A MARKOV OPERATOR AND A MARKOV SEMIGROUP ON $C(K)$ WITH NON-CYCLIC PERIPHERAL POINT SPECTRUM

JOCHEN GLÜCK

Abstract. In this note we construct an example of a Markov operator T on a $C(K)$-space such that the peripheral point spectrum of T is not cyclic. By a similar construction we obtain a strongly continuous semigroup of Markov operators on a $C(K)$-space whose generator has non-cyclic peripheral point spectrum.

1. Introduction

Spectra of positive operators have been studied since Perron and Frobenius firstly discovered the amazing spectral properties of positive matrices at the beginning of the 20th century. Many results of the finite-dimensional Perron Frobenius Theory can be generalized to large classes of positive operators on complex Banach classes. A number of well-known theorems provide criteria for the existence of positive eigenvectors, non-triviality of the spectrum, monotonicity of the spectral radius and the cyclicity of the peripheral (point) spectrum. For a detailed discussion of those results we refer to the survey article [2] as well as to the references therein.

In this note, we are concerned with the latter of those topics. Let E be a complex Banach lattice (see e.g. [4] for an introduction to the theory of Banach lattices) and let T be a positive linear operator on E with spectrum $\sigma(T)$, point spectrum $\sigma_{\text{pnt}}(T)$ and spectral radius $r(T)$. Recall that the *peripheral spectrum* and the *peripheral point spectrum* of T are defined by

$$\sigma_{\text{per}}(T) := \{\lambda \in \sigma(T) : |\lambda| = r(T)\} \quad \text{and}$$
$$\sigma_{\text{pnt,per}}(T) := \{\lambda \in \sigma_{\text{pnt}}(T) : |\lambda| = r(T)\}.$$

A set M of complex numbers is said to be *cyclic* if $re^{i\varphi} \in M$ for some $r, \varphi \in \mathbb{R}$ implies that $re^{in\varphi} \in M$ for all $n \in \mathbb{Z}$.

There are plenty of results that provide sufficient conditions for the peripheral spectrum and the peripheral point spectrum of a positive operator T to by cyclic (see for example Section 4 in [2] and Sections V.4 and V.5 in [4]). However it is still an open problem whether actually each positive operator on a Banach lattice has cyclic peripheral spectrum.

Particularly important are *Markov operators* on spaces of continuous functions. Let K be a compact Hausdorff space and let $C(K)$ be the complex Banach lattice of all complex valued continuous functions on K.

A linear operator T on $C(K)$ is called a *Markov operator* if $T \geq 0$ and $T\mathbb{1} = \mathbb{1}$, where $\mathbb{1}$ denotes the constant 1-function on K. Many questions about an arbitrary positive operator T (on an arbitrary Banach lattice) can be reduced to questions about a Markov operator by restricting T to an appropriate ideal which may be identified with a $C(K)$-space by means of Kakutani's Representation Theorem. See e.g. the proof of Theorem 4.6 in [2] for an example of this technique. See [4, Theorem II.7.4] for the Kakutani Representation Theorem on real Banach lattices

Date: March 1, 2014.

and note that this Theorem holds in a similar manner for complex Banach lattices [4, p. 138].

2. A Markov operators with non-cyclic peripheral point spectrum

From now on, let T be a Markov operator on $C(K)$. It is easy to see that $||T|| = r(T) = 1$. Moreover, one can show that the *peripheral spectrum* $\sigma_{\text{per}}(T)$ is cyclic. This follows for example from the much more general Theorem V.4.9 in [4]. It is also true that the peripheral point spectrum $\sigma_{\text{pnt,per}}(T)$ is cyclic under some additional assumptions. Those assumptions are closely related to the structure of the invariant ideals of T (see Section 5 in [3]) and to the existence of certain sub-fixed points of the adjoint operator T' (see e.g. [4, Proposition V.4.6]).
Moreover, it is known that the set of those peripheral eigenvalues of a Markov operator T which are associated to a unimodular eigenvector f (i.e. $|f| = \mathbb{1}$) is cyclic (see e.g. [2, Proposition 4.8]).
However, we now give an example which shows that the peripheral point spectrum of a Markov operator does not need to be cyclic in general.

