Evolutionsgleichungen: Exercise Sheet 6

Submission in pairs is possible. If you have any questions or need a hint, send a mail to henrik.kreidler@uni-ulm.de

Exercises marked with * are bonus exercises.

1. Let \((A, D(A))\) be an operator on a Banach space \(X\) with \(\rho(A) \neq \emptyset\). We equip \(D(A)\) with the graph norm and consider the operator \((A_1, D(A_1))\) on \(D(A)\), where

\[
D(A_1) := D(A^2), \\
A_1 x := Ax \text{ for } x \in D(A_1).
\]

Show that if \((A_1, D(A_1))\) generates a \(C_0\)-semigroup on \(D(A)\), then \((A, D(A))\) generates a \(C_0\)-semigroup on \(X\).

2. (i) Show with perturbation theory that \((A, D(A))\) with

\[
D(A) := \{f \in C^1(\mathbb{R}): f, f' \in C_0(\mathbb{R})\}, \\
Af(x) := f'(x) + f'(0)e^{-x^2} \text{ for } f \in D(A) \text{ and } x \in \mathbb{R},
\]

generates a \(C_0\)-semigroup on \(C_0(\mathbb{R})\). You may use that \((B, D(B))\) with \(D(B) := D(A)\) and \(Bf := f'\) for \(f \in D(B)\) is the generator of a \(C_0\)-semigroup (namely the shift semigroup).

(ii) For operators \((A, D(A))\) and \((B, D(B))\) on a Banach space \(X\) we define their sum \((A + B, D(A + B))\) by

\[
D(A + B) := D(A) \cap D(B), \\
(A + B)x := Ax + Bx \text{ for } x \in D(A + B).
\]

Show with an example that the sum of two generators does not need to be a generator.

3. Let \((A, D(A))\) be the generator of a \(C_0\)-semigroup \(T\) on a Banach space \(X\). Show that

\[
D(A^{n}) := \{x \in X: x \in D(A^n) \text{ for all } n \in \mathbb{N}\}
\]
is a core for \(A\) (i.e., \(D(A^{n})\) is dense in \(D(A)\) with respect to the graph norm).

(Hint: Show first that \(D(A^{n})\) is dense in \(X\) and then apply a result of the lecture. For the density in \(X\) choose \(\varphi \in C_c^\infty((0, \infty))\) with \(\|\varphi\|_{L^1((0, \infty))} = 1\) and for \(x \in X\) look at the sequence \((x_n)_{n \in \mathbb{N}}\), where

\[
x_n := \int_0^\infty n\varphi(nt)T(t)x \, dt
\]

for \(n \in \mathbb{N}\).)

4. Let \((A, D(A))\) be an operator on a complex Hilbert space \(H\). Show that the following assertions are equivalent.

(a) \(A\) is selfadjoint.

(b) \(A\) is densely defined, symmetric, closed and \(\pm i - A^*\) is injective.

5. Let \((A, D(A))\) be a densely defined, symmetric operator on a complex Hilbert space \(H\). Show that the following assertions are equivalent.
(a) $\mathcal{A}$ is selfadjoint.

(b) $(\pm i - A)D(A)$ is dense in $H$. 

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