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Exercise Sheet 11

Applied Analysis

Discussion on Thursday 16-1-2014 at 16ct

Exercise 1 (*Lebesgue measure of some sets*) (3+3+3+5)

In the following we write λ for the Lebesgue measure on $\mathbb{R}$.

The aim of this exercise: We already know from construction that $\lambda([a,b)) = b - a$ holds for every $a, b \in \mathbb{R}$ with $a \leq b$. But what is the Lebesgue measure of $[0,1]$ or $\{0\}$? One can derive those values directly from $\lambda([a,b)) = b - a$ and the fact that λ is a measure.

(a) If $M \subset \mathbb{R}$ consists of only one element, what is $\lambda(M)$?

(b) Calculate $\lambda(\mathbb{Q})$ and $\lambda([0,1])$.

(c) Calculate $\lambda(\mathbb{R})$ and use this to deduce that $\mathbb{R}$ is not countable.

(d) Let $C \subset \mathbb{R}$ be the following set of all real numbers

$$C = \left\{ \sum_{k=1}^{\infty} a_k 3^{-k} \,\middle|\, \text{for some sequence } (a_k) \text{ with } a_k \in \{0,2\} \right\}.$$

Show that C is Borel measurable (i.e. it lies in $\mathcal{B}(\mathbb{R})$) and calculate $\lambda(C)$.

Hint: One can use without a proof the following alternative description of C. We denote by $C_n \subset [0,1]$ (for every $n \in \mathbb{N}$) a finite union of closed disjoint intervals given by the following construction (see also figure 1):

(1) $C_0 = [0,1]$.

(2) The construction of C_{n+1} from C_n is given as follows: C_n is a finite union of closed disjoint intervals of the form $[a_k, a_k + 3l_k]$ for $k = 1, ..., 2^n$. Then C_{n+1} is the union of the intervals $[a_k, a_k + l_k]$ and $[a_k + 2l_k, a_k + 3l_k]$ for $k = 1, ..., 2^n$. In other words C_{n+1} is obtained from C_n by removing the middle third of each interval.

Then we get

$$C = \bigcap_{n \in \mathbb{N}} C_n.$$

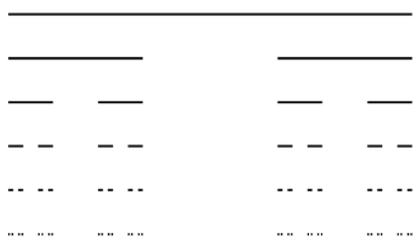

Figure 1: Alternative construction of C. The first line is C_0, below this is the picture of C_1 and so forth.

Exercise 2 (*Measurable functions*) (7+5)

We use the Euclidean norm on $\mathbb{R}^n$ and the usual Borel-σ-algebras on $\mathbb{R}^n$ and $\overline{\mathbb{R}}$.

(a) Which of the following functions f are measurable and which are continuous?

please turn over!

i. $f\colon \mathbb{R}^2 \to \mathbb{R}$ with $f(x,y) = e^{xy}x^2 + 2xy^2$.

ii. $f\colon \mathbb{R}^2 \to \mathbb{R}$ with

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^6} & \text{, for } (x,y) \neq (0,0) \\ 0 & \text{, for } x = y = 0 \end{cases}$$

Hint: This function is not continuous on all of $\mathbb{R}^2$. But why?

(b) Show that $f\colon \mathbb{R}^2 \to \overline{\mathbb{R}}$ given by

$$f(x,y) = \begin{cases} \frac{1}{(x+y)^2} & \text{, for } x + y \neq 0 \\ \infty & \text{, for } x + y = 0 \end{cases}$$

is measurable.

Exercise 3 (12+10+5)

(a) Calculate the following (Lebesgue) integrals, if they exist

i. $\displaystyle\int_{\mathbb{N}} \frac{(-1)^n}{n}\, d\zeta(n)$

ii. $\displaystyle\int_{[0,1]} f\, d\mu$

iii. $\displaystyle\int_{\mathbb{R}} \mathbb{1}_{\mathbb{Q}}\, d\lambda$

iv. $\displaystyle\int_{\mathbb{N}} 2^{-n}\, d\zeta(n)$

Here λ is the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, δ_0 the Dirac measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ supported in 0, ζ is the counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, μ is given by $\mu = 3\delta_0 + 7\lambda$ and the function f by

$$f(x) = \begin{cases} -2 & \text{, for } x = 0 \\ 2 & \text{, for } x = 1 \\ 1 & \text{, otherwise} \end{cases}.$$

(b) Calculate

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{\sin^n(x)}{x^2}\, d\lambda(x).$$

(c) Let $f\colon [0,1] \to \mathbb{R}$ be monotonically increasing (i.e. $f(x) \geq f(y)$ for $x \geq y$ and $x, y \in [0,1]$) function. Show that f is measurable and integrable.