Exercise Sheet 7

[Applied Analysis]

Discussion on Thursday 5-12-2013 at 16ct

**Exercise 1**

(5+5+5)

(a) Find a non-convergent Cauchy sequence in \((c_{00}, \|\cdot\|_2)\) with the norm

\[
\| (x_k) \|_2 = \sum_{k=1}^{\infty} |x_k|^2
\]

for \((x_k) \in c_{00}\).

(b) Let \((R^N, \|\cdot\|)\) be a normed vector space. If \((x^{(n)})_{n \in N}\) is a sequence in \(R^N\), then the following properties are equivalent:

(i) \((x^{(n)})\) converges in \((R^N, \|\cdot\|)\) to \(x\).

(ii) Every coordinate sequence \((x_k^{(n)})_{n \in N}\) converges to \(x_k\) for \(k = 1, ..., N\).

Prove this equivalence.

(c) Show that the above equivalence is wrong for all the spaces \((\ell^p, \|\cdot\|_p)\) (with \(p \in [1, \infty]\)) by giving counterexamples.

**Exercise 2** *(Continuous functions vanishing at infinity)*

(5+5+5)

We denote by

\[
C_0(\mathbb{R}) = \left\{ f: \mathbb{R} \to \mathbb{R} : \text{is continuous and } \lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 0 \right\}
\]

the space of continuous functions, which vanish at infinity and define

\[
\| f \|_\infty = \sup_{x \in \mathbb{R}} |f(x)|
\]

for every \(f \in C_0(\mathbb{R})\).

(a) Show that \((C_0(\mathbb{R}), \|\cdot\|_\infty)\) is a normed space.

(b) Is this space complete?

(c) Show that \(C_0(\mathbb{R})\) is separable.

**Exercise 3** *(Multiple Choice)*

Which of the following statements are true?

(a) The compact sets in a normed space \((\mathbb{R}^N, \|\cdot\|)\) are precisely the bounded and closed sets.

- true
- false

(b) The compact sets in the normed space \((\ell^1, \|\cdot\|_1)\) are precisely the bounded and closed sets.

- true
- false

(c) If \((K, d)\) is a compact metric space, then \(K\) is separable.

- true
- false

(d) If \((M, d)\) is a separable metric space, then \(M\) is compact.

- true
- false

(e) \((C(K), \|\cdot\|_\infty)\) for a compact metric space \((K, d)\) is a Polish metric space.

- true
- false

please turn over!
(f) \((C_b(M), \|\cdot\|_\infty)\) is separable for every metric space \((M, d)\).
\ □ true  \ □ false 

(g) \((\mathcal{F}_b(\Omega), \|\cdot\|_\infty)\) is a Polish metric space for every set \(\Omega\).
\ □ true  \ □ false 

(h) \((\mathcal{F}_b(\Omega), \|\cdot\|_\infty)\) is a Polish metric space for every countable set \(\Omega\).
\ □ true  \ □ false 

(i) The trigonometric polynomials are dense in \((C([0, 2\pi]), \|\cdot\|_\infty)\).
\ □ true  \ □ false 

(j) The polynomials are dense in \((C([0, 2\pi]), \|\cdot\|_\infty)\).
\ □ true  \ □ false