1. Let $(\Omega, \Sigma, \mu)$ be a probability space. Two sets $A, B \in \Sigma$ are called (stochastically) independent, if and only if

$$\mu(A \cap B) = \mu(A)\mu(B).$$

Let us suppose that $A \in \Sigma$ and $E \subset \Sigma$ are given. We say that $A$ is independent of $E$, if and only if $A, B$ are independent for all $B \in E$.

(a) Find a concrete example of the above situation such that $A$ is independent of $E$ but $A$ is not independent of $\sigma(E)$.

(b) Let us suppose that $E$ is stable under intersections. Prove that $A$ and $E$ are independent if and only if $A$ and $\sigma(E)$ are independent.

2. Let $F: \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function, i.e., $F(x) \leq F(y)$ if $x \leq y$. Define $F_+(t) := \inf\{F(s) : s > t\}$. Show that there exists a measure $\mu$ on $\mathcal{B}(\mathbb{R})$ such that $\mu((a, b]) = F_+(b) - F_+(a)$ for all $a, b \in \mathbb{R}$ with $a < b$.

3. Let $(\Omega, \Sigma, \mu)$ be a measure space and $f: \Omega \to [0, \infty)$ be a measurable function.

(a) Show that $\nu(A) = \int 1_A f \, d\mu$ defines a measure on $(\Omega, \Sigma)$.

(b) When is the measure $\nu$ finite?

4. Suppose $\mu$ is the counting measure on the measurable space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. Let $f: \mathbb{N} \to [0, \infty)$ be a function. Note that $f$ is measurable. Show that $f$ is integrable if and only if $f \in \ell^1$. 