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## § 14 Hille's proof

### (14.1) Proof of HY

Let $A$ be d.d., $(0,\infty) \subseteq \varrho(A)$, $\|\lambda R(\lambda, A)\| \leq 1 \ \forall \lambda > 0$.

Define

$$V_n(t) := (I - \tfrac{t}{n}A)^{-n} = (\tfrac{n}{t})^n R(\tfrac{n}{t}, A)^n \quad \forall t > 0 \ \forall n \in \mathbb{N}$$

and $V_n(0) := I \quad \forall n \in \mathbb{N}$.

Then $\|V_n(t)\| \leq 1 \quad \forall n \in \mathbb{N}, t \geq 0$.

Moreover: $\forall n \in \mathbb{N} \ \forall x \in X$: $V_n(t)x = (\tfrac{n}{t})^n R(\tfrac{n}{t}, A)^n x \xrightarrow{t \downarrow 0} x$ (*)

since $\lambda R(\lambda, A) x \xrightarrow{\lambda \to \infty} x$ by (3.6).

By (3.3) (Neumann series representation of $R(\lambda, A)$)

$$(0,\infty) \longrightarrow \mathcal{L}(X), \ \lambda \longmapsto R(\lambda, A)^n$$

is differentiable with

$$\frac{d}{d\lambda} R(\lambda, A)^n = (-1)\, n \cdot R(\lambda, A)^{n+1}$$

$$\Rightarrow \frac{d}{d\lambda} \lambda^n R(\lambda, A)^n = n \lambda^{n-1} R(\lambda, A)^n - n \cdot \lambda^n \cdot R(\lambda, A)^{n+1}$$

$$= ((\lambda - A) - \lambda)\, n \cdot \lambda^{n-1} R(\lambda, A)^{n+1}$$

$$= -A n\, \lambda^{n-1} R(\lambda, A)^{n+1} \quad \forall n \in \mathbb{N}, \lambda > 0$$

$\Rightarrow$ $V_n$ is differentiable on $(0,\infty)$ with

$$\left.\begin{aligned} \frac{d}{dt} V_n(t) &= -A\, n \cdot (\tfrac{n}{t})^{n-1} R(\tfrac{n}{t}, A)^{n+1} \cdot (-\tfrac{n}{t^2}) \\ &= A\, (\tfrac{n}{t})^{n+1} R(\tfrac{n}{t}, A)^{n+1} \end{aligned}\right\} (**)$$

(*) + (**) $\Rightarrow$ $V_n$ is strongly continuous $\forall n \in \mathbb{N}$.

Moreover:

$$V_n(t)x - V_m(t)x = \lim_{\varepsilon \downarrow 0} \int_\varepsilon^{t-\varepsilon} \frac{d}{ds} V_m(t-s) V_n(s) x \, ds$$

$$= \lim_{\varepsilon \downarrow 0} \int_\varepsilon^{t-\varepsilon} \underbrace{[-V_m'(t-s) V_n(s) x + V_m(t-s) V_n'(s) x]}_{(***)} ds$$

with

$$(***) = -A(I - \tfrac{t-s}{m}A)^{-m-1}(I - \tfrac{s}{n}A)^{-n} x$$
$$+ (I - \tfrac{t-s}{m}A)^{-m} A (I - \tfrac{s}{n}A)^{-n-1} x$$

$$= (I - \tfrac{t-s}{m} A)^{-m-1} [(I - \tfrac{s}{n} A) - (I - \tfrac{t-s}{m} A)] (I - \tfrac{s}{n} A)^{-n-1} A x$$

$$= (\tfrac{t-s}{m} - \tfrac{s}{n}) (I - \tfrac{t-s}{m} A)^{-m-1} (I - \tfrac{s}{n} A)^{-n-1} A^2 x$$

$\forall t > 0,\ x \in D(A^2),\ n, m \in \mathbb{N}$.

