Long term behaviour of positive operator semigroups

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Joint work with Moritz Gerlach (University of Potsdam)
Goal

Consider a positive one-parameter semigroup \((T_t)_{t \geq 0}\) on \(L^p\) (or on a Banach lattice).
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Problem

*Find conditions which ensure convergence of \(T_t\) (weak/strong/in operator norm) as \(t \to \infty\).*
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$$Tf = \int_{\Omega} k(\cdot, \omega)f(\omega) \, d\mu(\omega)$$

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There is a sensible generalisation of this notion to the case where $E$ is a Banach lattice.
A convergence theorem.

**Theorem (Greiner, 1982)**

Let \((T_t)_{t \in [0, \infty)}\) be a positive, contractive \(C_0\)-semigroup on \(E = L^p\) with a fixed point \(f_0 \gg 0\).

Simple generalisations:
- \(E\) is allowed to be a Banach lattice with order continuous norm.
- It suffices if \((T_t)_{t \geq 0}\) is bounded instead of contractive.
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Let \((T_t)_{t \geq 0}\) be a Markov \(C_0\)-semigroup on \(E = L^1\) with a fixed point \(f_0 \gg 0\).

If \((T_t)_{t \geq 0}\) dominates a positive kernel operator \(K \neq 0\) for some \(t_0 \geq 0\), and if the semigroup is irreducible, then \((T_t)_{t \geq 0}\) converges strongly as \(t \to \infty\).
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What about the irreducibility assumption?

- It suffices if \(K\) “interacts” with the entire semigroup.
- More precisely: It suffices that \(Kf \neq 0\) for every fixed point \(0 \neq f \geq 0\) of the semigroup (& that a weak technical assumption be fulfilled).
Domination of kernel operators

Example

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\[B := A + K\]
generates of positive \(C_0\)-semigroup \((S_t)_{t \geq 0}\) given by

\[S_t = \sum_{k=0}^{\infty} V(k)t\]

Here, \(V(k)t \geq 0\) and \(V(0)t = T_t\), \(V(1)t = \int_0^t T_t - sKtds\).
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\(S_t\) dominates \(V_t^{(1)}\) which is a kernel operator for \(t > 0\)!
Tic, Toc...

The above results fail in discrete time: the power-bounded positive matrix 
\( T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) is a kernel operator on \( \mathbb{R}^2 \), but \( T^n \) does not converge as \( n \to \infty \).

This is because \( T \) does not possess positive matrix roots of high order, right? Wrong!

Example: Let \( S, T \in \mathbb{R}^3 \times \mathbb{R}^3 \) be the permutation matrices for the cycles \( (123) \) and \( (132) \).

Then \( S^2 = T \) and \( T^2 = S \), so \( T \) has a positive matrix root \( T^{1/2} \) for each \( n \in \mathbb{N} \).

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Rest of the talk:
\textbf{No} time regularity is needed to prove convergence results.
Why should we dismiss time regularity?

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- even less continuity (for instance, liftings of semigroups to ultra products).

Don’t prove theorems for each of those single cases!

Simply prove theorems without any time regularity.
Convergence of semigroup representations.

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This makes every semigroup representation \((T_t)_{t \in S}\) on a Banach space \(E\) a net. Hence, we can speak about strong convergence of \((T_t)_{t \in S}\).
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**Example**

For \(S = (0, \infty)\) and \(S = \mathbb{N}\), this yields the usual convergence as \(t \to \infty\).
A convergence theorem without time regularity.

**Theorem (Gerlach and G., article in preparation)**

Let $(G, +)$ be a commutative group and let $S \subseteq G$ be a subsemigroup such that $\langle S \rangle = G$. 

Let $(T_t)_{t \in S}$ be a positive, bounded semigroup representation on a Banach lattice with o.c. norm. Suppose that $(T_t)_{t \in S}$ has a fixed point $f_0 \gg 0$ and that $T_{t_0}$ is a kernel operator for some $t_0 \in S$.

