Exercise Course in Functional Analysis: Problem Sheet 2

6. Let $K \in \{\mathbb{R}, \mathbb{C}\}$.
   
   (a) Let $V$ be a normed space over $K \in \{\mathbb{R}, \mathbb{C}\}$ and let $U \subseteq V$ be a dense vector subspace. We endow $U$ with the norm induced by $V$ and consider $U$ as a normed space in its own right.
   
   Define the mapping $T : V' \to U'$ by 
   
   $$T(\varphi) = \varphi|_U \quad \text{for all } \varphi \in V';$$
   
   here, $\varphi|_U$ denotes the restriction of the mapping $\varphi$ to $U$. Prove that $T$ is an isometric isomorphism.
   
   (b) We endow that space $c_0 := \{x = (x_n)_{n \in \mathbb{N}} \subseteq K : \text{all but finitely many } x_n \text{ are 0}\}$ with the norm $\|x\|_\infty = \sup_{n \in \mathbb{N}}|x_n|$. Show that the dual spaces $(c_0)'$ and $(c_{00})'$ are isometrically isomorphic.

7. Let $E$ be a Banach space over $\mathbb{C}$ and let $P \in \mathcal{L}(E)$ be a projection, meaning that $P^2 = P$.
   
   (a) Prove that either $P = 0$ or $\|P\| \geq 1$. \hfill (1)
   
   (b) Let $\lambda \in \mathbb{C}$ be such that $|\lambda| > \|P\|$. Show that $(\lambda I - P)$ is invertible and compute $(\lambda I - P)^{-1}$. \hfill (2)
   
   Hint: Neumann series!
   
   (c) For which $\lambda \in \mathbb{C}$ is $\lambda I - P$ invertible? \hfill (3)

8. Let $K = \mathbb{R}$ and let $K$ be a compact metric space. Consider a sequence of functions $(f_n)_{n \in \mathbb{N}} \subseteq C(K)$ which is decreasing in the sense that $f_{n+1}(x) \leq f_n(x)$ for all $x \in K$ and all $n \in \mathbb{N}$. Let $f \in C(K)$ and assume that, for every $x \in K$, $f_n(x) \to f(x)$ as $n \to \infty$. Prove that $f_n \to f$ with respect to the $\| \cdot \|_\infty$-norm.

   Remark: This assertion is called Dini’s Theorem.

9. Let $K = \mathbb{R}$ and let $P \subseteq C([-1, 1])$ be the space of all polynomial functions on $[-1, 1]$ with real coefficients. Let $m \in C([-1, 1])$ be given by $m(x) = |x|$ for all $x \in [-1, 1]$. Prove, without using the Weierstraß approximation theorem, that $m$ is contained in the closure of $P$ (with respect to the $\| \cdot \|_\infty$-norm).

   Hint: Let $\varepsilon \in (0, 1)$ and let $f_\varepsilon \in C([-1, 1])$ be given by $f_\varepsilon(x) = \sqrt{\varepsilon + x^2}$. First show that $f_\varepsilon$ is contained in the closure of $P$.

Definition. Let $K$ be a compact metric space.

(a) A mapping $T : C(K) \to C(K)$ is called an algebra homomorphism if $T$ is linear and if $T(f \cdot g) = (Tf) \cdot (Tg)$ for all $f, g \in C(K)$.

(b) A mapping $T : C(K) \to C(K)$ is called a unital algebra homomorphism if $T$ is an algebra homomorphism and if $T1_K = 1_K$.

10. Let $K$ be a compact metric space, let $K = \mathbb{R}$.
    
    (a) We write $f \geq 0$ for a function $f \in C(K)$ if $f(x) \geq 0$ for all $x \in K$.

    Let $T : C(K) \to C(K)$ be an algebra homomorphism. Prove that $Tf \geq 0$ whenever $f \geq 0$. \hfill (1)

    (b) Prove that every algebra homomorphism $T : C(K) \to C(K)$ is continuous. \hfill (2)

    (c) Let $S, T : C(K) \to C(K)$ be unital algebra homomorphisms, let $h \in C(K)$ be an injective function and assume that $Sh = Th$. Prove that $S = T$. \hfill (2)
(d) Let $T : C(K) \to C(K)$ be a unital algebra homomorphism, let $h \in C(K)$ be an injective function and assume that the sequence $(T^n h)_{n \in \mathbb{N}}$ converges (with respect to the $\| \cdot \|_\infty$-norm). Show that the sequence $(T^n f)_{n \in \mathbb{N}}$ converges for every $f \in C(K)$. 

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