Example 2.1. We construct a compact space K as follows: Let $\overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$ be the one point compactification of the discrete space $\mathbb{N}_0$ and let $\mathbb{Z}_4 := \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ be equipped with the discrete topology and the addition modulo 4. We set $K = \mathbb{Z}_4 \,\dot{\cup}\, \overline{\mathbb{N}}_0$.
Define an operator $T : C(K) \to C(K)$ by

$$(Tf)(k) = \begin{cases} f(k-1) & \text{if } k \in \mathbb{Z}_4 \\ f(k-1) & \text{if } k \in \mathbb{N} \cup \{\infty\} \\ \frac{1}{2}\big(f(\bar{1}) + f(\bar{3})\big) & \text{if } k = 0. \end{cases}$$

It is easy to see that T indeed maps $C(K)$ to $C(K)$ and that T is a Markov operator.
Now we show that the peripheral point spectrum of T is not cyclic. Let $g \in C(K)$ be defined by

$$g(k) = \begin{cases} (-i)^k & \text{if } k \in \mathbb{Z}_4 \\ 0 & \text{else.} \end{cases}$$

Then $Tg = ig$ and thus i is an eigenvalue of T. On the other hand, -1 is not an eigenvalue of T. To see this, assume for a contradiction that $Th = -h$ for a non-zero function $h \in C(K)$. Note that $h(\bar{0}) \neq 0$, since $h(\bar{0}) = 0$ would imply $h(k) = 0$ for all $k \in K$. Hence, we may assume that $h(\bar{0}) = 1$. Then,

$$h(\bar{0}) = h(\bar{2}) = 1 \quad \text{and} \quad h(\bar{1}) = h(\bar{3}) = -1,$$

which implies $h(0) = 1$. Therefore, we obtain $h(k) = (-1)^k$ for all $k \in \mathbb{N}_0$, which contradicts the continuity of h at ∞.
We showed that $i \in \sigma_{\text{pnt,per}}(T)$, but $i^2 = -1 \notin \sigma_{\text{pnt,per}}(T)$. Thus, $\sigma_{\text{pnt,per}}(T)$ is not cyclic.

The following remarks on the preceding example might be useful:

Remarks 2.2.

(i) Although -1 is not an eigenvalue of T, we know that $-1 \in \sigma_{\text{per}}(T)$ as $\sigma_{\text{per}}(T)$ is cyclic. This means that -1 is an approximate eigenvalue of T, since the peripheral spectrum is contained in the topological boundary of the spectrum $\sigma(T)$.

Actually, it is easy to concretely construct an approximate eigenvector for -1: For $n \in \mathbb{N}$, define $h_n \in C(K)$ by

$$h_n(k) = \begin{cases} (-1)^k & \text{if } k \in \mathbb{Z}_4 \\ (-1)^k \cdot (1 - \frac{1}{n})^k & \text{else.} \end{cases}$$

Then $h_n \in C(K)$ for all $n \in \mathbb{N}$ and the sequence (h_n) is an approximate eigenvector for the spectral value -1.

(ii) On the other hand, note that $i^3 = -i$ is again an eigenvalue of T. The corresponding eigenvector is the complex conjugate $\overline{g}$ of the eigenvector g of i. We can directly see this by computing $T\overline{g}$, but note that – from a more general point of view – it also follows from the fact that T is a real operator (i.e. Tf is a real function whenever f is so). Indeed, it is easy to see that non-real eigenvalues and the associated eigenvectors of real operators always occur in complex conjugate pairs.

3. A strongly continuous Markov semigroup with non-cyclic peripheral point spectrum

Instead of a single positive operator T on a Banach lattice E, we now consider strongly continuous semigroups $(T(t))_{t\geq 0}$ of positive operators on E. For a detailed treatment of those semigroups we refer to [1].
It is possible to develop a Perron Frobenius Theory for positive semigroups which is in many points similar to the Perron Frobenius Theory of positive operators (see [1], Sections B-III and C-III). For our purposes, we only recall some basic notions which differ slightly from those used in the spectral theory of single operators:
Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on a complex Banach space. Let A be the generator of $(T(t))_{t\geq 0}$ with spectrum $\sigma(A)$, point spectrum $\sigma_{\mathrm{pnt}}(A)$ and with spectral bound $s(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$. In contrast to the preceding sections we now define the *peripheral spectrum* and the *peripheral point spectrum* of A by

$$\sigma_{\mathrm{per}}(A) := \{\lambda \in \sigma(A) : \operatorname{Re}\lambda = s(A)\} \quad \text{and}$$
$$\sigma_{\mathrm{pnt,per}}(A) := \{\lambda \in \sigma_{\mathrm{pnt}}(A) : \operatorname{Re}\lambda = s(A)\}.$$