Then $(T_t)_{t \in S}$ is strongly convergent, provided that $G$ is “good”.

Idea: Kernel operators map order intervals to relatively compact sets $\Rightarrow$ the JdLG machinery can be applied. This reduces the theorem to Lemma.
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Let \((T_t)_{t \in G}\) be a positive and bounded group representation on an atomic Banach lattice with o.c. norm. If \(G\) is “good” and if the representation has a fixed point \(f_0 \gg 0\), then \(T_t = \text{id}\) for all \(t \in G\).
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**Lemma**

Let \((T_t)_{t \in G}\) be a positive and bounded group representation on an atomic Banach lattice with o.c. norm. If \(G\) is “good” and if the representation has a fixed point \(f_0 \gg 0\), then \(T_t = \text{id}\) for all \(t \in G\).
What does “good” mean?

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- Think of the space as \(\ell^p(\mathbb{N})\). Consider a canonical unit vector \(e_k\).
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- The orbit \(\{T_t e_k : t \in G\}\) has to stay below a multiple of \(f_0\), but it has to be bounded below.
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- The orbit \(\{T_t e_k : t \in G\}\) has to stay below a multiple of \(f_0\), but it has to be bounded below \(\Rightarrow\) the orbit is supported on a finite subset of \(\mathbb{N}\).
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Lemma

Let \((T_t)_{t \in G}\) be a positive and bounded \textit{group} representation on an \textit{atomic} Banach lattice with o.c. norm. If \(G\) is “good” and if the representation has a fixed point \(f_0 \gg 0\), then \(T_t = \text{id}\) for all \(t \in G\).

- Think of the space as \(\ell^p(\mathbb{N})\). Consider a canonical unit vector \(e_k\).
- \(T_t e_k\) is itself a multiple of a canonical unit vector for each \(t \in G\).
- The orbit \(\{T_t e_k : t \in G\}\) has to stay below a multiple of \(f_0\), but it has to be bounded below \(\Rightarrow\) the orbit is supported on a finite subset of \(\mathbb{N}\).
- We thus obtain a group action of \(G\) on a finite subset of \(\mathbb{N}\).
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If \(G\) is divisible (meaning that \(\forall g \in G \\forall n \in \mathbb{N} \exists h \in G\) s.t. \(nh = g\),

Jochen Glück (Ulm University) Convergence of positive semigroups April 2017 10 / total
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\(\Rightarrow\) The Lemma holds for every divisible group \(G\).
A convergence theorem without time regularity.

**Theorem (Gerlach and G., article in preparation)**

Let \((G, +)\) be a commutative group and let \(S \subseteq G\) be a subsemigroup such that \(\langle S \rangle = G\).

Let \((T_t)_{t \in S}\) be a positive, bounded semigroup representation on a Banach lattice with o.c. norm. Suppose that the semigroup has a fixed point \(f_0 \gg 0\) and that \(T_{t_0}\) is a kernel operator for some \(t_0 \in S\).

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A similar result holds if \(T_{t_0}\) only dominates a kernel operator.
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Examples

(a) $R = \langle [0, \infty) \rangle$ is divisible.

(b) More generally, $R^d = \langle [0, \infty)^d \rangle$ is divisible $\Rightarrow$ convergence theorems for multi-parameter semigroups.

(c) $Q = \langle Q^+ \rangle$ is divisible.

(d) $Z = \langle N \rangle$ is not divisible.

(e) The dyadic numbers $D := \{ k2^n : k \in \mathbb{Z}, n \in \mathbb{N}_0 \} = \langle D \cap [0, \infty) \rangle$ are not divisible.

Remark $Q$ and $D$ are homeomorphic, but not algebraically isomorphic $\Rightarrow$ the algebraic structure is relevant, not the topological structure. The existence of some roots is not sufficient. We need roots of every order.
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