University of Ulm

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Winter term 2014/15

Discussion: Friday, 21.11.2014

Exercises Applied Analysis: Sheet 5

1. (a) Let $T \in \mathcal{L}(X,Y)$. Show that $\ker T := \{x \in X : Tx = 0\}$ is a closed vectors subspace of X.

(b) Show that if two linear operators $T, S \in \mathcal{L}(X,Y)$ are equal on a dense set $A \subset X$, then $T = S$.

2. Let $1 \le p < \infty$ and $y \in \ell^q$. Show that

$$\varphi_y(x) := \sum_{k=1}^{\infty} y_k x_k$$

defines an element of $(\ell^p)'$ such that $\|\varphi_y\|_{(\ell^p)'} = \|y\|_q$.

3. (a) Let X, Y be two normed spaces. Suppose there exists an isomorphism between X and Y. Show that X is separable (complete) if and only if Y is separable (complete). Deduce that ℓ^∞ is not isomorphic to ℓ^p for $1 \le p < \infty$.

(b) Let X be a vector space and $\|\cdot\|_1, \|\cdot\|_2$ two norms on X. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if and only if the spaces $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are isomorphic.

(c) Let X be a normed space over $\mathbb{K}$ such that $N := \dim X < \infty$. Show that X is isomorphic to $(\mathbb{K}^N, \|\cdot\|_\infty)$.

4. (a) Give an example of an operator that does not have closed range.

(b) Give an example of an isometric operator that is not an isomorphism.

(c) Give an example of a linear map between two normed spaces that is not bounded.