This change of notation is appropriate for the spectral theory of strongly continuous semigroups and in our setting it should not cause any ambiguity, since from now on we will only deal with the spectra of generators of semigroups.
We also have to adjust the notion of a cyclic subset of $\mathbb{C}$: A set M of complex numbers is said to be *imaginary additively cyclic* if $\alpha + i\beta \in M$ for some $\alpha, \beta \in \mathbb{R}$ implies that $\alpha + in\beta \in M$ for all $n \in \mathbb{Z}$. From now on, when we discuss spectra of semigroup generators, the term "cyclic" will always be understood as "imaginary additively cyclic".

Many results in the spectral theory of positive operators have analogues in the spectral theory of positive semigroups (see again [1], Sections B-III and C-III). In particular, one can show that the peripheral spectrum $\sigma_{\mathrm{per}}(A)$ of the generator of a positive semigroup is imaginary additively cyclic if some additional assumptions are fulfilled (see [1], Theorem 2.10 and Corollary 2.12 in Section C-III). As in the single operator case, it is still an open problem, whether the generator of each positive semigroup has imaginary additively cyclic peripheral spectrum.

Now, we want to focus on the peripheral point spectrum again. In [1, Section B-III, Example 2.13], one can find a positive semigroup whose generator has non-cyclic peripheral point spectrum. However, the semigroup in this example does not consist of Markov operators on a $C(K)$-space.

Here, we want to adapt our example 2.1 to construct a semigroup of Markov operators whose generator has non-cyclic peripheral point spectrum. As a preparation we briefly discuss two shift semigroups that will play a role in our construction:

Remarks 3.1.

(i) Let $\mathbb{T}$ be the complex unit circle and let $(R(t))_{t\geq 0}$ be the rotation semigroup on $C(\mathbb{T})$ with angular velocity 1, i.e. $R(t)f(x) = f(e^{-it}x)$ for all $x \in \mathbb{T}$ and all $t \geq 0$.

The generator A_R of $(R(t))_{t\geq 0}$ is defined on the space $D(A_R) = C^1(\mathbb{T})$, where $\mathbb{T}$ is equipped with the usual manifold structure. For each $f \in D(A_R)$ we have

$$A_R f(x) = -\frac{d}{d\theta} f(xe^{i\theta})|_{\theta=0} \quad \text{for all } x \in \mathbb{T}.$$

(ii) Let $[0,\infty]$ be the one point compactification of $[0,\infty)$. Note that we can identify the space $C([0,\infty])$ with the space all continuous functions f on $[0,\infty)$ such that $\lim_{x\to\infty} f(x)$ exists. Moreover, we define

$$C^1([0,\infty]) := \{f \in C([0,\infty]) \cap C^1([0,\infty)) : f' \in C([0,\infty])\}.$$

Now, consider the shift semigroup $(S(t))_{t\geq 0}$ on $C([0,\infty])$ which is given by

$$S(t)f(x) = \begin{cases} f(x-t) & \text{if } x \in [t,\infty] \\ f(0) & \text{if } x \in [0,t). \end{cases}$$

The generator A_S of $(S(t))_{t\geq 0}$ is defined on $D(A_S) = \{f \in C^1([0,\infty]) : f'(0) = 0\}$. We have

$$A_S f(x) = -f'(x) \quad \text{for all } x \in [0,\infty)$$

whenever $f \in D(A_S)$.

Now we want to adapt Example 2.1 to a time continuous setting. Therefore, we define $K = \mathbb{T} \,\dot\cup\, [0,\infty]$ and consider the space $C(K) \simeq C(\mathbb{T}) \oplus C([0,\infty])$. We intend to construct a semigroup $(T(t))_{t\geq 0}$ on $C(K)$ by using a rotation semigroup on $C(\mathbb{T})$ and a shift semigroup on $C([0,\infty])$. Moreover, we want to connect both semigroups by transferring some information from the torus $\mathbb{T}$ to the left end of the real line $[0,\infty]$.

