1. Let $X$ be a Banach space and $A$ an operator on $X$ with domain $D(A)$. Equipped with the graph norm $\| \cdot \|_A := \| x \| + \| Ax \|$ for $x \in D(A)$ the domain $D(A)$ is a normed space.

(i) Show that $A$ is a closed operator if and only if $(D(A), \| \cdot \|_A)$ is a Banach space. (2)

(ii) The operator $A$ is called closable if there is a closed operator $B$ on $X$ such that $A$ is a restriction of $B$ (in symbols: $A \subseteq B$), i.e., $D(A) \subseteq D(B)$ and $Ax = Bx$ for all $x \in D(A)$. Show that if $A$ is closable, then there is a smallest closed operator $\overline{A}$ (called the closure of $A$) with $A \subseteq \overline{A}$. (2)

3. Consider a $C_0$-semigroup $T$ on a Banach space $X$ with generator $(A, D(A))$. Show that the following semigroups $S$ are strongly continuous and determine their generators.

(i) $S(t) := e^{\alpha t}T(\beta t)$ for all $t \geq 0$ where $\alpha \in \mathbb{C}$ and $\beta > 0$ are fixed parameters. (1)

(ii) $S(t) := V^{-1}T(t)V$ for all $t \geq 0$ where $V : Y \rightarrow X$ is an isomorphism from a Banach space $Y$ to $X$. (1)

(iii) $S(t) := T(t)|_Z$ for all $t \geq 0$ where $Z \subseteq X$ is a closed subspace of $X$ with $T(t)Z \subseteq Z$ for all $t \geq 0$. (1)

4. Given two functions $f, g \in L^2(\mathbb{R})$ we define their convolution $f \ast g$ by

$$f \ast g(x) := \int_{\mathbb{R}} f(x - y)g(y) \, dy \quad \text{for } x \in \mathbb{R}.$$ 

Moreover, we denote the Fourier transformation by $\mathcal{F} \in \mathcal{L}(L^2(\mathbb{R}))$.

(i)* Show that for two Schwartz functions $f, g \in \mathcal{S}(\mathbb{R})$ the convolution $f \ast g$ is also contained in $\mathcal{S}(\mathbb{R})$ and satisfies

$$\mathcal{F}(f \ast g) = \sqrt{2\pi} \cdot \mathcal{F}f \cdot \mathcal{F}g.$$

(ii)* Show that the function $\gamma \in \mathcal{S}(\mathbb{R})$ given by $\gamma(x) = e^{-\frac{x^2}{2}}$ for $x \in \mathbb{R}$ is a fixed point of $\mathcal{F}$, i.e.,

$$\mathcal{F}\gamma = \gamma.$$ 

You may use that $\gamma$ is a Schwartz function and that

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \gamma(x) \, dx = 1$$

holds. (Hint: Consider the linear ordinary differential equation $y'(x) + xy(x) = 0$ for $x \in \mathbb{R}$ and show that $\gamma$ and $\mathcal{F}\gamma$ both solve this equation with initial condition $y(0) = 1$. Uniqueness of the solution of this initial value problem then implies the claim.)

The Gauss semigroup $T$ on $L^2(\mathbb{R})$ is defined by

$$T(t)f(x) := \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f(y) \, dy$$

for $x \in \mathbb{R}$, $f \in L^2(\mathbb{R})$ and $t > 0$ as well as $T(0) := I$. Consequently, for each $t > 0$ we have

$$T(t)f = k_t \ast f$$

for $f \in L^2(\mathbb{R})$, if we set

$$k_t(x) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

for all $x \in \mathbb{R}$. 

(iii) Show that $T(t) \in \mathcal{L}(L^2(\mathbb{R}))$ for each $t > 0$.

(iv) Show that the semigroup given by $S(t) := \mathcal{F}T(t)\mathcal{F}^{-1}$ for $t \geq 0$ is the diagonal semigroup on $L^2(\mathbb{R})$ induced by the function

$$q: \mathbb{R} \rightarrow \mathbb{C}, \quad x \mapsto -x^2.$$ 

(Hint: Show this on the dense subspace $\mathcal{F}\mathcal{S}(\mathbb{R}) = \mathcal{S}(\mathbb{R})$ first (as in the lecture) using (i)* and (ii)*. Then use (iii) to show equality on all of $L^2(\mathbb{R})$.)

(v) Conclude that $T$ is a $C_0$-semigroup with generator $A$ where

$$D(A) = W^{2,2}(\mathbb{R}), \quad Af = f'' \text{ for all } f \in D(A).$$