Eventual Positivity of Operator Semigroups

Jochen Glück

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Joint work with Daniel Daners (University of Sydney) and James B. Kennedy (University of Lisbon)
The Spectrum of Positive Matrices

Theorem (Perron–Frobenius)

Let $T \in \mathbb{R}^{d \times d}$ be such that $T \geq 0$.

(a) The spectral radius $r(T)$ is an element of the spectrum $\sigma(T)$.

(b) There exists a positive eigenvector for $r(T)$.

(c) The peripheral spectrum $\sigma_{\text{per}}(T) := \{ \lambda \in \sigma(T) : |\lambda| = r(T) \}$ is cyclic, i.e. if $r(T)e^{i\theta} \in \sigma_{\text{per}}(T)$, then $r(T)e^{in\theta} \in \sigma_{\text{per}}(T)$ for all $n \in \mathbb{Z}$.

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(d) And many more...
Generalisations

Similar results remain true (under appropriate technical assumptions)

(a) if $T$ is only eventually positive, i.e. $T_n \geq 0$ for all sufficiently large $n$ (extensive literature, cf. [Glü17b, Section 1]) or

(b) if $T$ is a positive operator on a Banach lattice (cf. [Sch74, Chapter 5]).
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Observation

Nobody has combined these two approaches, yet.
Notions of Eventual Positivity

Definition

A bounded linear operator $T$ on a Banach lattice $E$ is called...

(a) uniformly eventually positive if the inequality $T_n \geq 0$ holds whenever $n$ is larger than an appropriate $n_0$.

(b) individually eventually positive if, for all $x \in E^+$, the inequality $T_n x \geq 0$ holds whenever $n$ is larger than an appropriate $n_0$ (where $n_0$ might depend on $x$).
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Remarks

There are even more interesting notions, e.g.

- Weak eventual positivity: consider the inequality $\langle x', T_n x \rangle \geq 0$ for $x, x' \geq 0$ and let $n_0$ depend on both $x$ and the functional $x'$.

- Asymptotic positivity: consider the condition $\text{dist}(T_n x, E^+^n) \to 0$ for $x \in E^+$.

In infinite dimensions: ind. eventual positivity $\not\Rightarrow$ unif. eventual positivity.

Counterexample (idea)

Let $E = C([0,1])$ and construct $T$ non-positive such that for each $f \in E$ $T_n f \to \int_0^1 f(x) \, dx \cdot 1$ ($n \to \infty$).

If $f \geq 0$, then $T_n f \geq 0$ for all large $n$, but this might happen very late if $\int_0^1 f(x) \, dx$ is small compared to $\|f\|_\infty$.
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The Spectrum of Eventually Positive Operators

Theorem (G., in [Glü17b])

Let $T$ be a bounded linear operator on a non-zero Banach lattice $E$. Assume that $T$ is individually eventually positive and that $r(T) > 0$. 

(a) We have $r(T) \in \sigma(T)$. 

(b) If $T$ is compact, then $r(T)$ is an eigenvalue of $T$ with a positive eigenvector. 

(c) If $T$ is even uniformly eventually positive and if $T/r(T)$ is power-bounded, then the peripheral spectrum of $T$ is cyclic.

Ideas for the proof.

(a) A (subtle) resolvent estimate. 

(b) Laurent expansion of the resolvent about $r(T)$. 

(c) Associate a positive operator $S$ to the operator $T$ by means of an ultra power argument.
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**Theorem (Daners, G., Kennedy, [DGK16a])**

Let \(e^{tA}\) be an analytic \(C_0\)-semigroup on \(E\). Assume that \(e^{tA}\) is compact for one (equivalently all) \(t > 0\)
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(a) The semigroup has the following eventual positivity property:

$\forall f \in E^+ \{0\} \exists t_0 \geq 0 \forall t \geq t_0$:

$e^{tA}f \gg u_0$.

(b) The spectral bound $s(A)$ is a dominant spectral value of $A$; moreover, $\ker(s(A) - A)$ is spanned by a vector $v \gg u_0$ and $\ker(s(A) - A')$ contains a strictly positive functional.
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Remarks.

(a) One can also relate the above properties (a) and (b) to properties of the resolvent and to the spectral projection associated with $A$.

(b) One can vary the assumptions of the theorem (e.g. analyticity) in several ways. That’s all certainly nice – but is it useful?
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Let \( \Omega \subseteq \mathbb{R}^d \) be the unit ball.
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**Example 1.**

Let $\Omega \subseteq \mathbb{R}^d$ be the unit ball. Consider the Cauchy problem

$$\begin{cases}
\dot{w} = -\Delta^2 w & \text{in } B, \\
w|_{\partial \Omega} = \frac{\partial}{\partial \nu} w = 0 \\
+ \text{ initial condition.}
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Then the associated semigroup on \( L^p(\Omega) \) (\( 1 < p < \infty \)) fulfils (a), where \( u(x) = \text{dist}(x, \partial \Omega)^2 \) for all \( x \in \Omega \).
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**Proof.**

It follows from work of Grunau and Sweers [GS98] that \( -\Delta^2 \) (with the given boundary conditions) fulfils the spectral condition (b) in the above theorem.
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Example 2.
Consider the following heat equation with non-local boundary conditions:

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\begin{align*}
\dot{w} &= \Delta w \quad \text{in} \quad (0,1), \\
\lim_{t \to 0} w(t) &= \lim_{t \to 1} w(t) = w'(0) = -w'(1) + \text{initial condition}.
\end{align*}
\]

Then the associated semigroup on \( L^2((0,1)) \) fulfills (a), where \( u = 1((0,1)) \).

But the semigroup is not positive.

Sketch of the proof.
Let \( \Delta \) denote the Laplace operator with the above boundary conditions.
Explicit computation:
\[ (\Delta)^{-1}f \gg u \quad \text{whenever} \quad 0 \neq f \geq 0. \]

Kre˘ın–Rutman type argument \( \Rightarrow \) condition (b) in the theorem holds.
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Develop the perturbation theory of eventually positive semigroups until it reaches a satisfactory state.

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Non-positivity of the semigroup generated by the Dirichlet-to-Neumann operator.

Daniel Daners and Jochen Glück.
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F. Shakeri and R. Alizadeh.  
Nonnegative and eventually positive matrices.  

Helmut H. Schaefer.  
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