1. Let \((\Omega, \Sigma, \mu)\) be a measure space and \(f: \Omega \to [0, \infty]\) measurable. Define \(\nu(A) := \int_A f \, d\mu\) for \(A \in \Sigma\). We know that \(\nu(A)\) is a measure. Show that \(g \in L^1(\Omega, \Sigma, \nu)\) if and only if \(gf \in L^1(\Omega, \Sigma, \mu)\). Moreover, show that then \(\int g \, d\nu = \int gf \, d\mu\).

2. Let \(\alpha > 0\).
   (a) Show that the following measures \(\nu(A) = \int_A f \, d\lambda\), where \(\lambda\) is the Lebesgue measure and
      \[
      (i.) f(x) = \frac{1}{\pi(1 + x^2)}, \quad (ii.) f(x) = 1_{[0,\infty)}(x)e^{-\alpha x},
      \]
   are probability measures.
   (b) Find some \(c > 0\) (depending on \(\alpha > 0\)) such that a measure \(\nu(A) = \int_A f \, d\zeta\), where \(\zeta\) is the counting measure on \(\mathbb{N}_0\) and
       \[
       f(k) = c\frac{\alpha^k}{k!}
       \]
   defines a probability measure.
   (c) Calculate the expected value of random variables with the above densities \(f\), i.e.
       \[
       \int_{\mathbb{R}} \frac{x}{\pi(1 + x^2)} \, d\lambda(x), \quad \int_{[0,\infty)} x e^{-\alpha x} \, d\lambda(x), \quad \text{and} \quad \int_{\mathbb{N}} kc\frac{\alpha^k}{k!} \, d\zeta(k),
       \]
   where \(\lambda\) is the Lebesgue measure and \(\zeta\) the counting measure.

3. Consider the measure space \((\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)\). Give a sequence of integrable functions \(f_n: \mathbb{R} \to [0,1]\) such that \(f_n(x) \to 0\) as \(n \to \infty\) for all \(x \in \mathbb{R}\) and \(\int f_n = 1\) for all \(n \in \mathbb{N}\). Does such an example also exist if \(\mathbb{R}\) is replaced by \([-1,1]\)?

4. Find a probability space \((\Omega, \Sigma, \mathbb{P})\) that supports a sequence of stochastically independent, identically distributed random variables \((X_n)_{n \in \mathbb{N}}\) such that \(\mathbb{P}(X_n = 1) = 3/5\), \(\mathbb{P}(X_n = 0) = 1/5\) and \(\mathbb{P}(X_n = 2) = 1/5\). Describe the construction of such a sequence for your choice of \((\Omega, \Sigma, \mathbb{P})\). Prove that the first two functions in your sequence are indeed stochastically independent.