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$$\Rightarrow \|V_n(t)x - V_m(t)x\| \leq \int_0^t \frac{s}{n} + \frac{t-s}{m}\, ds \cdot \|A^2 x\|$$

$$= \frac{t^2}{2} \left(\frac{1}{n} + \frac{1}{m}\right) \|A^2 x\| \leq \frac{T^2}{2} \left(\frac{1}{n} + \frac{1}{m}\right) \|A^2 x\|$$

$\forall t \in (0, T],\ x \in D(A^2),\ n, m \in \mathbb{N},\ T > 0$.

$D(A^2)$ dicht in $X$

$\Rightarrow$ $(V_n(\cdot)x)_{n \in \mathbb{N}}$ is Cauchy sequence in $C([0,T], X)$ $\forall x \in X$ $\forall T > 0$.

$\Rightarrow \exists\ T(t)x := \lim_{n \to \infty} V_n(t)x \quad \forall t \geq 0,\ \forall x \in X$

and the limit is uniform on $[0,T]$ $\forall T > 0$.

In particular: $t \mapsto T(t)x$ is continuous $\forall x \in X$. and $\|T(t)\| \leq 1\ \forall t \geq 0$

Moreover:

(a) $T(0) = I$

$$\left(\tfrac{n}{t}\right)^n R\left(\tfrac{n}{t}, A\right)^n A x = A \left(\tfrac{n}{t}\right)^n R\left(\tfrac{n}{t}, A\right)^n x \qquad \forall x \in D(A)$$

$$\downarrow_{n \to \infty}$$

$$T(t) A x$$

$A$ closed

$\Rightarrow$ (b) $T(t)x \in D(A)$ and $T(t)Ax = AT(t)x \quad \forall t \geq 0$

$$(\ast\ast) \Rightarrow V_n(t)x - x = \int_0^t \left(\tfrac{n}{s}\right)^{n+1} R\left(\tfrac{n}{s}, A\right)^{n+1} A x\, ds$$

$$= \int_0^t \left(\tfrac{n}{s}\right) R\left(\tfrac{n}{s}, A\right) V_n(s) A x\, ds$$

$$\downarrow_{n \to \infty}$$

$$T(t)x - x = \int_0^t T(s) A x\, ds \qquad \forall x \in D(A),\ t > 0.$$

$$\Rightarrow \text{(c)} \quad \frac{d}{dt} T(t)x = AT(t)x \qquad \forall t > 0,\ x \in D(A)$$

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**(14.2) Lemma**

Let $A$ be d.d. + closed and

$T : [0,\infty) \to \mathcal{L}(X)$ strongly cont. with $\|T(t)\| \leq 1$ $\forall t \geq 0$

satisfying (a), (b), (c).

$\Rightarrow$ $T$ is contractive $C_0$-sgr. with generator $A$.

Proof:

Given $x \in D(A)$ $u(t) := T(t)x$ $\forall t \geq 0$ is the unique solution of

$$(CP) \begin{cases} u \in C^1([0,\infty), X),\ u(t) \in D(A)\ \forall t \geq 0 \\ \dot{u}(t) = Au(t) \quad \forall t > 0 \\ u(0) = x. \end{cases}$$

(uniqueness is proved as in (2.3)).

Now let $s > 0$ and $v(t) := T(t+s)x$ for $x \in D(A)$.

$\Rightarrow$ $v$ is solution of (CP) with $v(0) = T(s)x$.

$\Rightarrow$ $v(t) = T(t)T(s)x$.

$\overset{D(A)\text{ dense}}{\Rightarrow}$ $T(t)T(s) = T(t+s)$ $\forall t,s \geq 0$.

Let $B$ the generator of $T$.

$\overset{(c)}{\Rightarrow}$ $A \subseteq B$

Since $D(A)$ is dense and invariant, it is a core for $B$ (see (12.11)).

$\Rightarrow$ $\overline{D(A)}^{\|\cdot\|_B} = D(B)$ $\overset{A \text{ closed}}{\Rightarrow}$ $A = B$. $\square$

**(14.3) Corollary** (of (14.1))

Let $A$ be the generator of a contractive $C_0$-sgr. $T$.

