Eventual Positivity of Operator Semigroups

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based on joint work with W. Arendt, D. Daners and J. Kennedy
Assumptions throughout the talk:

(i) Let $E$ be a complex Banach lattice, e.g. $E = C(K)$ for a compact space $K$, or $E = L^p(\Omega, \Sigma, \mu)$.

(ii) Let $(e^{tA})_{t \geq 0}$ be a $C_0$-semigroup on $E$. 
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(ii) Let $(e^{tA})_{t\geq 0}$ be a $C_0$-semigroup on $E$.

**Definition**

The semigroup $(e^{tA})_{t\geq 0}$ is called...

(i) ... *positive*, if $e^{tA}x \geq 0$ for all $x \geq 0$ and for all $t \geq 0$.

(ii) ... *uniformly eventually positive* if there is a $t_0 \in [0, \infty)$ such that $e^{tA}x \geq 0$ for all $x \geq 0$ and for all $t \geq t_0$.

(iii) ... *individually eventually positive* if for each $x \geq 0$ there is a $t_0 \in [0, \infty)$ such that $e^{tA}x \geq 0$ whenever $t \geq t_0$. 
Example

Let $E = \mathbb{C}^3$ and let $B = (u_1, u_2, u_3)$ be the orthonormal basis given by

\[
    u_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad u_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.
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Let the representation matrix of $e^{tA}$ with respect to the basis $B$ be given by

$$\exp(t \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{pmatrix}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-t} \cos t & -e^{-t} \sin t \\ 0 & e^{-t} \sin t & e^{-t} \cos t \end{pmatrix}.$$ 

Then $(e^{tA})_{t \geq 0}$ is individually eventually positive.
Remark

Let $E = \mathbb{C}^n$ and let $(e^{tA})_{t \geq 0}$ be individually eventually positive. For large $t$, we have $e^{tA}e_1 \geq 0, \ldots, e^{tA}e_n \geq 0$.
Thus, $e^{tA}$ is uniformly eventually positive.
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Example

Let \( E = C([-1, 1]) \) and \( F := \{ f \in E : \int f \, d\lambda = 0 \} \). Then \( E = \langle 1 \rangle \oplus F \).
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Example

Let $E = C([-1, 1])$ and $F := \{f \in E : \int f \, d\lambda = 0\}$. Then $E = \langle 1 \rangle \oplus F$.
Let $R$ be the reflection operator on $F$, i.e.

$$Rf(\omega) = f(-\omega) \quad \text{for all } f \in E \text{ and for all } \omega \in [-1, 1].$$

Then $\sigma(R) = \{-1, 1\}$. 
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Then $\sigma(R) = \{-1, 1\}$. The operator

$$A = 0\langle 1 \rangle \oplus (R - 2 \text{id}_F)$$

generates an individually eventually positive semigroup on $E$. 
The following theorem is well-known for positive semigroups.

**Theorem**

Let \((e^{tA})_{t \geq 0}\) be individually eventually positive with growth bound \(\omega\) and spectral bound \(s(A) := \sup \{\text{Re } \lambda : \lambda \in \sigma(A)\}\).

(i) We always have \(s(A) \in \sigma(A)\).

(ii) If \(E = C(K)\) or \(E = L^1(\Omega, \Sigma, \mu)\) or \(E\) is a Hilbert space, then \(s(A) = \omega\).
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**Question**

For positive semigroups, (ii) is also true on \(E = L^p(\Omega, \Sigma, \mu)\) and on \(E = C_0(L)\) for a locally compact space \(L\).

Does this remain true for (individually or uniformly) eventually positive semigroups?
Let $E = C(K)$.

(i) We write $f > 0$ if $f \geq 0$ and $f \neq 0$.

(ii) We write $f \gg 0$ and say that $f$ is strongly positive if $f(\omega) > 0$ for all $\omega \in K$. 

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**Definition**

Let $E = C(K)$. The semigroup $(e^{tA})_{t \geq 0}$ is called individually eventually strongly positive if for each $f > 0$ there is a $t_0 \in [0, \infty)$ such that $e^{tA}f \gg 0$ for all $t \geq t_0$. 
Theorem

If $e^{tA}$ is compact for large $t$, then the following assertions are equivalent:

(i) $(e^{tA})_{t \geq 0}$ is individually eventually strongly positive.

(ii) $s(A)$ is a simple and dominant eigenvalue of $A$ and $\ker(s(A) - A) = \langle u \rangle$ for some $u \gg 0$. 

A glimpse of the proof.

"(ii) $\Rightarrow$ (i)"

Assertion (ii) implies that the spectral projection $P$ corresponding to $s(A)$ is strongly positive and that $e^{tA} \to P$ as $t \to \infty$.

"(i) $\Rightarrow$ (ii)"

To see that $s(A)$ is dominant:

Split off the peripheral spectrum.

Show that the corresponding restriction of the semigroup is positive.

Apply Perron-Frobenius theory of positive semigroups.
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A characterization

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- Split off the peripheral spectrum.
- Show that the corresponding restriction of the semigroup is positive.
- Apply Perron-Frobenius theory of positive semigroups.
Remark

(i) *Further characterizations involve the resolvent of $A$ or the spectral projection corresponding to $s(A)$.*

(ii) *A generalization to arbitrary Banach lattices is possible under additional regularity assumptions on $(e^{tA})_{t \geq 0}$ and on the domain $D(A)$.*
Remark

(i) Further characterizations involve the resolvent of $A$ or the spectral projection corresponding to $s(A)$.

(ii) A generalization to arbitrary Banach lattices is possible under additional regularity assumptions on $(e^{tA})_{t \geq 0}$ and on the domain $D(A)$.

(iii) This generalization can be applied to study e.g. the semigroup generated by the bi-Laplacian on the disk in $\mathbb{R}^2$. 
For $x \in E$, let $d_+(x) := \text{dist}(x, E_+)$ be the distance of $x$ to the positive cone.

**Definition**

Suppose that $s(A) = 0$. The semigroup $(e^{tA})_{t \geq 0}$ is called...

(i) ...*uniformly asymptotically positive* if for each $\varepsilon > 0$ there is a $t_0 \in [0, \infty)$ such that $d_+(e^{tA}x) \leq \varepsilon \|x\|$ for all $x \geq 0$ and for all $t \geq t_0$.

(ii) ...*individually asymptotically positive* if $\lim_{t \to \infty} d_+(e^{tA}x) = 0$ for all $x \geq 0$. 
Suppose that $s(A) = 0$ and that $(e^{tA})_{t \geq 0}$ is bounded and eventually compact. Then the following assertions are equivalent:

(i) $(e^{tA})_{t \geq 0}$ is individually asymptotically positive.

(ii) $(e^{tA})_{t \geq 0}$ is uniformly asymptotically positive.

(iii) $s(A)$ is a dominant eigenvalue and the corresponding spectral projection $P$ is positive.

(iv) $e^{tA}$ converges (in operator norm) to a positive mapping as $t \to \infty$. 

Theorem
Literature

For the finite dimensional case, see e.g.


For the Dirichlet-to-Neumann operator which motivated this work, see


For eventual positivity of the bi-Laplacian, see e.g.