Exercise 1 (Measure with density) (3+2+5)
Let \((\Omega, \Sigma, \mu)\) be a measure space and \(f: \Omega \rightarrow [0, \infty)\) an integrable function.
(a) We construct a measure, which we call \(f \cdot \mu\), on \((\Omega, \Sigma)\) by setting
\[
(f \cdot \mu)(A) := \int_A f \, d\mu
\]
for every \(A \in \Sigma\). Show that \(f \cdot \mu\) is indeed a measure (see Exercise 3.65).
(b) Show that \(f \cdot \mu\) is a probability measure if and only if
\[
\int_{\Omega} f \, d\mu = 1.
\]
(c) Given another measurable function \(g: \Omega \rightarrow [0, \infty)\) show (see Exercise 3.72 and follow the proof of Theorem 3.73)
\[
\int_{\Omega} g \, d(f \cdot \mu) = \int_{\Omega} g f \, d\mu.
\]
We say that a measure \(\nu\) has density \(f: \Omega \rightarrow [0, \infty)\) with respect to \(\mu\) if \(\nu = f \cdot \mu\).

Exercise 2 (Some probability distributions) (5+5+5+5)
(a) Show that the following measures \(f \cdot \lambda\) with \((\alpha > 0\) fixed and see the exercise before)
   i. \(f(x) = \frac{1}{\pi(1 + x^2)}\)
   ii. \(f(x) = 1_{[0, \infty)}(x)ae^{-\alpha x}\)
are probability measures.
(b) Find some \(c > 0\) (depending on \(\alpha > 0\)) such that a measure \(f \cdot \zeta\) which has the density
   \(f(k) = c\alpha \frac{k^\alpha}{k!}\) with respect to the counting measure \(\zeta\) on \(\mathbb{N}_0\) defines a probability measure.

Exercise 3 (5+5+5+5+5)
Calculate the following integrals if they exist \((\alpha > 0)\)
(a) \(\int_{\mathbb{N}} \frac{k^\alpha}{k!} e^{-\alpha} \, d\zeta(k)\)
(b) \(\int_{[0, \infty)} x\alpha e^{-\alpha x} \, d\lambda(x)\)
(c) \(\int_{\mathbb{R}} \frac{x}{\pi(1 + x^2)} \, d\lambda(x)\)
(d) \(\int_{[0,2]} g \, d\lambda\)
Here \(g: [0, 2] \rightarrow \mathbb{R}\) is given by
\[
g(x) = \begin{cases} 
  x^2, & \text{for } x < 1 \\
  x(2 - x), & \text{for } x \geq 1
\end{cases}
\]
Remark: In the last exercise of this sheet we interpret these integrals as expected values of certain probability distributions.

please turn over!
Exercise 4 (Two basic formulas for expected values)

Let a probability space \((\Omega, \Sigma, P)\) and a real valued random variable \(X : \Omega \rightarrow \mathbb{R}\) be given. We say that \(X\) has a finite expected value, if

\[
\int_{\Omega} |X| \, dP < \infty
\]

and call in this case

\[
\mathbb{E}X = \int_{\Omega} X \, dP
\]

the expected value.

(a) Prove (use Theorem 3.73) that \(X\) has finite expected value, iff

\[
\int_{\mathbb{R}} |x| \, dP_X(x) < \infty
\]

and that in this case one has

\[
\mathbb{E}X = \int_{\mathbb{R}} x \, dP_X(x).
\]

Here \(P_X\) is the push forward of \(P\) under \(X\) on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) given by \(P_X(A) = P(X^{-1}(A))\) for all \(A \in \mathcal{B}(\mathbb{R})\) (compare this with the lecture).

(b) Let us suppose that \(X\) has a density \(f\) with respect to \(\mu\), i.e. \(P_X = f \cdot \mu\) for some measure \(\mu\) on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) and some integrable function \(f : \mathbb{R} \rightarrow \mathbb{R}\). Prove that \(X\) has finite expected value, iff

\[
\int_{\mathbb{R}} |x| f(x) \, d\mu(x) < \infty
\]

and that in this case one has

\[
\mathbb{E}X = \int_{\mathbb{R}} x f(x) \, d\mu(x).
\]

(c) Interpret the integrals in Exercise 3 as the expected values of some random variables. In particular, what is the probability distribution for the last integral (Is it a probability distribution?)?