$\Rightarrow$ $T(t)x = \lim_{n\to\infty} \frac{n}{t} R(\frac{n}{t}, A)x$

uniformly on $[0,J]$ $\forall x \in X$ $\forall J > 0$.

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## § 15 Numerical range

Let $H$ be a complex Hilbert space.

**(15.1) Def.**

Let $A$ be an operator on $H$.

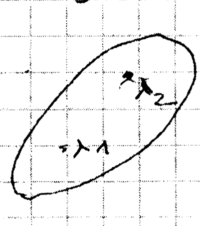
$W(A) := \{(Ax|x) : x \in D(A), \|x\| = 1\} \subseteq \mathbb{C}$

is the *numerical range of $A$*.

**(15.2) Example**

$H = \mathbb{C}^2$, $A \in \mathcal{L}(H)$, $\sigma(A) = \{\lambda_1, \lambda_2\}$

$\Rightarrow$ $W(A)$ is a (possibly degenerate) elliptical disc with foci $\lambda_1, \lambda_2$.




Pf: Exercise (maybe).

**(15.3) Proposition**

$W(A)$ is convex.

Proof: Let $w_1, w_2 \in W(A)$, $w_i = (Ax_i|x_i)$, $i = 1,2$.

Set $H_1 := \operatorname{span}\{x_1, x_2\} \cong \mathbb{C}^2$ and let $P : H \longrightarrow H$ the orthogonal proj. onto $H_1$.

Then $B := PA|_{H_1} \in \mathcal{L}(H_1)$ and

$w_i = (Ax_i|x_i) = (Ax_i|Px_i) = (Bx_i|x_i)$, $i = 1,2$,

$\overset{(15.2)}{\Rightarrow}$ $\lambda w_1 + (1-\lambda) w_2 \in W(B)$ $\forall \lambda > 0$

Let $\lambda \in (0,1)$ and $y \in H_1$ with $(By|y) = \lambda w_1 + (1-\lambda) w_2$, $\|y\| = 1 = \|Py\|$.

$\Rightarrow$ $(By|y) = (PAy|y) = (Ay|Py) = (APy|Py)$

$\Rightarrow$ $\lambda w_1 + (1-\lambda) w_2 \in W(A)$. $\square$

~~[illegible]~~

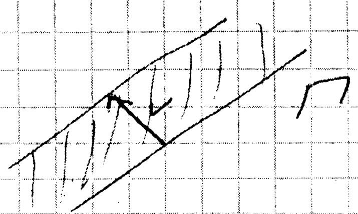
**(15.4) Proposition**

Let $\Gamma \subseteq \mathbb{C}$ be closed + convex. Then

(a) $\mathbb{C} \setminus \Gamma$ is connected or

(b) $\Gamma$ is a closed strip, i.e. $\exists L \subseteq \mathbb{C}$ line, $c \geq 0$:

(5) $\Gamma = \bigcup\limits_{t\in[0,c]} (L + t\cdot v)$




with $v$ normed vector orthogonal to $L$.

($c=0$: $\Gamma$ = line)

Proof:

Suppose $\mathbb{C}\setminus\Gamma$ is not connected.

1) Assume $\exists$ line $L\subseteq\Gamma$.

$\Rightarrow$ $\forall z\in\Gamma$: the line parallel to $L$ through $z$

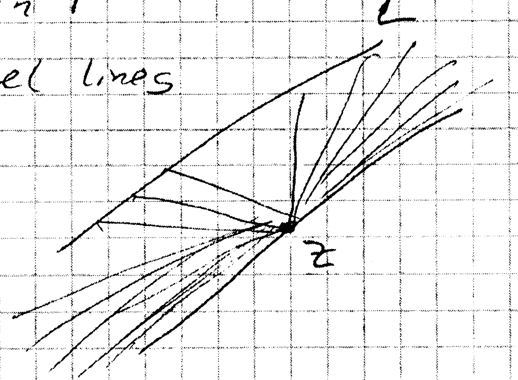
$\Gamma$ closed

also is contained in $\Gamma$

$\Rightarrow$ $\Gamma$ is union of parallel lines

$\Rightarrow$ $\Gamma$ is a strip.