The difficulty is to construct this transfer in such a way that continuous functions are still mapped to continuous ones. This explains the somewhat complicated "transfer term" that arises in the construction of the following semigroup:

Example 3.2. Let $K = \mathbb{T} \,\dot\cup\, [0,\infty]$. For each $f \in C(K)$ and each $t \geq 0$ let

$$T(t)f(x) = \begin{cases} f(e^{-it}x) & \text{if } x \in \mathbb{T} \\ f(x-t) & \text{if } x \in [t,\infty] \\ e^{-(t-x)} f(0) + e^{-(t-x)} \int_0^{t-x} e^s \, \langle \mu, R(s) f|_{\mathbb{T}} \rangle \, ds & \text{if } x \in [0,t), \end{cases}$$

where $(R(t))_{t\geq 0}$ is the rotation semigroup on $C(\mathbb{T})$ from Remark 3.1 (i) and $\mu \in C(\mathbb{T})'$ is the functional on $C(\mathbb{T})$ that is defined by $\langle \mu, f \rangle = \frac{1}{2}\big(f(i) + f(-i)\big)$.

It can be checked that $(T(t))_{t\geq 0}$ is a strongly continuous semigroup of Markov operators on $C(K)$. The domain of its generator A is given by

$$\begin{aligned} D(A) = \{f \in C(K) : f|_{\mathbb{T}} \in C^1(\mathbb{T}),\ f|_{[0,\infty]} \in C^1([0,\infty]), \\ f'(0) = f(0) - \langle \mu, f|_{\mathbb{T}} \rangle\} \end{aligned}$$

and for all $f \in D(A)$ we have

$$Af(x) = \begin{cases} -\frac{d}{d\theta} f(xe^{i\theta})|_{\theta=0} & \text{if } x \in \mathbb{T} \\ -f'(x) & \text{if } x \in [0,\infty). \end{cases}$$

Now we can immediately check that i is an eigenvalue of A. A corresponding eigenfunction is given by

$$g(x) = \begin{cases} x^{-1} & \text{if } x \in \mathbb{T} \\ 0 & \text{if } x \in [0, \infty]. \end{cases}$$

Since $(T(t))_{t \geq 0}$ is a semigroup of Markov operators, we have $s(A) = 0$ and thus, $i \in \sigma_{\mathrm{pnt,per}}(A)$.

However, $2i$ is not an eigenvalue of A. Indeed, if we assume $Ah = 2ih$ for a function $0 \neq h \in D(A)$, then there are scalars $a, b \in \mathbb{C}$ such that

$$h(x) = \begin{cases} ax^{-2} & \text{if } x \in \mathbb{T} \\ be^{-2ix} & \text{if } x \in [0, \infty). \end{cases}$$

Since h must be continuous at ∞, we conclude that $b = 0$. Now it follows from the equation $h'(0) = h(0) - \langle \mu, h|_{\mathbb{T}} \rangle$ that $\langle \mu, h|_{\mathbb{T}} \rangle = 0$. Since $\langle \mu, h|_{\mathbb{T}} \rangle = \frac{1}{2}\big(h(i) + h(-i)\big) = -a$, we also have $a = 0$. This contradicts $h \neq 0$.

Therefore, $2i$ is not an eigenvalue of A, which shows that the peripheral point spectrum of A is not imaginary additively cyclic.

References

[1] W. Arendt, A. Grabosch, G. Greiner, U. Groh, H. P. Lotz, U. Moustakas, R. Nagel, F. Neubrander, and U. Schlotterbeck. *One-parameter semigroups of positive operators*, volume 1184 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.

[2] J. J. Grobler. Spectral theory in Banach lattices. In *Operator theory in function spaces and Banach lattices*, volume 75 of *Oper. Theory Adv. Appl.*, pages 133–172. Birkhäuser, Basel, 1995.

[3] H. H. Schaefer. Invariant ideals of positive operators in $C(X)$. II. *Illinois J. Math.*, 12:525–538, 1968.

[4] Helmut H. Schaefer. *Banach lattices and positive operators*. Springer-Verlag, New York, 1974. Die Grundlehren der mathematischen Wissenschaften, Band 215.

E-mail address: jochen.glueck@uni-ulm.de

Jochen Glück, Institute of Applied Analysis, Ulm University, 89069 Ulm, Germany