$

where Γ is a small circle around λ contained in $\rho(\widetilde{D}_A) = \rho(D_A)$. By the finite dimensionality of $P(\mathcal{X}^p)$ and the denseness of $\widetilde{\mathcal{X}}^p$ in $\mathcal{X}^p$ (by Lemma 2.5 and (8.1)), it follows that $P(\mathcal{X}^p) = P(\widetilde{\mathcal{X}}^p)$. But $P(\widetilde{\mathcal{X}}^p) = \widetilde{P}(\widetilde{\mathcal{X}}^p)$ since $R(\zeta, \widetilde{D}_A)$ and $R(\zeta, D_A)$ coincide on $\widetilde{\mathcal{X}}^p$ by Remark 8.1 for $A - \zeta I$. This shows that $\widetilde{m} = m$. ∎

By combining the above with Sections 6 and 7 (with (W, X) replaced by $(\widetilde{W}, \widetilde{X})$), it is straightforward to obtain corresponding theorems in the spaces $\widetilde{\mathcal{W}}^{1,p}_{per}$ and $\widetilde{\mathcal{X}}^{1,p}$.

It is also worth pointing out that the results of this section can be extended to spaces $\widehat{\mathcal{W}}^p_{per}$ and $\widehat{\mathcal{X}}^p$ associated with spaces $\widehat{X}$ and $\widehat{W}$ such that

$$\widehat{W} \hookrightarrow \widetilde{W} \hookrightarrow \widehat{X} \hookrightarrow \widetilde{X}. \tag{8.4}$$

The point here is that (8.1) and (8.4) do not imply $\widehat{W} \hookrightarrow W \hookrightarrow \widehat{X} \hookrightarrow X$ (the example of Sobolev spaces given earlier shows that the second embedding may fail). If so, the results of this section may still be true when $(\widetilde{W}, \widetilde{X})$ is replaced by $(\widehat{W}, \widehat{X})$, even though the proofs requires two consecutive applications of the theorems. Of course, even more general results follow by using the theorems any finite number of times (abstract "bootstrapping").

9. Isomorphism theorems

For the definition of an r-sectorial operator and related concepts (r-angle) used below, see Section 5. If A_0 is an r-sectorial operator on X with domain W and r-angle $\phi^r_{A_0} < \frac{\pi}{2}$, it is by now well-known that the Cauchy problem

$$\begin{cases} D_{A_0} u = f, \\ u(0) = 0, \end{cases}$$

has a unique solution $u \in \mathcal{W}^p := W^{1,p}(0, 2\pi; X) \cap L^p(0, 2\pi; W)$ for every $f \in \mathcal{X}^p = L^p(0, 2\pi; X)$ and every $p \in (1, \infty)$. For instance, this follows at once from [11, Theorem 4.4], where $(0, 2\pi)$ is replaced by $(0, \infty)$. Since the multiplication by $e^{\mu t}$ is an isomorphism of both the spaces X^p and W^p, it follows that the same uniqueness property holds when it is only assumed that $A_0 + \mu I$ is r-sectorial with r-angle $\phi^r_{A_0+\mu I} < \frac{\pi}{2}$ for some $\mu \geq 0$. This will be used in the proof of Lemma 9.1.

Now, let

$$\mathcal{T}^p := \{u(0) : u \in \mathcal{W}^p\} \subset X,$$

denote the space of traces of elements of $\mathcal{W}^p$. It is readily checked that $\mathcal{T}^p$ is a Banach space for the norm

$$||x||_{\mathcal{T}^p} := \inf_{u \in \mathcal{W}^p, u(0)=x} ||u||_{\mathcal{W}^p}.$$

For future use, note that $u(2\pi) \in \mathcal{T}^p$ for every $u \in \mathcal{W}^p$ since $v(t) := u(2\pi - t)$ is also in $\mathcal{W}^p$.

Lemma 9.1. *Suppose that* (H2) *holds and that, for every $t \in [0, 2\pi]$, there is $\mu_t \geq 0$ such that $A(t) + \mu_t I$ is r-sectorial with r-angle $\phi^r_{A(t)+\mu_t I} < \frac{\pi}{2}$. Then, for every $p \in (1, \infty)$, there is a constant $C_p > 0$ such that $||u(2\pi)||_{\mathcal{T}^p} \leq C_p ||u(0)||_{\mathcal{T}^p}$ for every $u \in \mathcal{W}^p$ such that $D_A u = 0$.*

Proof. Obviously, $\mathcal{K}^p := \{u \in \mathcal{W}^p : D_A u = 0\}$ is a closed subspace of $\mathcal{W}^p$ and the mapping $u \in \mathcal{K}^p \mapsto u(0) \in \mathcal{T}^p$ is linear and continuous. We claim that it is in fact bijective, which in turn follows from the existence and uniqueness of a solution $u \in \mathcal{W}^p$ of the Cauchy problem

$$\begin{cases} D_A u = 0 \text{ in } (0, 2\pi), \\ u(0) = x \in \mathcal{T}^p. \end{cases}$$

This is a special case of Arendt *et al.* [4, Theorem 2.7]. Both (H2) and the preliminary discussion at the beginning of this section with $A_0 = A(t)$ and $t \in [0, 2\pi]$ are relevant to the applicability of this result. The fact that $\mathcal{T}^p$ coincides with the real interpolation space $(X, W)_{\frac{1}{p'}, p}$ (Lunardi [21, Chapter 1]) must also be used.

