1. The proof of Banach’s fixed point theorem shows that for any initial function \( u \in C(I) \) the sequence \( \Phi^u \) converges to the unique fixed-point \( u^* \) of \( \Phi \) that is the unique solution of (ODE). This allows to approximate the solution \( u^* \) by simply iterating \( \Phi \). This method is called Picard iteration.

For the ordinary differential equation \( u'(t) = tu(t) \) on \([0, 1]\), use Picard iteration to construct the solution of the equation for the initial value \( u(0) = 1 \).

2. Calculate the value of the limit

\[
\lim_{N \to \infty} \sum_{k=1}^{N} \frac{N}{N^2 + k^2}
\]

by using the Riemann integral.

3. In the lecture it was shown that \((C[0, 1], \|\cdot\|_{L^1})\) with \( \|f\|_{L^1} := \int_0^1 |f(s)| \, ds \) is a normed space.

(a) Show that \((C[0, 1], \|\cdot\|_{L^1})\) is not complete.

(b) Is \((C[0, 1], \|\cdot\|_{L^1})\) separable?

4. Urysohn’s lemma:

Let \((X, \|\cdot\|)\) be a normed space and \( A, B \subset X \) be closed sets such that \( A \cap B = \emptyset \). Show that there exists a continuous function \( f \in C(X) \) such that \( f(x) = 0 \) for all \( x \in A \) and \( f(x) = 1 \) for all \( x \in B \).

[Hint: First show that \( x \mapsto \inf \{ \|x - y\| : y \in A \} \) is Lipschitz continuous if \( A \neq \emptyset \). For \( f \) use a suitable combination of these functions.]