1. (a) Let $T \in \mathcal{L}(X, Y)$. Show that $\ker T := \{ x \in X : Tx = 0 \}$ is a closed vectors subspace of $X$.

(b) Show that if two linear operators $T, S \in \mathcal{L}(X, Y)$ are equal on a dense set $A \subset X$, then $T = S$.

2. Let $1 \leq p < \infty$ and $y \in \ell^q$. Show that
   $$\varphi_y(x) := \sum_{k=1}^{\infty} y_k x_k$$
   defines an element of $(\ell^p)'$ such that $||\varphi_y||_{(\ell^p)'} = ||y||_q$.

3. (a) Let $X, Y$ be two normed spaces. Suppose there exists an isomorphism between $X$ and $Y$. Show that $X$ is separable (complete) if and only if $Y$ is separable (complete). Deduce that $\ell^\infty$ is not isomorphic to $\ell^p$ for $1 \leq p < \infty$.

(b) Let $X$ be a vector space and $||\cdot||_1, ||\cdot||_2$ two norms on $X$. Then $||\cdot||_1$ and $||\cdot||_2$ are equivalent if and only if the spaces $(X, ||\cdot||_1)$ and $(X, ||\cdot||_2)$ are isomorphic.

(c) Let $X$ be a normed space over $K$ such that $N := \dim X < \infty$. Show that $X$ is isomorphic to $(K^N, ||\cdot||_{\infty})$.

4. (a) Give an example of an operator that does not have closed range.

(b) Give an example of an isometric operator that is not an isomorphism.

(c) Give an example of a linear map between two normed spaces that is not bounded.