Therefore, by the open mapping theorem, there is a constant $C_p > 0$ such that $||u||_{\mathcal{W}^p} \leq C_p||u(0)||_{\mathcal{T}^p}$ for every $u \in \mathcal{K}^p$. Since the mapping $u \in \mathcal{W}^p \mapsto u(2\pi) \in \mathcal{T}^p$ is continuous with norm 1, the inequality $||u(2\pi)||_{\mathcal{T}^p} \leq C_p||u(0)||_{\mathcal{T}^p}$ follows. ∎

Theorem 9.2. *Suppose that* (H1) *and* (H2) *hold and that, for every* $t \in [0, 2\pi]$, *there is* $\mu_t \geq 0$ *such that* $A(t) + \mu_t I$ *is r-sectorial with r-angle* $\phi^r_{A(t)+\mu_t I} < \frac{\pi}{2}$. *Then, there is* $c > 0$ *such that* $D_{\pm(A-\lambda I)}$ *is an isomorphism of* $\mathcal{W}^p_{per}$ *onto* $\mathcal{X}^p$ *for every* $p \in (1, \infty)$ *if* $\operatorname{Re}\lambda > c$. *In particular,* $D_{\pm A}$ *has compact resolvent and (hence) index* 0.

Proof. By Corollary 5.3, D_A is Fredholm, so that $D_{A-\lambda I} = D_A + \lambda I$ is Fredholm for every $\lambda \in \mathbb{C}$. In a first step, we show that $D_{A-\lambda I}$ is one to one if $\operatorname{Re}\lambda$ is large enough.

To see this, let $u \in \mathcal{W}^p_{per}$ be such that $D_A u + \lambda u = 0$ and set $v(t) := e^{\lambda t}u(t)$. Then, $v \in \mathcal{W}^p$ and $D_A v = 0$ (obviously, v is not periodic), so that $||v(2\pi)||_{\mathcal{T}^p} \leq C_p||v(0)||_{\mathcal{T}^p}$ by Lemma 9.1. But $v(2\pi) = e^{2\lambda\pi}u(2\pi) = e^{2\lambda\pi}u(0) = e^{2\lambda\pi}v(0)$ by the periodicity of u. Hence, $e^{2\operatorname{Re}\lambda}||v(0)||_{\mathcal{T}^p} \leq C_p||v(0)||_{\mathcal{T}^p}$, so that $v(0) = 0$ if $2\operatorname{Re}\lambda > \ln C_p$. If so, $v = 0$ by [4, Theorem 2.7], already used in the proof of Lemma 9.1, whence $u = 0$.

Next, the hypotheses of the theorem are unchanged after replacing $A(t)$ by $\check{A}(t) := A(2\pi - t)$, so that $D_{\check{A}-\lambda I}$ is one to one if $\operatorname{Re}\lambda$ is large enough. Since the change of variable $t \mapsto 2\pi - t$ induces isomorphisms of $\mathcal{W}^p_{per}$ and $\mathcal{X}^p$, it follows that $D_{-A+\lambda I}$ is also one to one if $\operatorname{Re}\lambda$ is large enough.

We now claim that the hypotheses of the theorem are also unchanged upon replacing A by A^* and exchanging the roles of X and W^* and of W and X^*, respectively. In Section 4, we already noticed that (H1) and (H2) (and even (H3)) are unchanged. It remains only to check that the r-sectoriality condition still holds. For simplicity of notation, we assume $\mu_t = 0$, which merely amounts to replacing $A(t)$ by $A(t) + \mu_t I$ everywhere.

First, by the argument of Lemma 2.2, the r-boundedness of $\{\zeta R(-\zeta, A(t)) : |\arg\zeta| \leq \theta\}$ in $\mathcal{L}(X)$ for some $\theta \in (0, \pi)$ is equivalent to the r-boundedness of $\{R(-\zeta, A(t)) : |\arg\zeta| \leq \theta\}$ in $\mathcal{L}(W, X)$. That $A(t)$ is invertible (so that no problem arises for ζ near 0) is important for this point and is part of the sectoriality assumption. In turn, by [25, Lemma 2.3 and Remark 3.1], this implies that $\{R(-\zeta, A(t))^* : |\arg\zeta| \leq \theta\} = \{R(-\zeta, A^*(t)) : |\arg\zeta| \leq \theta\}$ is r-bounded in $\mathcal{L}(X^*, W^*)$. By the equivalence noted above, this amounts to saying that $\{\zeta R(-\zeta, A^*(t)) : |\arg\zeta| \leq \theta\}$ is r-bounded in $\mathcal{L}(W^*)$, so that indeed $A(t)$ and $A^*(t)$ have the same r-angle. This completes the verification of the hypotheses of the theorem for A^*. Accordingly, from the above, $D_{-A^*+\lambda I}$ is one to one if $\operatorname{Re}\lambda$ is large enough. In this statement, $D_{-A^*+\lambda I}$ acts between the spaces $\mathcal{X}^q_{*per}$ and $\mathcal{W}^q_*$ for any $q \in (1, \infty)$; see (4.1) and the comments following (4.5).