$\Gamma$ convex




2) Suppose that no line is contained in $\Gamma$.

First case: $\Gamma$ is unbounded.

Choose $x_0\in\Gamma$

$\Rightarrow \forall n\in\mathbb{N}: \exists x_n\in\Gamma: |x_n - x_0| \geq n$

Set $v_n := \dfrac{x_n - x_0}{|x_n - x_0|}$ $\forall n\in\mathbb{N}$

$|v_n|=1\ \forall n\in\mathbb{N}$

$\Rightarrow$ $\exists$ Subsequence $(v_{n_k})_{k\in\mathbb{N}}$ with

$\lim\limits_{k\to\infty} v_{n_k} = v\in\mathbb{C},\ |v|=1.$

For each $\tilde{n}\in\mathbb{N}$ we have

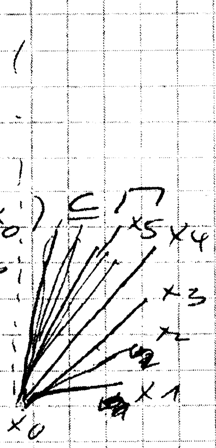
$\lim\limits_{k\to\infty} x_0 + n_{\tilde{n}} v_{n_k} = x_0 + n_{\tilde{n}} v$

Since $x_0 + n_{\tilde{n}} v_{n_k} \in x_0 + [0,1]\cdot(x_{n_k} - x_0) \subseteq \Gamma$

we obtain: $x_0 + n_{\tilde{n}} v \in \Gamma$ $\forall \tilde{n}\in\mathbb{N}$

(since $\Gamma$ is closed)

$\Rightarrow$ $\Gamma$ contains a halfline.



Define

$L(x) := \{y \in \mathbb{R} : x + iy \in \Gamma\}$

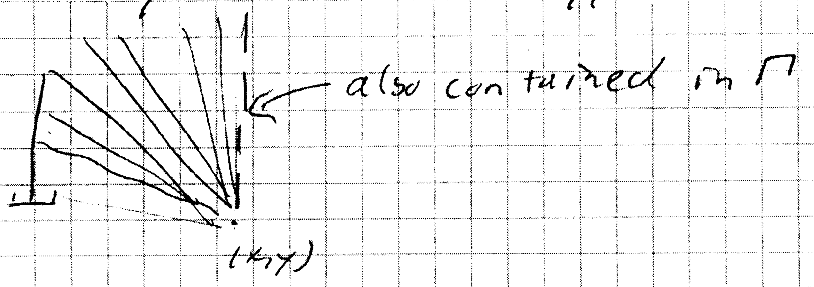
By rotating we may assume

$L(x_0) = [b, \infty)$

for some $b \in \mathbb{R}$.

~~$\Rightarrow$ ... $\{x \in \mathbb{R} : \exists y \in \mathbb{R} : x + iy \in \Gamma\}$~~

$\Rightarrow \quad \forall x + iy \in \Gamma : \quad x + i[y, \infty) \in \Gamma$




Set $Q := \{x \in \mathbb{R} : \exists y \in \mathbb{R} : x + iy \in \Gamma\}$

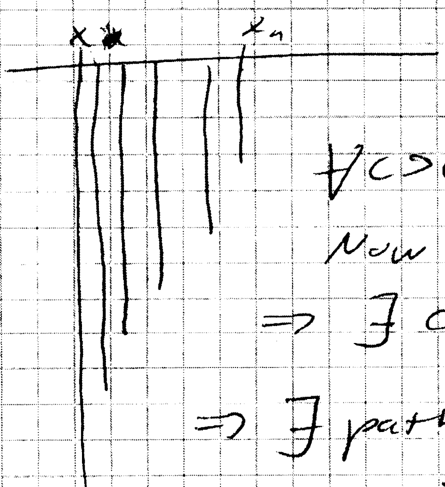
and

$$\beta(x) := \begin{cases} \min \{y \in \mathbb{R} : x + iy \in \Gamma\}, & x \in Q \\ \infty, & x \notin \end{cases}$$

Then

$$\Gamma = \bigcup_{x \in Q} (x + i[\beta(x), \infty)).$$

Suppose $\exists x_n \in Q$ with $x_n \to x$, $\beta(x_n) \to -\infty$.




