ulm university universität uulm

Manfred Sauter
Manuel Bernhard
winter semester 2013/2014

Exercise Sheet 5

Applied Analysis

Discussion on Thursday 21-11-2013 at 16ct

Exercise 1 (*Banach's fixed point theorem - assumptions*) (1+1+1+1+2)

Can one directly apply Banach's fixed point theorem to $f\colon M \to M$? In other words are the assumptions of Banach's fixed point theorem valid?

(a) $M = \mathbb{R}$ and $f(x) = x$
(b) $M = \mathbb{R}$ and $f(x) = \frac{1}{2}x$
(c) $M = \mathbb{Q}$ and again $f(x) = \frac{1}{2}x$
(d) $M = (0, \infty)$ and $f(x) = \frac{1}{2}x$
(e) $M = [0, 1]$ and $f(x) = (x + 1)^{-1}$

Exercise 2 (*A concrete application of Banach's fixed point theorem*) (2+2)

We are interested in $x, y \in [-1, 1]$ which solve

$$50x = x^2 + y^2 + x + 1$$
$$50y = x^3 + y^2 + y.$$

(a) Prove the existence of a unique solution (use Banach's fixed point theorem).
(b) Find an approximate solution with error $< 10^{-4}$ (for each value) by using the fixed point iteration starting with $x_0 = 0$ and $y_0 = 0$.

Exercise 3 (*Multiple Choice*) (10)

Decide which statements are true. Give counterexamples or proofs.
Most of the questions are repetitions or simple restatements of propositions mentioned in the lecture. This is the reason why each question gives fewer points than last time.

(a) $[0, 1)$ is compact (with the Euclidean metric).
□ true □ false
(b) $[0, 1] \times [-1, 1]$ is compact.
□ true □ false
(c) Given a metric space (M, d) and let $C \subset M$ be compact. Then C is closed.
□ true □ false
(d) Given a metric space (M, d) and let $C \subset M$ be compact. Then C is not open.
□ true □ false
(e) $\mathbb{Q} \times \mathbb{Z}$ is a countable set.
□ true □ false
(f) $\{1, 4, 5, 9\}$ is a countable set.
□ true □ false
(g) $\mathbb{Q} \times \mathbb{R}$ is a countable set.
□ true □ false
(h) The (arbitrary) union of countable sets is countable.
□ true □ false
(i) The countable product of finite sets is countable.
□ true □ false
(j) $\{(x, y) \in \mathbb{R}^2 | x^6 y^4 < 1\}$ is open and not compact.
□ true □ false

please turn over!

(k) Every normed space $(X, \|\cdot\|)$ with $X = \mathbb{R}^n$ is complete.
☐ true ☐ false

(l) A continuous function $f\colon F \to \mathbb{R}$ on a bounded and closed subset F of $\mathbb{R}^n$ attains its infimum and supermum in F.
☐ true ☐ false

(m) A continuous function $f\colon F \to \mathbb{R}$ on a bounded and closed subset F of ℓ^2 attains its infimum and supermum in F.
☐ true ☐ false

(n) Every convergent sequence has only one accumulation point.
☐ true ☐ false

(o) Every sequence with only one accumulation point is convergent.
☐ true ☐ false

(p) Every bounded sequence in $\mathbb{R}^n$ with only one accumulation point is convergent.
☐ true ☐ false

(q) Every bounded sequence (in an arbitrary normed space) with only one accumulation point is convergent.
☐ true ☐ false

(r) If (a_n) converges to zero in $\mathbb{R}$, then $\sum_{k=1}^{\infty} a_k$ is convergent.
☐ true ☐ false

(s) Given a sequence (a_n) in $\mathbb{R}$ such that $\sum_{k=1}^{\infty} a_k$ is convergent. Then (a_n) and $(\sum_{k=n}^{\infty} a_k)$ are convergent to zero.
☐ true ☐ false

(t) $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ is convergent for every $x \in \mathbb{R}$ fixed.
☐ true ☐ false