At this stage, observe that the formula (4.5) for the index yields

$$\operatorname{index} D_{A-\lambda I} = \dim\ker D_{A-\lambda I} - \dim\ker D_{-A^*+\lambda I}.$$

Thus, if $\operatorname{Re}\lambda$ is large enough, $D_{A-\lambda I}$ is one to one with index 0 and hence an isomorphism. Note that in the above arguments, "large enough" depends upon p. However, Theorem 5.2 (p-independence of $\sigma(D_A)$) shows that this is not the case.

That D_A has compact resolvent now follows from Corollary 3.7 and the corresponding properties for D_{-A} are obtained by first changing A into $\check{A}$ ant then t into $2\pi - t$, as was done earlier in the proof. ∎

Under an additional condition, we obtain an isomorphism theorem for $D_{\pm A}$:

Corollary 9.3. *In* Theorem 9.2*, assume also that $A(t)+\varepsilon(t)I$ is dissipative for a.e. $t\in[0,2\pi]$, where $\varepsilon\in L^q(0,2\pi)$ for some $q>1$ and $\int_0^{2\pi}\varepsilon(s)ds>0$. Then, $D_{\pm A}$ is an isomorphism of $\mathcal{W}^p_{per}$ onto $\mathcal{X}^p$ for every $p\in(1,\infty)$.*

Proof. By Theorem 9.2, D_A has index 0, so that it suffices to prove that D_A is one to one and, by the p-independence of $\sigma(D_A)$ (Theorem 5.2), it suffices to consider the case $p=q$.

Let $u\in\mathcal{W}^q_{per}$ be such that $D_Au=0$ and set $v(t):=e^{\int_0^t\varepsilon(s)ds}u(t)$. Then $v\in\mathcal{W}^q$ and $D_{(A+\varepsilon I)}v=0$. Now, since $A(t)+\varepsilon(t)I$ is dissipative, it follows from[9] [4, Proposition 3.2] that $||v(\cdot)||_X$ is nonincreasing. In particular, $||v(2\pi)||_X=e^{\int_0^{2\pi}\varepsilon(s)ds}||u(2\pi)||_X\leq||v(0)||_X=||u(0)||_X$. Since $u(2\pi)=u(0)$, this shows that $e^{\int_0^{2\pi}\varepsilon(s)ds}||u(0)||_X\leq||u(0)||_X$, whence $u(0)=0$ since $e^{\int_0^{2\pi}\varepsilon(s)ds}>1$. Thus, $u=0$ follows once again from [4, Theorem 2.7].

The analogous result for D_{-A} follows by first replacing A and ε by $\check{A}(t):=A(2\pi-t)$ and $\check{\varepsilon}(t):=\varepsilon(2\pi-t)$, respectively and next changing t into $2\pi-t$. ∎

Remark 9.1. *When* $\dim X<\infty$*, it is an easy by-product of Floquet's theory that $\sigma(D_A)\neq\mathbb{C}$ and even that $D_{\pm A}$ has compact resolvent. To put* Corollary 9.3 *in perspective, note that this also follows from part (ii) of that corollary, whose assumptions are always satisfied in the finite dimensional case.*

By using Section 7, the results of this section yield similar properties in the $\mathcal{W}^{1,p}_{per}$ - $\mathcal{X}^{1,p}_{per}$ setting under suitable smoothness and periodicity assumptions about A.

Also, if $\widetilde{X}$ and $\widetilde{W}$ are Banach spaces satisfying (8.1) and if, in addition, ($\tilde{\text{H}}$1) to ($\tilde{\text{H}}$3) (i. e., (H1) to (H3) for $\widetilde{X}$ and $\widetilde{W}$) hold, it follows from Theorem 8.2 that the results of this section remain valid with the spaces $\widetilde{\mathcal{W}}^p_{per}$ and $\widetilde{\mathcal{X}}^p$ obtained by replacing W and X by $\widetilde{W}$ and $\widetilde{X}$, respectively, in the definitions of $\mathcal{W}^p_{per}$ and $\mathcal{X}^p$. In practice, it is important to notice that the r-sectoriality or the dissipativity (in Corollary 9.3) needs to be retained in *either* (i.e. (W,X) or $(\widetilde{W},\widetilde{X})$) setting, but is not needed in both.

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