Exercise 1 (Construction of random variables) (5+5+10)
We construct now a random variable with a given cumulative distribution function \( F : \mathbb{R} \to [0, 1] \). Here \( F \) is an arbitrary given monotonically increasing function, with
\[
\lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to \infty} F(x) = 1
\]
and such that \( F \) is right-continuous (compare this to Sheet 9 Exercise 1).

(a) Show that \([0, 1], \mathcal{B}([0, 1]), \lambda\) is a probability space, \( X : [0, 1] \to \mathbb{R} \) given by \( X(\omega) = \omega \) is a random variable and calculate the cumulative distribution function \( F_X \).

(b) Let us define the quantile function \( Q_F : [0, 1] \to \mathbb{R} \) by
\[
Q_F(p) = \inf \{ x \in \mathbb{R} : p \leq F(x) \}.
\]
Prove that \( Q_F \) is measurable.

(c) Prove that \( Y = Q_F \circ X \) (see (a) and (b)) is an almost surely finite random variable on the probability space \([0, 1], \mathcal{B}([0, 1]), \lambda\) and that \( Y \) has \( F \) as its cumulative distribution function (i.e. \( F = F_Y \)).

Exercise 2 (Fubini’s theorem) (5+5+5+5)
Calculate the following integrals, if they exist.

(a) \( \int_0^1 \int_x^1 \frac{y}{y^3 + 1} \, d\lambda(y) \, d\lambda(x) \)

(b) \( \int_{\{(k,l) \in \mathbb{N}^2 : l \leq k\}} \frac{1}{2k} \, d(\zeta \otimes \zeta)(k,l) \)

(c) \( \int_{-1}^1 \int_{-1}^1 \frac{x(y + 2y^2)}{ey + |y| + y^2 + |\sin y|} \, d\lambda(y) \, d\lambda(x) \)

(d) \( \int_{[0,1]^2} \frac{x-y}{(x+y)^3} \, d\lambda^2(x,y) \)

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