Okubo's hypergeometric system

Werner Balser http://cantor.mathematik.uni-ulm.de/m5/index.php?file=balser/index.htmlt

Institut für Angewandte Analysis, Universität Ulm

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$$\Lambda' \;=\; \operatorname{diag}[\lambda'_1, \ldots, \lambda'_n]\,, \qquad A \;=\; \begin{bmatrix} 0 & a_{12} & \ldots & a_{1n} \\ a_{21} & 0 & \ldots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & 0 \end{bmatrix}$$

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Are the solutions of the hypergeometric system, or the entries in their Stokes multipliers, special functions in the weak sense? Reinhard and myself believe so!

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For fixed Λ and Λ', V is an entire function of the entries in A, hence is an entire map from the $n(n-1)$-dimensional complex vector space into itself. At “most” points, this map is locally injective.

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Question: Are the functions $v_{jk}(\Lambda, \Lambda', A)$ interrelated, or are they independent?

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Proof: For every pair (j, k) with $j \neq k$, $1 \leq j, k \leq n$, we can find a permutation matrix P so that

$$v_{jk}(\Lambda, \Lambda', A) \;=\; v_{21}(P^{-1}\Lambda\, P, P^{-1}\Lambda'\, P, P^{-1}A\, P)\,.$$

The entry $v(\Lambda, \Lambda', A) := v_{21}(\Lambda, \Lambda', A)$ shall here be called *the Stokes function*.

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Prenormalizations: From now on, let $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_1' = 0$.

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This situation can be made to hold by means of some elementary transformations which do not change the Stokes multipliers!

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Assume that the above prenormalizations hold. Let α, β be so that

$$\alpha + \beta \;=\; \lambda_2', \quad \alpha\,\beta \;=\; -a_{12}\,a_{21}\,.$$

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The number γ in this formula is the relevant quantity in the asymptotic of the coefficients of a formal solution of the confluent system.

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Further analysis of the coefficients of this power series is work in progress!

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Open question: Who is going to write this book?