$\Rightarrow x + i\mathbb{R} \subseteq \Gamma$ ↯

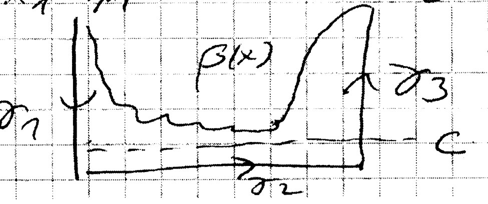
Consequently:

$\forall c > 0 : \inf\{\beta(x) : x \in [-c, c]\} > -\infty$

Now let $x_1 + iy_1, x_2 + iy_2 \in \mathbb{C} \setminus \Gamma$.

$\Rightarrow \exists c \in \mathbb{R} : \beta(x) \geq c \quad \forall x \in [x_1, x_2]$

$\Rightarrow \exists$ path connecting $x_1 + iy_1$ and $x_2 + iy_2$:




$\Rightarrow \mathbb{C} \setminus \Gamma$ connected ↯.

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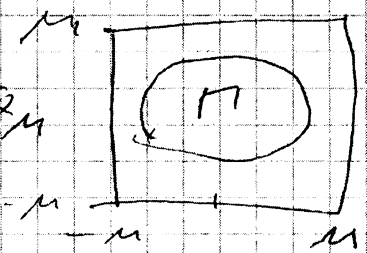
<u>Second case:</u> $\Gamma$ is bounded

Choose $M>0$ with

$$\Gamma \subseteq [-M,M]+i[-M,M] =: Q_M$$




Clearly $\mathbb{C}\setminus Q_M$ is connected.

Take $x \in \mathbb{C}\setminus\Gamma$ and consider the line

$$L := x + i\mathbb{R}.$$

If there are $t_1, t_2 > 0$ with

$$x + it_1 \in \Gamma$$

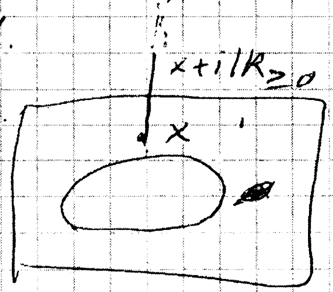
and $x - it_2 \in \Gamma$,

then also $x \in \Gamma$, a contradiction.

Consequently $x + i\mathbb{R}_{\geq 0} \subseteq \mathbb{C}\setminus\Gamma$ or $x - i\mathbb{R}_{\geq 0} \subseteq \mathbb{C}\setminus\Gamma$.

$\Rightarrow$ There is a line segment ~~connecting~~ from $x$ to $\mathbb{C}\setminus Q_M$.

$\Rightarrow \mathbb{C}\setminus\Gamma$ is connected. $\square$



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(15.5) **Lemma**

$\|\lambda x - Ax\| \geq \operatorname{dist}(\overline{W(A)}, \lambda) \cdot \|x\| \quad \forall x \in D(A),\ \lambda \in \mathbb{C}$

**Proof**

Let $x \in D(A) \setminus \{0\}$, $\lambda \in \mathbb{C}$.

$$\Rightarrow \operatorname{dist}(\overline{W(A)}, \lambda) \cdot \|x\|^2 \leq \left(\lambda - \left(A\tfrac{x}{\|x\|} \middle| \tfrac{x}{\|x\|}\right)\right)\|x\|^2 = (\lambda x|x) - (Ax|x) \leq \|\lambda x - Ax\| \cdot \|x\| \quad \square$$

