On the structure of semigroups on $L_p$ with a bounded $H^\infty$-calculus

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Notation: $\Sigma_\varphi := \{ z \in \mathbb{C} : |\arg(z)| < \varphi \}$.

**Definition (Sectorial Operator)**

$(A, D(A))$ densely defined operator with $\omega \in (0, \pi)$ such that

$(S_\omega) \quad \sigma(A) \subseteq \overline{\Sigma_\omega}$ and $\sup_{\lambda \notin \Sigma_{\omega+\epsilon}} \|\lambda R(\lambda, A)\| < \infty \quad \forall \epsilon > 0$.

Then $\omega(A) := \inf\{ \omega : (S_\omega) \text{ holds} \}$. 
Definition (Analytic $C_0$-semigroup)

Family of operators $(T(z))_{z \in \Sigma_\delta}$ $(\delta \in (0, \frac{\pi}{2}))$ satisfying

(i) $z \mapsto T(z)$ is analytic

(ii) $T(z_1 + z_2) = T(z_1)T(z_2)$ $\forall z_1, z_2 \in \Sigma_\delta$

(iii) $\lim_{z \to 0} T(z)x = x$ $\forall \delta' \in (0, \delta), \forall x \in X$

It is called bounded if $\sup_{z \in \Sigma_{\delta'}} \|T(z)\| < \infty$ for all $\delta' \in (0, \delta)$.

One has 1:1 correspondence

bounded analytic $C_0$-semigroups $\leftrightarrow$ A sectorial with $\omega(A) < \frac{\pi}{2}$

At least formally $T(z) = e^{-zA}$. 
Given \( f \in H_0^\infty(\Sigma_\sigma) := \left\{ f : \Sigma_\sigma \to \mathbb{C} \text{ analytic} : |f(\lambda)| \leq \frac{|\lambda|^\varepsilon}{(1+|\lambda|)^{2\varepsilon}} \right\} \) define
\[
f(A) := \int_{\partial \Sigma'_\sigma} f(\lambda) R(\lambda, A) \, d\lambda \quad (\omega(A) < \sigma' < \sigma).
\]

**Definition (Bounded \( H^\infty \)-calculus)**

\((A, D(A)) \) sectorial has bounded \( H^\infty(\Sigma_\sigma) \)-calculus if for some \( C \geq 0 \)

\((H_\sigma) \) \[ \|f(A)\| \leq C \sup_{\lambda \in \Sigma_\sigma} |f(\lambda)| \quad \forall f \in H_0^\infty(\Sigma_\sigma). \]

Then \( \omega_{H^\infty}(A) := \inf\{\sigma : (H_\sigma) \text{ holds}\} \).
Theorem (C. Le Merdy)

\[ -A \sim (T(z))_{z \in \Sigma} \text{ bounded analytic } C_0 \text{-semigroup on Hilbert space } H. \]

Equivalent:

(i) A has a bounded \( H^\infty \)-calculus

(ii) there exists \( S \in \mathcal{B}(H) \) invertible such that

\[ \|S^{-1}T(t)S\| \leq 1 \quad \forall t \geq 0. \]

Put differently: contractive semigroups are generic for all semigroups with a bounded \( H^\infty \)-calculus.
Can this be generalized to $L_p$ ($1 < p < \infty$)? In one direction, one has

**Theorem (L. Weis)**

$-A \sim (T(z))_{z \in \Sigma}$ bounded analytic $C_0$-semigroup on $L_p$, positive and contractive on the real line.

Then $A$ has a bounded $H^\infty$-calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$.

Can all semigroups on $L_p$ with a bounded $H^\infty$-calculus be obtained from such semigroups?
Theorem (S.F.)

$-A \sim (T(z))_{z \in \Sigma}$ bounded analytic $C_0$-semigroup on $L_p(\Omega)$ ($1 < p < \infty$).

Equivalent:

(i) $A$ has a bounded $H^\infty$-calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$. 
Theorem (S.F.)

$-A \sim (T(z))_{z \in \Sigma}$ bounded analytic $C_0$-semigroup on $L_p(\Omega)$ ($1 < p < \infty$).

