Multisummation and its application to integrability of Hamiltonian systems

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While the power series for $\check g(z,x)$ may diverge for every $x \neq 0$, we require the coefficients $g_p(z)$ to be holomorphic functions in a fixed disc $\mathcal{D}_\rho$ of radius $\rho > 0$ about the origin.

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The formal series $\hat{F}(z)$ is known to be multi-summable in every non-singular multi-direction $d = (d_1, \ldots, d_r)$. Its corresponding sum shall be denoted as $F(z; d)$, and we set

$$X(z; d) \;:=\; F(z; d)\,z^L\,e^{Q(z)}\,.$$

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Every system of the form considered here has a semi-formal solution of the form

$$\check{x}(z,c) \;=\; \sum_{|p|\geq 1} x_p(z)\, c^p \;=\; X(z)\, c \;+\; \sum_{|p|\geq 2} x_p(z)\, c^p\,,$$

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where $X(z)$ is a fundamental solution of the corresponding linear system, and for every multi-index $p \neq 0$

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Every other semi-formal solution is of the form $\check{x}(z, v(c))$, with suitable

$$v(c) \;=\; \sum_{|p|\geq 1} v_p\, c^p \;=\; V\, c \;+\; \sum_{|p|\geq 2} v_p\, c^p\,.$$

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$$q(z,p) \;=\; \sum_{j=1}^{\nu} p_j \, q_j(z) \;, \qquad \lambda(p) \;=\; r \;+\; \sum_{j=1}^{\nu} p_j \, (\lambda_j - r) \;.$$

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Theorem 1. *For every non-singular multi-direction d there exists a unique semi-formal solution*

$$\check{x}(z,c;d) \;=\; \sum_{|p|\ge 1} x_p(z;d)\, c^p$$

whose linear part is the normal solution $X(z;d)$ of the corresponding linear equation. The coefficients $x_p(z;d)$ all are logarithmic-exponential expressions of type $(q(z,p),\lambda(p))$ which are recursively obtained by means of the identity

$$x_p(z;d) \;=\; X(z;d)\int X^{-1}(z;d)\Big[\sum_{2\le|q|\le|p|} x_{qp}(z;d)\, g_q(z)\Big]\frac{dz}{z^{r+1}}\;.$$

Stokes functions

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Given two non-singular multi-directions d and $\tilde{d}$, there exists a unique formal expression $\check{v}(c;\tilde{d},d) = \sum_p v_p(\tilde{d},d)\, c^p$ for which

$$\check{x}(z,c;d) \;=\; \check{x}(z,\check{v}(c;\tilde{d},d);\tilde{d})\,.$$

This $\check{v}(c;\tilde{d},d)$ will be referred to as the (formal) *Stokes function* corresponding to d and $\tilde{d}$.

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Theorem 2. *For any normalized system, all Stokes functions are invertible with respect to formal composition, and for any three non-singular multi-directions* d, $\tilde{d}$, $\hat{d}$ *we have* $v(c;d,d) = c$, *i. e. is the identity element of* $\mathbb{G}(c)$, *while* $\check{v}(c;\tilde{d},d)$ *and* $\check{v}(c;d,\tilde{d})$ *are inverse to one another. Moreover,*

$$\check{v}(c;\hat{d},d) \;=\; \check{v}(\check{v}(c;\tilde{d},d);\hat{d},\tilde{d})\,.$$

Altogether, this shows that the the Stokes functions form a subgroup of the group $\mathbb{G}(c)$.

A Hamiltonian system

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In a joint article by *B. and Yoshino*, and a later one by *Yoshino*, a Hamiltonian system has been investigated, corresponding to the Hamilton function

$$\mathcal{H} = -q_2 p_2 \partial_{q_1} r + \left(r^2 + q_2 \partial_{q_2} r \right) p_1,$$

with

$$r \equiv r(q_1, q_2, p_1, p_2) = q_1^{2\sigma} + a(q_1^{2\sigma}) q_2^2 + \tilde{r}(q_1, q_2, p_1, p_2) q_2^3.$$

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Under some additional assumption, it has been shown that this system is not analytically integrable at the origin. It has, however, a first integral in form of a transseries, which is closely related to the semiformal normal solutions discussed before!

Literature

For a list of literature, refer to my abstract!