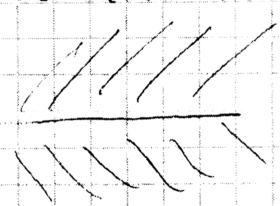
(15.6) **Proposition**

Let $D$ be a connected component of $\mathbb{C} \setminus \overline{W(A)}$. If $\exists \lambda_0 \in D$: $\lambda_0 - A$ surjective then $D \subseteq \rho(A)$ and

$$\|R(\lambda, A)\| \leq \frac{1}{\operatorname{dist}(\lambda, \overline{W(A)})} \quad \forall \lambda \in D.$$

**Pf.**: Apply (15.4) to $\Gamma = \overline{W(A)}$ and use (15.5).

(15.7) **Corollary**



Let $A$ be symmetric.

1. Then a) $\sigma(A) \subseteq \mathbb{R}$ or

b) $\sigma(A) \subseteq \{z \in \mathbb{C}\ \operatorname{Im} z \geq 0\}$ or

c) $\sigma(A) \subseteq \{z \in \mathbb{C}: \operatorname{Im} z \leq 0\}$ or

d) $\sigma(A) = \mathbb{C}$.

2. $A$ is s.a. $\Leftrightarrow$ $\sigma(A) \subseteq \mathbb{R}$.

29.05.17 **Proof**: $A$ symmetric $\Rightarrow \overline{W(A)} \subseteq \mathbb{R}$.

(1) $\overline{W(A)} = \mathbb{R}$.

If $\exists \lambda_0 \in \mathbb{C}$: $\operatorname{Im} \lambda_0 > 0$: $\lambda_0 - A$ surj $\Rightarrow$ c)

" " $\operatorname{Im} \lambda_0 < 0$ " $\Rightarrow$ b)

If both of the above $\Rightarrow$ a)

(2) $\overline{W(A)} \subsetneq \mathbb{R}$.

If $\exists \lambda \in \mathbb{C} \setminus \overline{W(A)}$: $\lambda - A$ surj $\Rightarrow$ a)

Else: $\mathbb{C} \setminus \mathbb{R} \subseteq \sigma(A) \xRightarrow{\sigma(A) \text{ closed}} \sigma(A) = \mathbb{C} \Rightarrow$ d)

For 2. apply (13.10).

<u>Remark</u> $A$ s.a. $\Rightarrow \overline{\text{conv}(A)} = \overline{W(A)}$. 9

(15.8) <u>Corollary</u>

$A \in \mathcal{L}(H) \Rightarrow \sigma(A) \subseteq \overline{W(A)}$

<u>Proof</u>: $\overline{W(A)}$ is bounded

$\Rightarrow \overline{W(A)}$ cannot contain a line

$\Rightarrow \mathbb{C} \setminus \overline{W(A)}$ connected.

Let $|\lambda| > \|A\|$. $\Rightarrow \lambda \in \rho(A)$

$\underset{(15.6)}{\Rightarrow} \mathbb{C} \setminus \overline{W(A)} \subseteq \rho(A)$

$\Rightarrow \sigma(A) \subseteq \overline{W(A)}$. □

(15.9) <u>Proposition</u>

~~Let $\overline{W(A)} \neq H$~~

If $A$ is d.d. and $\overline{W(A)} \neq \mathbb{C}$,

then $A$ is closable. (and $\overline{W(\bar{A})} = \overline{W(A)}$).

<u>proof</u>:

Take $\lambda \in \mathbb{C} \setminus \overline{W(A)}$. By the Hahn-Banach sep. theorem for convex sets (applied to the Bspace $\mathbb{C}$) we find a halfplane which contains $\overline{W(A)}$ but not $\lambda$.

Replacing $A$ bei $c + e^{i\theta}A$ we may assume that

$$\overline{W(A)} \subseteq \{(x+iy) : x \leq 0\}$$

$\Rightarrow \text{Re}(Ax|x) \leq 0 \quad \forall x \in H$.

$\Rightarrow A$ is dissipative

$\Rightarrow A$ is closable □