Equivalent:

(i) $A$ has a bounded $H^\infty$-calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$.

(ii) There exists a bounded holomorphic $C_0$-semigroup $(R(z))_{z \in \tilde{\Sigma}}$ in some $L_p(\tilde{\Omega})$, positive and contractive on the real line with

- $N \subset M \subset L_p(\tilde{\Omega})$ closed subspaces invariant under $(R(z))$
- $S \in \mathcal{B}(L_p(\Omega), M/N)$ isomorphism

such that

$$T(z) = S^{-1}R_{M/N}(z)S \quad \forall z \in \tilde{\Sigma}.$$
Theorem (S.F. (Reminder))

\[ -A \sim (T(z))_{z \in \Sigma} \] bounded analytic \( C_0 \)-semigroup on \( L_p(\Omega) \). Equivalent:

(i) \( A \) has a bounded \( H^\infty \)-calculus with \( \omega_{H^\infty}(A) < \frac{\pi}{2} \).

(ii) \( T(z) = S^{-1}R_{M/N}(z)S \quad \forall z \in \tilde{\Sigma} \).

- On Hilbert spaces \( \omega_{H^\infty}(A) < \frac{\pi}{2} \) holds automatically
- This seems to be open for \( L_p \)-spaces, but false for general subspaces of \( L_p \)-spaces (N.J. Kalton)
- (i) \( \Rightarrow \) (ii) holds on every UMD-Banach lattice \( ((R(z)) \) lives on another UMD-Banach lattice)
Problem

Does the result hold without factorizing through a subspace-quotient as in the Hilbert space case?
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Can the constructed semigroup \((R(z))\) be chosen to be contractive on a whole sector as in the Hilbert space case?
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Problem

Does every positive contractive \(C_0\)-semigroup on a UMD Banach lattice have a bounded \(H^\infty\)-calculus?
Main ideas of the proof (I):

- For $\alpha > 1$ give $H_0^\infty(\Sigma \frac{\pi}{2\alpha} +)$ a $p$-operator space structure as follows:

  $$H^\infty(\Sigma \frac{\pi}{2\alpha} +) \hookrightarrow \mathcal{B}(L_p(\mathbb{R}; Y))$$

  $$f \mapsto f(B^{\frac{1}{\alpha}}),$$

where $-B$ generates the shift semigroup $V(t)g(s) = g(s - t)$ on $L_p(\mathbb{R}; Y)$ for some vector-valued $L_p$-space $Y$. 
Main ideas of the proof (I):

- For $\alpha > 1$ give $H_0^\infty(\sum \frac{\pi}{2\alpha} +)$ a $p$-operator space structure as follows:

$$H^\infty(\sum \frac{\pi}{2\alpha} +) \hookrightarrow \mathcal{B}(L_p(\mathbb{R}; Y))$$

$$f \mapsto f(B^{\frac{1}{\alpha}}),$$

where $-B$ generates the shift semigroup $V(t)g(s) = g(s - t)$ on $L_p(\mathbb{R}; Y)$ for some vector-valued $L_p$-space $Y$.

- $p$-complete boundedness of the functional calculus, e.g. mappings

$$\mathcal{B}(\ell^n_p(H^\infty(\Sigma))) \supset M_n(H^\infty) \to M_n(\mathcal{B}(L_p(\mathbb{R}; Y))) \cong \mathcal{B}(\ell^n_p(L_p(\mathbb{R}; Y)))$$

$$[f_{ij}] \mapsto [f_{ij}(B)]$$

are uniformly bounded in $n$. 
Main ideas of the proof (II):

- A factorization theorem of G. Pisier yields a semigroup as asserted, except for strong continuity (ultraproduct construction).
- Reduce to the strongly continuous part.
Every bounded analytic $C_0$-semigroup on $L_p(\Omega)$ with generator $-A$ satisfying $\omega_{H\infty}(A) < \frac{\pi}{2}$ can be obtained

- from a bounded analytic $C_0$-semigroup on $L_p(\tilde{\Omega})$, positive and contractive on the real line
- after passing to invariant subspace-quotients and similarity transforms
Je vous remercie de votre attention.
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Et bon appétit!